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Periodic Solutions for Non-Linear Systems of Boundary Value Problems

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Abstract

In this work, we investigate the periodic solutions for new non-linear system of boundary value problems by using the numerical analytic method, which was introduced by Samoilenko. These investigations lead us to improving and extending the results of Samoilenko.

Keywords: *Numerical-analytic methods, Existence of periodic solutions, Nonlinear system, Boundary value problem.*

I Introduction

Many results about the existence and approximation of periodic solutions for system of non–linear differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko [6,7] which had been later applied in many studies [1, 2, 3, 4, 5].

Samoilenko [6, 7] has used the numerical-analytic methods of periodic solutions for ordinary differential equation with boundary and boundary integral conditions which has the form:

$$\frac{dx}{dt} = f(t,x)$$

$$\int_0^T x(t)\,dt = d\,, \quad d \in R^n,$$

Where $x \in D$, D is closed and bounded subset of R^n, the vector function.

$f(t,x)$ is defined on the domain:

$$(t,x) \in R^1 \times D = (-\infty,\infty) \times D\,,$$

Which is continuous in t and x and periodic in t of period T.

Our work, we investigate the periodic solution for new non-linear system of differential equations with boundary integral conditions which has the form:

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax + f(t,x,y,z) \\ \int_0^T x(t)\,dt &= d_1 \\ \frac{dy}{dt} &= By + g(t,x,y,z) \\ \int_0^T y(t)\,dt &= d_2 \\ \frac{dz}{dt} &= Cz + h(t,x,y,z) \\ \int_0^T z(t)\,dt &= d_3 \end{aligned}\right\} \quad \cdots \text{(BVP)}$$

Where $x \in D \subset R^n, y \in D_1 \subset R^m$ and $z \in D_2 \subset R^k$.The domains D, D_1 and D_2 are closed and bounded.

Let the vector functions $f(t,x,y,z)$, $g(t,x,y,z)$ and $h(t,x,y,z)$ are defined and continuous on the domain:

$$(t,x,y,z) \in R^1 \times D \times D_1 \times D_2 \qquad \cdots (1)$$

And periodic in t of period T, and $A = (A_{ij})$, $A_1 = (A_{1ij}), A_2 = (A_{2ij})$, $B = (B_{ij}), B_1 = (B_{1ij})$, $B_2 = (B_{2ij})$, $C = (C_{ij})$, $C_1 = (C_{1ij})$, $C_2 = (C_{2ij})$ are

$n \times n$ non-negative matrices, also $e_1 = (e_{11}, e_{12}, \dots), e_2 = (e_{21}, e_{22}, \dots)$, $e_3 = (e_{31}, e_{32}, \dots)$ are positive constant vectors.

Suppose that the functions f, g and h satisfy the following inequalities:

$$\|f(t,x,y,z)\| \le M_1,\ \|g(t,x,y,z)\| \le M_2,\ \|h(t,x,y,z)\| \le M_3 \qquad \cdots (2)$$

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + \\ + K_3\|z_1 - z_2\| \qquad \cdots (3)$$

$$\|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)\| \le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| + \\ + L_3\|z_1 - z_2\| \qquad \cdots (4)$$

$$\|h(t,x_1,y_1,z_1) - h(t,x_2,y_2,z_2)\| \le P_1\|x_1 - x_2\| + P_2\|y_1 - y_2\| + \\ + P_3\|z_1 - z_2\| \qquad \cdots (5)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$, $z, z_1, z_2 \in D_2$.

Where $M_1, M_2, M_2, K_1, K_2, K_3, L_1, L_2, L_3, P_1, P_2, P_3$ are positive constants.

Provided that:

$$\left\|e^{A(t-s)}\right\| \le \frac{\gamma_1}{\lambda_1}, \quad \left\|e^{B(t-s)}\right\| \le \frac{\gamma_2}{\lambda_2}, \quad \left\|e^{C(t-s)}\right\| \le \frac{\gamma_3}{\lambda_3} \qquad \cdots (6)$$

Where $\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2, \lambda_3$ are positive conistants and $\|.\| = \max_{t\in[0,T]}|.|$.

We define the non-empty sets as follows:

$$\left.\begin{aligned} D_\rho &= D - \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 - \frac{\gamma_1}{\lambda_1}b(x_0)T \\ D_\sigma &= D_1 - \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 - \frac{\gamma_2}{\lambda_2}c(y_0)T \\ D_\varphi &= D_2 - \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 - \frac{\gamma_3}{\lambda_3}e(z_0)T \end{aligned}\right\} \qquad \cdots (7)$$

Where

$$b(x_0) = R_1 \left\| \frac{x_0}{A}(e^{AT} - E) - d_1 + \int_0^T Lf(t,x,y,z)ds \right\|,$$
$$c(y_0) = R_2 \left\| \frac{y_0}{B}(e^{BT} - E) - d_2 + \int_0^T Lg(t,x,y,z)dt \right\|,$$
$$e(y_0) = R_3 \left\| \frac{z_0}{C}(e^{CT} - E) - d_3 + \int_0^T Lh(t,x,y,z)dt \right\|,$$

$$R_1 = \left\| \frac{\mathrm{A}^2}{(e^{\mathrm{AT}} - \mathrm{TAE} - \mathrm{E})} \right\|, \quad R_2 = \left\| \frac{\mathrm{B}^2}{(e^{\mathrm{BT}} - \mathrm{TBE} - \mathrm{E})} \right\|, R_3 = \left\| \frac{\mathrm{C}^2}{(e^{\mathrm{CT}} - \mathrm{TCE} - \mathrm{E})} \right\|$$

Furthermore, we suppose that the largest eigen- value of the matrix

$$\Omega = \begin{pmatrix} \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_1r_1 & \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_2r_1 & \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_3r_1 \\ \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_1r_2 & \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_2r_2 & \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_3r_2 \\ \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1r_3 & \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2r_3 & \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_3r_3 \end{pmatrix} \text{ Does not exceed unity}$$

$$\frac{\omega_1}{3} + \frac{\omega_4}{6} + \frac{2\omega_2}{3} + \frac{2{\omega_1}^2}{3} < 1 \quad , \qquad \cdots \ (8)$$

where $\omega_1 = \frac{T}{2}(K_1r_1\frac{\gamma_1}{\lambda_1} + L_2r_2\frac{\gamma_2}{\lambda_2} + P_3r_3\frac{\gamma_3}{\lambda_3})$

$$\omega_2 = \frac{T^2}{2}(\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}r_1r_2(K_2L_1 - K_1L_2) + \frac{\gamma_1}{\lambda_1}\frac{\gamma_3}{\lambda_3}r_1r_3(K_3P_1 - K_1P_3) + \frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}r_2r_3(L_3P_2 - L_2P_3)),$$

$$\omega_3 = \frac{T^3}{2}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}r_1r_2r_3(K_1(L_2P_3 - L_3P_2) + K_2(L_3P_1 - L_1P_3) + K_3(L_1P_2 - L_2P_1)),$$

$$\omega_4 = 8{\omega_1}^3 + 180\omega_3 + 36\omega_1\omega_2 + 12(81{\omega_3}^2 + 12\omega_3{\omega_1}^3 + 54\omega_1\omega_2\omega_3 - 3{\omega_1}^2{\omega_2}^2 - 12{\omega_2}^3)^{(1/2)})^{(1/3)},$$

Also

$$r_1 = 1 + \frac{\gamma_1}{\lambda_1}T^2R_1\,, \qquad r_2 = 1 + \frac{\gamma_2}{\lambda_2}T^2R_2\,, \qquad r_3 = 1 + \frac{\gamma_3}{\lambda_3}T^2R_3\,.$$

Define a sequence of functions:

$\{x_m(t,x_0,y_0,z_0), y_m(t,x_0,y_0,z_0), z_m(t,x_0,y_0,z_0)\}_{m=0}^{\infty}$ by

$$x_{m+1}(t,x_0,y_0,z_0) = x_0 e^{\mathrm{At}} + \int_0^t e^{A(t-s)}\,[f(s,x_m(s,x_0,y_0,z_0).$$

$$y_m(s,x_0,y_0,z_0), z_m(s,x_0,y_0,z_0)) -$$

$$\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),$$

$$z_m(s,x_0,y_0,z_0))ds - \rho]ds \qquad \cdots (9)$$

with $x_0(t,x_0,y_0,z_0) = x_0 e^{\mathrm{At}}$

where

$$\rho = \frac{\mathrm{A}^2}{(e^{\mathrm{AT}}-\mathrm{TAE}-\mathrm{E})}\left[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}}-\mathrm{E}) - d_1 + \int_0^T Lf(t,x,y,z)dt\right];\ det\mathrm{A} \neq 0$$

and $det(e^{\mathrm{AT}}-\mathrm{TAE}-\mathrm{E}) \neq 0$.

and $Lf(t,x,y,z) = \int_0^T e^{A(t-s)}\,[f(s,x,y,z) -$

$$\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x,y,z)ds]ds$$

And

$$y_{m+1}(t,x_0,y_0,z_0) = y_0 e^{Bt} + \int_0^t e^{B(t-s)}\,[g(s,x_m(s,x_0,y_0,z_0),$$

$$y_m(s,x_0,y_0,z_0), z_m(s,x_0,y_0,z_0)) -$$

$$\frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,g(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),$$

$$z_m(s,x_0,y_0,z_0))ds - \sigma]ds \qquad \cdots (10)$$

with $y_0(t,x_0,y_0,z_0) = y_0 e^{Bt}$

where

$$\sigma = \frac{\mathrm{B}^2}{(e^{\mathrm{BT}}-\mathrm{TBE}-\mathrm{E})}\left[\frac{y_0}{\mathrm{B}}(e^{\mathrm{BT}}-\mathrm{E}) - d_2 + \int_0^T Lg(t,x,y,z)ds\right]; det\mathrm{B} \neq 0,$$

$det(e^{\mathrm{BT}}-\mathrm{TBE}-\mathrm{E}) \neq 0$.

$$Lg(t,x,y,z) = \int_0^t e^{B(t-s)} [g(s,x,y,z) -$$
$$\frac{\mathrm{B}}{(e^{\mathrm{BT}} - \mathrm{E})} \int_0^T e^{B(T-s)} g(s,x,y,z)ds]ds$$

Also

$$z_{m+1}(t,x_0,y_0,z_0) = z_o e^{Ct} + \int_0^t e^{C(t-s)} [h(s, x_m(s,x_0,y_0,z_0),$$
$$y_m(s,x_0,y_0,z_0), z_m(s,x_0,y_0,z_0)) -$$
$$\frac{\mathrm{C}}{(e^{\mathrm{CT}} - \mathrm{E})} \int_0^T e^{C(T-s)} h(s, x_m(s,x_0,y_0,z_0), y_m(s,x_0,y_0,z_0),$$
$$z_m(s,x_0,y_0,z_0))ds - \varphi]ds \qquad \cdots (11)$$

with $z_0(t,x_0,y_0,z_0) = z_0 e^{Ct}$

Where

$$\varphi = \frac{\mathrm{C}^2}{(e^{\mathrm{CT}} - \mathrm{TCE} - \mathrm{E})} \left[\frac{z_0}{\mathrm{C}}(e^{\mathrm{CT}} - \mathrm{E}) - d_3 + \int_0^T Lh(t,x,y,z,)dt\right]; det\mathrm{C} \neq 0,$$

$$det(e^{\mathrm{CT}} - \mathrm{TCE} - \mathrm{E}) \neq 0$$

$$Lh(t,x,y,z) = \int_0^t e^{C(t-s)} [h(s,x,y,z) -$$
$$\frac{\mathrm{C}}{(e^{\mathrm{CT}} - \mathrm{E})} \int_0^T e^{C(T-s)} h(s,x,y,z)ds]ds, \text{ m=0, 1, 2, …}$$

By using lemma3.1 [6], we can state and proof the following lemma:

Lemma 1: *Suppose that the functions f, g and f be vectors which are defined in the interval $[0,T]$, then the following inequality holds:*

$$\begin{pmatrix} \|F_1(t,x_0,y_0,z_0)\| \\ \|F_2(t,x_0,y_0,z_0)\| \\ \|F_3(t,x_0,y_0,z_0)\| \end{pmatrix} \leq \begin{pmatrix} \alpha_1(t)\dfrac{\gamma_1}{\lambda_1}M_1 \\ \alpha_2(t)\dfrac{\gamma_2}{\lambda_2}M_2 \\ \alpha_3(t)\dfrac{\gamma_3}{\lambda_3}M_3 \end{pmatrix}, \qquad \ldots (12)$$

for $0 \leq t \leq T$, $\alpha_1(t) \leq \frac{T}{2}$, $\alpha_2(t) \leq \frac{T}{2}$, $\alpha_3(t) \leq \frac{T}{2}$, Where

$$F_1(t,x_0,y_0,z_0) = \int_0^t e^{A(t-s)}\,[f(s,x_0,y_0,z_0) -$$
$$-\frac{A}{(e^{AT}-E)}\int_0^T e^{A(T-s)}\,f(s,x_0,y_0,z_0)ds]ds$$

$$F_2(t,x_0,y_0,z_0) = \int_0^t e^{B(t-s)}\,[g(s,x_0,y_0,z_0) -$$
$$-\frac{B}{(e^{BT}-E)}\int_0^T e^{B(T-s)}\,g(s,x_0,y_0,z_0)ds]ds$$

$$F_3(t,x_0,y_0,z_0) = \int_0^t e^{C(t-s)}\,[h(s,x_0,y_0,z_0) - \delta -$$
$$-\frac{C}{(e^{CT}-E)}\int_0^T e^{C(T-s)}\,h(s,x_0,y_0,z_0)ds\,]ds$$

And

$$\alpha_1(t) = \frac{t\left(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|\right) + T\left(e^{\|A\|T} - e^{\|A\|(T-t)}\right)}{(e^{\|A\|T} - \|E\|)}$$

$$\alpha_2(t) = \frac{t\left(2e^{\|B\|(T-t)} - e^{\|B\|T} - \|E\|\right) + T\left(e^{\|B\|T} - e^{\|B\|(T-t)}\right)}{(e^{\|B\|T} - \|E\|)}$$

$$\alpha_3(t) = \frac{t\left(2e^{\|C\|(T-t)} - e^{\|C\|T} - \|E\|\right) + T\left(e^{\|C\|T} - e^{\|C\|(T-t)}\right)}{(e^{\|C\|T} - \|E\|)}$$

Proof:

$$\|F_1(t,x_0,y_0,z_0)\| \le$$
$$\left(\|E\| - \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_0^t \left\|e^{A(t-s)}\right\| \|f(s,x_0,y_0,z_0)\|ds +$$
$$+\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T \left\|e^{A(t-s)}\right\| \|f(s,x_0,y_0,z_0)\|\,ds$$
$$\le \alpha_1(t)\frac{\gamma_1}{\lambda_1}M_1 \qquad \ldots(13)$$

And similarly

$$\|F_2(t,x_0,y_0,z_0)\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2}M_2 \qquad \ldots(14)$$

$$\|F_3(t,x_0,y_0,z_0)\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}M_3 \qquad \ldots(15)$$

from (13), (14) and (15) we conclude that the inequality (12) holds. □

II Approximation of Periodic Solution for (BVP)

The investigation of approximate solution of (BVP) will be introduced by the following theorem:

Theorem 1: *Let the vector functions* f, g *and* h *are defined and continuous on the domain (1) and periodic in t of period T. Suppose that these functions satisfy the inequalities (2), (3), (4), (5) and the conditions (6), (7) and (8), then there exist a sequences of functions (9), (10) and (11), converges uniformly on the domain:*

$$(t, x_0, y_0, z_0) \in [0, T] \times D_\rho \times D_\sigma \times D_\varphi \qquad \cdots (16)$$

To the limit function $\begin{pmatrix} x^0(t, x_0, y_0, z_0) \\ y^0(t, x_0, y_0, z_0) \\ z^0(t, x_0, y_0, z_0) \end{pmatrix}$ which is continuous in the domain

(16) and periodic in t of period T and satisfies the following vector form:

$$\begin{pmatrix} x(t, x_0, y_0, z_0) \\ y(t, x_0, y_0, z_0) \\ z(t, x_0, y_0, z_0) \end{pmatrix} =$$

$$\begin{pmatrix} x_0 e^{\mathrm{At}} + \int_0^t e^{A(T-s)} \left[f(s, x, y, z) - \frac{\mathrm{A}}{(e^{\mathrm{AT}} - \mathrm{E})} \int_0^T e^{A(T-s)} f(s, x, y, z) ds - \rho \right] ds \\ y_o e^{Bt} + \int_0^t e^{B(t-s)} \left[g(s, x, y, z) - \frac{\mathrm{B}}{(e^{\mathrm{BT}} - \mathrm{E})} \int_0^T e^{B(T-s)} g(s, x, y, z) ds - \sigma \right] ds \\ z_o e^{Ct} + \int_0^t e^{C(t-s)} \left[h(s, x, y, z) - \frac{\mathrm{C}}{(e^{\mathrm{CT}} - \mathrm{E})} \int_0^T e^{C(T-s)} h(s, x, y, z) ds - \varphi \right] ds \end{pmatrix} \qquad \cdots (17)$$

And it is a unique solution of (BVP) which satisfies the following inequality:

$$\begin{pmatrix} \|x^0(t, x_0, y_0, z_0) - x_m(t, x_0, y_0, z_0)\| \\ \|y^0(t, x_0, y_0, z_0) - y_m(t, x_0, y_0, z_0)\| \\ \|z^0(t, x_0, y_0, z_0) - z_m(t, x_0, y_0, z_0)\| \end{pmatrix} \leq \Omega^m (E - \Omega)^{-1} \Psi_1$$

where $\Psi_1 = \begin{pmatrix} \frac{T}{2} \frac{\gamma_1}{\lambda_1} M_1 + \frac{\gamma_1}{\lambda_1} b(x_0) T \\ \frac{T}{2} \frac{\gamma_2}{\lambda_2} M_2 + \frac{\gamma_2}{\lambda_2} c(y_0) T \\ \frac{T}{2} \frac{\gamma_3}{\lambda_3} \mathrm{M}_3 + \frac{\gamma_3}{\lambda_3} e(z_0) T \end{pmatrix}$, for all t ∈ [0,T] and $x_0 \in D_\rho$, $y_0 \in D_\sigma$, $z_0 \in D_\varphi$.

Provided that:

$$\begin{pmatrix} \|x(t,x_0,y_0,z_0)-x_0\| \\ \|y(t,x_0,y_0,z_0)-y_0\| \\ \|z(t,x_0,y_0,z_0)-z_0\| \end{pmatrix} \le \begin{pmatrix} \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1+\frac{\gamma_1}{\lambda_1}b(x_0)T \\ \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2+\frac{\gamma_2}{\lambda_2}c(y_0)T \\ \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3+\frac{\gamma_3}{\lambda_3}e(z_0)T \end{pmatrix} \qquad \cdots (18)$$

Proof: Setting m=0 in (9), (10) and (11), we have

$$\|x_1(t,x_0,y_0,z_0)-x_0\|$$

$$\le \left\| \int_0^t [e^{A(t-s)} f(s,x_0,y_0,z_0) - \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)} f(s,x_0,y_0,z_0)ds]ds \right\|$$

$$+\int_0^t \|e^{A(t-s)}\| \left\| \frac{\mathrm{A}^2}{(e^{\mathrm{AT}}-\mathrm{TAE}-\mathrm{E})}\Big[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}}-\mathrm{E})-d_1+\right.$$

$$\left. +\int_0^T Lf(t,x_0,y_0,z_0)dt\Big]\right\| ds$$

By using Lemma1 we get

$$\|x_1(t,x_0,y_0,z_0)-x_0\| \le \alpha_1(t)\frac{\gamma_1}{\lambda_1}M_1+\frac{\gamma_1}{\lambda_1}b(x_0)T.$$

Hence $x_1(t,x_0,y_0,z_0) \in D_\rho$ for all t ∈ [0,T],

$$\|y_1(t,x_0,y_0,z_0)-y_0\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2}M_2+\frac{\gamma_2}{\lambda_2}c(y_0)T$$

Hence $y_1(t,x_0,y_0,z_0) \in D_\sigma$ for all t ∈ [0,T],

$$\|z_1(t,x_0,y_0,z_0)-z_0\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}M_3+\frac{\gamma_3}{\lambda_3}e(z_0)T$$

Hence $z_1(t,x_0,y_0,z_0) \in D_\varphi$ for all t ∈ [0,T]

Now by mathematical induction, we can prove the following inequalities for $m = 0, 1, 2, \ldots,$

$$\|x_m(t,x_0,y_0,z_0)-x_0\| \le \alpha_1(t)\frac{\gamma_1}{\lambda_1}M_1+\frac{\gamma_1}{\lambda_1}b(x_0)T, \qquad \cdots (19)$$

$$\|y_m(t,x_0,y_0,z_0)-y_0\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2}M_2+\frac{\gamma_2}{\lambda_2}c(y_0)T, \qquad \cdots (20)$$

$$\|z_m(t,x_0,y_0,z_0)-z_0\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}M_3+\frac{\gamma_3}{\lambda_3}e(z_0)T. \quad \cdots (21)$$

That is $x_m(t,x_0,y_0,z_0)\in D_\rho$ $y_m(t,x_0,y_0,z_0)\in D_\sigma$ and $z_m(t,x_0,y_0,z_0)\in D_\varphi$ for all t ∈ [0,T]

Next, we shall prove that the sequence of functions (9), (10) and (11) are convergent uniformly on the domain (16).Then by mathematical induction we can prove the following inequalities:

$$\|x_{m+1}(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\|$$

$$\le \alpha_1(t)\frac{\gamma_1}{\lambda_1}K_1r_1\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_1(t)\frac{\gamma_1}{\lambda_1}K_2r_1\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_1(t)\frac{\gamma_1}{\lambda_1}K_3r_1\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|, \quad \cdots (22)$$

$$\|y_{m+1}(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\|\le$$

$$\alpha_2(t)\frac{\gamma_2}{\lambda_2}L_1r_2\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_2(t)\frac{\gamma_2}{\lambda_2}L_2r_2\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_2(t)\frac{\gamma_2}{\lambda_2}L_3r_2\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|, \quad \cdots (23)$$

$$\|z_{m+1}(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\|$$

$$\le \alpha_3(t)\frac{\gamma_3}{\lambda_3}P_1r_3\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_3(t)\frac{\gamma_3}{\lambda_3}P_2r_3\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|+$$

$$+\alpha_3(t)\frac{\gamma_3}{\lambda_3}P_3r_3\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\| \quad \cdots (24)$$

Rewrite (22), (23) and (24) in a vector form i. e.

$$\Psi_{m+1}(t)\le \Omega(t)\Psi_m(t) \quad \cdots (25)$$

$$\Psi_{m+1} = \begin{pmatrix} \|x_{m+1}(t,x_0,y_0,z_0) - x_m(t,x_0,y_0,z_0)\| \\ \|y_{m+1}(t,x_0,y_0,z_0) - y_m(t,x_0,y_0,z_0)\| \\ \|z_{m+1}(t,x_0,y_0,z_0) - z_m(t,x_0,y_0,z_0)\| \end{pmatrix}$$

$$\Psi_m = \begin{pmatrix} \|x_m(t,x_0,y_0,z_0) - x_{m-1}(t,x_0,y_0,z_0)\| \\ \|y_m(t,x_0,y_0,z_0) - y_{m-1}(t,x_0,y_0,z_0)\| \\ \|z_m(t,x_0,y_0,z_0) - z_{m-1}(t,x_0,y_0,z_0)\| \end{pmatrix}$$

And

$$\Omega(t) = \begin{pmatrix} \alpha_1(t)\frac{\gamma_1}{\lambda_1}K_1r_1 & \alpha_1(t)\frac{\gamma_1}{\lambda_1}K_2r_1 & \alpha_1(t)\frac{\gamma_1}{\lambda_1}K_3r_1 \\ \alpha_2(t)\frac{\gamma_2}{\lambda_2}L_1r_2 & \alpha_2(t)\frac{\gamma_2}{\lambda_2}L_2r_2 & \alpha_2(t)\frac{\gamma_2}{\lambda_2}L_3r_2 \\ \alpha_3(t)\frac{\gamma_3}{\lambda_3}P_1r_3 & \alpha_3(t)\frac{\gamma_3}{\lambda_3}P_2r_3 & \alpha_3(t)\frac{\gamma_3}{\lambda_3}P_3r_3 \end{pmatrix}$$

Now, we take the maximum value for the both sides of the inequalities (25) we get

$$\Psi_{m+1} \leq \Omega\Psi_m \qquad \cdots \ (26)$$

Where $\Omega = \max_{t\in[0,T]} \Omega(t)$

$$\Omega = \begin{pmatrix} \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_1r_1 & \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_2r_1 & \frac{\gamma_1}{\lambda_1}\frac{T}{2}K_3r_1 \\ \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_1r_2 & \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_2r_2 & \frac{\gamma_2}{\lambda_2}\frac{T}{2}L_3r_2 \\ \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1r_3 & \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2r_3 & \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_3r_3 \end{pmatrix}$$

And by repetition (26) we find $\Psi_{m+1} \leq \Omega^m\Psi_1$ and also we get

$$\sum_{i=1}^{m} \Psi_i \leq \sum_{i=1}^{m} \Omega^{i-1}\,\Psi_1 \ . \qquad \cdots (27)$$

By condition (8) then the sequence (27) is uniformly convergent that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Omega^{i-1}\,\Psi_1 = \sum_{i=1}^{\infty} \Omega^{i-1}\,\Psi_1 = (E-\Omega)^{-1}\Psi_1 \qquad \cdots (28)$$

Let

$$\lim_{m\to\infty} \begin{pmatrix} x_m(t,x_0,y_0,z_0) \\ y_m(t,x_0,y_0,z_0) \\ z_m(t,x_0,y_0,z_0) \end{pmatrix} = \begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix} \qquad \cdots \ (29)$$

Since the sequence of functions (3), (4) and (5) is defined and continuous in the domain (1) then the limiting function $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ is also defined and continuous in the same domain.

Moreover, by using Lemma1, the relation (29) and proceeding (9), (10) and (11) to the limit $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ when m→ ∞ , the equality (28) satisfied for all $m \geq 0$, and this show that the limiting function $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ is the solution of (BVP).

Finally, we have to show that $\begin{pmatrix} x(t,x_0,y_0,z_0) \\ y(t,x_0,y_0,z_0) \\ z(t,x_0,y_0,z_0) \end{pmatrix}$ is a unique solution of (BVP).

Let $\begin{pmatrix} \overline{x}(t,x_0,y_0,z_0) \\ \overline{y}(t,x_0,y_0,z_0) \\ \overline{z}(t,x_0,y_0,z_0) \end{pmatrix}$ be another solution of (BVP) where

$$\overline{x}(t,x_0,y_0,z_0) = x_0 e^{\mathrm{At}} + \int_0^t e^{A(t-s)} [f(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$,\overline{z}(s,x_0,y_0,z_0)) - \frac{\mathrm{A}}{(e^{\mathrm{AT}} - \mathrm{E})} \int_0^T e^{A(T-s)} f(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$,\overline{z}(s,x_0,y_0,z_0))ds - \frac{\mathrm{A}^2}{(e^{\mathrm{AT}} - \mathrm{TAE} - \mathrm{E})} \Big[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}} - \mathrm{E}) - d_1 +$$
$$+ \int_0^T Lf(t,\overline{x},\overline{y},\overline{z})dt\Big]]ds,$$

$$\overline{y}(t,x_0,y_0,z_0) = y_0 e^{\mathrm{Bt}} + \int_0^t e^{B(t-s)} [g(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$,\overline{z}(s,x_0,y_0,z_0)) - \frac{\mathrm{B}}{(e^{\mathrm{BT}} - \mathrm{E})} \int_0^T e^{B(T-s)} g(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$,\overline{z}(s,x_0,y_0,z_0))ds - \frac{\mathrm{B}^2}{(e^{\mathrm{BT}} - \mathrm{TBE} - \mathrm{E})} \Big[\frac{y_0}{\mathrm{B}}(e^{\mathrm{BT}} - \mathrm{E}) - d_2 +$$

$$+ \int_0^T Lg(t,\overline{x},\overline{y},\overline{z})dt\Big]]ds,$$

$$\overline{z}(t,x_0,y_0,z_0) = z_0 e^{Ct} + \int_0^t e^{C(t-s)}\,[h(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$,\overline{z}(s,x_0,y_0,z_0) - \frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\,h(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),$$

$$\overline{z}(s,x_0,y_0,z_0))ds - \frac{\mathrm{C}^2}{(e^{\mathrm{CT}}-\mathrm{TCE}-\mathrm{E})}\Big[\frac{z_0}{\mathrm{C}}(e^{\mathrm{CT}}-\mathrm{E}) - d_3 +$$

$$+\int_0^T Lh(t,\overline{x},\overline{y},\overline{z})dt\Big]]ds$$

$$\therefore \|x(t,x_0,y_0,z_0) - \overline{x}(t,x_0,y_0,z_0)\|$$

$$\leq \alpha_1(t)\frac{\gamma_1}{\lambda_1}K_1 r_1\|x(t,x_0,y_0,z_0) - \overline{x}(t,x_0,y_0,z_0)\| +$$

$$+\alpha_1(t)\frac{\gamma_1}{\lambda_1}K_2 r_1\|y(t,x_0,y_0,z_0) - \overline{y}(t,x_0,y_0,z_0)\| +$$

$$+\alpha_1(t)\frac{\gamma_1}{\lambda_1}K_3 r_1\|z(t,x_0,y_0,z_0) - \overline{z}(t,x_0,y_0,z_0)\| \quad \cdots (30)$$

Similarly

$$\|y(t,x_0,y_0,z_0) - \overline{y}(t,x_0,y_0,z_0)\|$$

$$\leq \alpha_2(t)\frac{\gamma_2}{\lambda_2}L_1 r_2\|x(t,x_0,y_0,z_0) - \overline{x}(t,x_0,y_0,z_0)\| +$$

$$+\alpha_2(t)\frac{\gamma_2}{\lambda_2}L_2 r_2\|y(t,x_0,y_0,z_0) - \overline{y}(t,x_0,y_0,z_0)\|$$

$$+\alpha_2(t)\frac{\gamma_2}{\lambda_2}L_3 r_2\|z(t,x_0,y_0,z_0) - \overline{z}(t,x_0,y_0,z_0)\| \quad \cdots (31)$$

$$\|z(t,x_0,y_0,z_0) - \overline{z}(t,x_0,y_0,z_0)\|$$

$$\leq \alpha_3(t)\frac{\gamma_3}{\lambda_3}P_1 r_3\|x(t,x_0,y_0,z_0) - \overline{x}(t,x_0,y_0,z_0)\| +$$

$$+\alpha_3(t)\frac{\gamma_3}{\lambda_3}P_2 r_3\|y(t,x_0,y_0,z_0) - \overline{y}(t,x_0,y_0,z_0)\| +$$

$$+\alpha_3(t)\frac{\gamma_3}{\lambda_3}P_3 r_3\|z(t,x_0,y_0,z_0) - \overline{z}(t,x_0,y_0,z_0)\| \cdots (32)$$

Then we can rewrite the inequalities (30) , (31) and(32) by the vector form:

$$\begin{pmatrix} \|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\| \\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\| \end{pmatrix} \leq \Omega \begin{pmatrix} \|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\| \\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\| \end{pmatrix} \quad \cdots (33)$$

Now by the condition (8), we get

$$\begin{pmatrix} \|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\| \\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\| \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x(t,x_0,y_0,z_0) \\ y(t,x_0,y_0,z_0) \\ z(t,x_0,y_0,z_0) \end{pmatrix} = \begin{pmatrix} \overline{x}(t,x_0,y_0,z_0) \\ \overline{y}(t,x_0,y_0,z_0) \\ \overline{z}(t,x_0,y_0,z_0) \end{pmatrix}.$$

This proves that the solution is a unique and this completes the proof. □

III Existence of Periodic Solution for (BVP) [7]

The problem of the existence solution for (BVP) is uniquely connected with existence of zero of the functions

$\Delta_1(x_0,y_0,z_0)$, $\Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$, which defined by:

$$\Delta_1(x_0,y_0,z_0) = \frac{\mathrm{A}^2}{(e^{\mathrm{AT}}-\mathrm{TAE}-\mathrm{E})}\Big[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}}-\mathrm{E})-d_1 + \\ + \int_0^T Lf(t,x^0,y^0,z^0)dt\Big] + \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,[f(s,x^0(s,x_0,y_0,z_0), \\ ,y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0))]ds \quad \cdots (34)$$

$\Delta_1 \colon D_\rho \times D_\sigma \times D_\varphi \to R^n$

$$\Delta_2(x_0,y_0,z_0) = \frac{\mathrm{B}^2}{(e^{\mathrm{BT}}-\mathrm{TBE}-\mathrm{E})}\Big[\frac{y_0}{\mathrm{B}}(e^{\mathrm{BT}}-\mathrm{E})-d_2 + \\ + \int_0^T Lg(t,x^0,y^0,z^0)dt\Big] +$$

$$+\frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,[g(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),$$

$$,z^0(s,x_0,y_0,z_0))]ds \qquad \cdots(35)$$

$\Delta_2\colon D_\rho \times D_\sigma \times D_\varphi \to R^n$

And

$$\Delta_3(x_0,y_0,z_0)=\frac{\mathrm{C}^2}{(e^{\mathrm{CT}}-\mathrm{TCE}-\mathrm{E})}\left[\frac{z_0}{\mathrm{C}}(e^{\mathrm{CT}}-\mathrm{E})-d_3+\right.$$

$$\left.+\int_0^T Lh(t,x^0,y^0,z^0)dt\right]+$$

$$+\frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\,[h(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0)$$

$$z^0(s,x_0,y_0,z_0)) \qquad \cdots(36)$$

$\Delta_3\colon D_\rho \times D_\sigma \times D_\varphi \to R^n$

Since the functions are approximately determined from the sequence of functions $\Delta_1(x_0,y_0,z_0)$, $\Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$:

$$\Delta_{1m}(x_0,y_0,z_0)=\frac{\mathrm{A}^2}{(e^{\mathrm{AT}}-\mathrm{TAE}-\mathrm{E})}\left[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}}-\mathrm{E})-d_1+\right.$$

$$\left.+\int_0^T Lf(t,x_m,y_m,z_m)dt\right]+$$

$$+\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),$$

$$,z_m(s,x_0,y_0,z_0))ds \qquad \cdots(37)$$

$\Delta_{1m}\colon D_\rho \times D_\sigma \times D_\varphi \to R^n$

$$\Delta_{2m}(x_0,y_0,z_0)=\frac{\mathrm{B}^2}{(e^{\mathrm{BT}}-\mathrm{TBE}-\mathrm{E})}\left[\frac{y_0}{\mathrm{B}}(e^{\mathrm{BT}}-\mathrm{E})-d_2+\right.$$

$$\left.+\int_0^T Lg(t,x_m,y_m z_m)dt\right]+$$

$$+\frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\, g(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),$$

$$,z_m(s,x_0,y_0,z_0))ds \qquad \cdots(38)$$

$\Delta_{2m}: D_\rho \times D_\sigma \times D_\varphi \to R^n$

$$\Delta_{3m}(x_0,y_0,z_0) = \frac{\mathrm{C}^2}{(e^{\mathrm{CT}}-\mathrm{TCE}-\mathrm{E})}\left[\frac{z_0}{\mathrm{C}}(e^{\mathrm{CT}}-\mathrm{E}) - d_3 + \right.$$

$$\left. + \int_0^T Lh(t,x_m,y_m,z_m)dt\right] +$$

$$+\frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\, h(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0)$$

$$,z_m(s,x_0,y_0,z_0))ds \qquad \cdots(39)$$

$\Delta_{3m}: D_\rho \times D_\sigma \times D_\varphi \to R^n$

Theorem 2: *Let all assumptions and conditions of Theorem1 were given, then the following inequality holds:*

$$\begin{pmatrix} \|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\| \\ \|\Delta_2(x_0,y_0,z_0)-\Delta_{2m}(x_0,y_0,z_0)\| \\ \|\Delta_3(x_0,y_0,z_0)-\Delta_{3m}(x_0,y_0,z_0)\| \end{pmatrix}$$

$$\le \begin{pmatrix} S_1\frac{\gamma_1}{\lambda_1}K_1 & S_1\frac{\gamma_1}{\lambda_1}K_2 & S_1\frac{\gamma_1}{\lambda_1}K_3 \\ S_2\frac{\gamma_2}{\lambda_2}L_1 & S_2\frac{\gamma_2}{\lambda_2}L_2 & S_2\frac{\gamma_2}{\lambda_2}L_3 \\ S_3\frac{\gamma_3}{\lambda_3}P_1 & S_3\frac{\gamma_3}{\lambda_3}P_2 & S_3\frac{\gamma_3}{\lambda_3}P_3 \end{pmatrix} \Omega^m(E-\Omega)^{-1}\Psi_1 \qquad \cdots(40)$$

Where $S_1 = \frac{\|\mathrm{A}\|\mathrm{T}}{(e^{\|\mathrm{A}\|\mathrm{T}}-\|\mathrm{E}\|)} + \alpha_1(t)R_1T$, $S_2 = \frac{\|\mathrm{B}\|\mathrm{T}}{(e^{\|\mathrm{B}\|\mathrm{T}}-\|\mathrm{E}\|)} + \alpha_2(t)R_2T$

$$S_3 = \frac{\|\mathrm{C}\|\mathrm{T}}{(e^{\|\mathrm{C}\|\mathrm{T}}-\|\mathrm{E}\|)} + \alpha_3(t)R_3T$$

Proof: By the equations (34) and (37), we have

$\|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\|$

$$\leq \frac{\|A\|T}{(e^{\|A\|T}-\|E\|)}\frac{\gamma_1}{\lambda_1}K_1\|x^0(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\|+$$

$$+\frac{\|A\|T}{(e^{\|A\|T}-\|E\|)}\frac{\gamma_1}{\lambda_1}K_2\|y^0(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\|+$$

$$+\frac{\|A\|T}{(e^{\|A\|T}-\|E\|)}\frac{\gamma_1}{\lambda_1}K_3\|z^0(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\|+$$

$$+\frac{\gamma_1}{\lambda_1}\alpha_1(t)R_1TK_1\|x^0(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\|+$$

$$+\frac{\gamma_1}{\lambda_1}\alpha_1(t)R_1TK_2\|y^0(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\|+$$

$$+\frac{\gamma_1}{\lambda_1}\alpha_1(t)R_1TK_3\|z^0(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\|$$

$\therefore \|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\|$

$$\leq \langle (S_1\frac{\gamma_1}{\lambda_1}K_1 \quad S_1\frac{\gamma_1}{\lambda_1}K_2 \quad S_1\frac{\gamma_1}{\lambda_1}K_3\ ),\ \Omega^m(E-\Omega)^{-1}\Psi_1\rangle \ = q_m \qquad \cdots (41)$$

And from the equation (35) and (38), we have

$\|\Delta_2(x_0,y_0,z_0)-\Delta_{2m}(x_0,y_0,z_0,z_0)\|$

$$\leq \langle (S_2\frac{\gamma_2}{\lambda_2}L_1 \quad S_2\frac{\gamma_2}{\lambda_2}L_2 \quad S_2\frac{\gamma_2}{\lambda_2}L_3\ ),\Omega^m(E-\Omega)^{-1}\Psi_1\ \rangle \ = v_m \qquad \cdots (42)$$

Also from the equation (36) and (39), we have

$\|\Delta_3(x_0,y_0,z_0)-\Delta_{3m}(x_0,y_0,z_0,z_0)\|$

$$\leq \langle (S_3\frac{\gamma_3}{\lambda_3}P_1 \quad S_3\frac{\gamma_3}{\lambda_3}P_2 \quad S_3\frac{\gamma_3}{\lambda_3}P_3\ ),\Omega^m(E-\Omega)^{-1}\Psi_1\ \rangle \ = w_m \qquad \cdots (43)$$

Then we rewrite (41), (42) and(43) by the vector form, then we get (40). □

Now, we prove the following theorem taking into account that the inequality (41), (42) and(43) will be satisfied for all $m \geq 0$.

Theorem 3[6] *Let (BVP) be defined in the* $[a,b],[c,d]$ *and* $[i,j]$ *on* R^1, *and periodic in* t *of period* T. *Suppose that for* $m \geq 0$ *the sequences of functions* $\Delta_{1m}(x_0,y_0,z_0)$, $\Delta_{2m}(x_0,y_0,z_0)$ *and* $\Delta_{3m}(x_0,y_0,z_0)$ *which are defined in (37), (38) and (39) satisfy the inequalities:*

$$\left.\begin{array}{c} min\Delta_{1m}(x_0,y_0,z_0) \leq -q_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \\ max\ \Delta_{1m}(x_0,y_0,z_0) \geq q_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \end{array}\right\} \quad \cdots (44)$$

$$\left.\begin{array}{c} min\Delta_{2m}(x_0,y_0,z_0) \leq -v_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \\ max\Delta_{2m}(x_0,y_0,z_0) \geq v_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \end{array}\right\} \quad \cdots (45)$$

$$\left.\begin{array}{c} min\Delta_{3m}(x_0,y_0,z_0) \leq -w_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \\ max\Delta_{3m}(x_0,y_0,z_0) \geq w_m \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \end{array}\right\} \quad \cdots (46)$$

Then the system (BVP) has a periodic solution $x = x(t,x_0,y_0,z_0)$, $y = y(t,x_0,y_0,z_0)$ and $z = z(t,x_0,y_0,z_0)$ such that

$$x_0 \in J_1 = \left[a + \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 + \frac{\gamma_1}{\lambda_1}b(x_0)T\ ,\quad b - \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 - \frac{\gamma_1}{\lambda_1}b(x_0)T\right],$$

$$y_0 \in J_2 = \left[c + \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 + \frac{\gamma_2}{\lambda_2}c(y_0)T\ ,\quad d - \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 - \frac{\gamma_2}{\lambda_2}c(y_0)T\right] \text{ and}$$

$$z_0 \in J_3 = [\,i + \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 + \frac{\gamma_3}{\lambda_3}e(z_0)T,\ \ j - \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 - \frac{\gamma_3}{\lambda_3}e(z_0)T]$$

Proof: Let x_1, x_2 be any points in the interval J_1, y_1, y_2 be any points in the interval J_2, and z_1, z_2 be any points in the Interval J_3, then:

$$\left.\begin{array}{c} \Delta_{1m}(x_1,y_1,z_1) = min\Delta_{1m}(x_0,y_0,z_0) \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \\ \Delta_{1m}(x_2,y_2,z_2) = max\ \Delta_{1m}(x_0,y_0,z_0) \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \end{array}\right\} \quad \cdots (47)$$

$$\left.\begin{array}{c} \Delta_{2m}(x_1,y_1,z_1) = min\Delta_{2m}(x_0,y_0,z_0) \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \\ \Delta_{2m}(x_2,y_2,z_2) = max\ \Delta_{2m}(x_0,y_0,z_0) \\ x_0 \in J_1,\ y_0 \in J_2\ ,\ z_0 \in J_3 \end{array}\right\} \quad \cdots (48)$$

$$\left.\begin{array}{c}\Delta_{3m}(x_1,y_1,z_1)=min\Delta_{3m}(x_0,y_0,z_0)\\ x_0\in J_1,\ y_0\in J_2\ ,\ z_0\in J_3\\ \Delta_{3m}(x_2,y_2,z_2)=max\ \Delta_{3m}(x_0,y_0,z_0)\\ x_0\in J_1,\ y_0\in J_2\ ,\ z_0\in J_3\end{array}\right\}\quad\cdots(49)$$

By using the inequalities(44), (45), (46), (47), (48) and (49) we have

$$\left.\begin{array}{l}\Delta_1(x_1,y_1,z_1)=\Delta_{1m}(x_1,y_1,z_1)+\big(\Delta_1(x_1,y_1,z_1)-\Delta_{1m}(x_1,y_1,z_1)\big)<0\\ \Delta_1(x_2,y_2,z_2)=\Delta_{1m}(x_2,y_2,z_2)+\big(\Delta_1(x_2,y_2,z_2)-\Delta_{1m}(x_2,y_2,z_2)\big)>0\end{array}\right\}\quad\cdots(50)$$

$$\left.\begin{array}{l}\Delta_2(x_1,y_1,z_1)=\Delta_{2m}(x_1,y_1,z_1)+\big(\Delta_2(x_1,y_1,z_1)-\Delta_{2m}(x_1,y_1,z_1)\big)<0\\ \Delta_2(x_2,y_2,z_2)=\Delta_{2m}(x_2,y_2,z_2)+\big(\Delta_2(x_2,y_2,z_2)-\Delta_{2m}(x_2,y_2,z_2)\big)>0\end{array}\right\}\quad\cdots(51)$$

$$\left.\begin{array}{l}\Delta_3(x_1,y_1,z_1)=\Delta_{3m}(x_1,y_1,z_1)+\big(\Delta_3(x_1,y_1,z_1)-\Delta_{3m}(x_1,y_1,z_1)\big)<0\\ \Delta_3(x_2,y_2,z_2)=\Delta_{3m}(x_2,y_2,z_2)+(\Delta_3(x_2,y_2,z_2)-\Delta_{3m}(x_2,y_2,z_2))>0\end{array}\right\}\quad\cdots(52)$$

From the continuity of the functions $\Delta_1(x_1,y_1,z_1)$, $\Delta_2(x_2,y_2,z_2)$ and $\Delta_3(x_2,y_2,z_2)$ and the inequalities (50), (51) and (52), then there exist an isolated points $(x^0,y^0,z^0)=(x_0,y_0,z_0)$ and $x^0\in[x_1\,,x_2],y^0\in[y_1\,,y_2]$ $z^0\in[z_1\,,z_2]$ where $\Delta_1(x_0,y_0,z_0)$ = $\Delta_2(x_0,y_0,z_0)$ = $\Delta_3(x_0,y_0,z_0)$ =0

This means that (17) is a periodic solution

$x=x(t,x_0,y_0,z_0),y=y(t,x_0,y_0,z_0)$ and $z=z(t,x_0,y_0,z_0)$. □

Theorem 4: *Suppose that the vector functions* $\Delta_1(x_0,y_0,z_0)$, , $\Delta_2(x_0,y_0,z_0)$ *and* $\Delta_3(x_0,y_0,z_0)$ *be defined by (34), (35) and (36), and then the following inequality holds:*

$$\begin{pmatrix}||\Delta_1(x_0,y_0,z_0)||\\ ||\Delta_2(x_0,y_0,z_0)||\\ ||\Delta_3(x_0,y_0,z_0)||\end{pmatrix}\le\begin{pmatrix}N_1\dfrac{\gamma_1}{\lambda_1}M_1+b(x_0)\\ N_2\dfrac{\gamma_2}{\lambda_2}M_2+c(y_0)\\ N_3\dfrac{\gamma_3}{\lambda_3}M_3+e(z_0)\end{pmatrix}\quad\cdots(53)$$

Where $N_1=\dfrac{||\mathrm{A}||\mathrm{T}}{(e^{||\mathrm{A}||\mathrm{T}}-||\mathrm{E}||)}$, $\quad N_2=\dfrac{||\mathrm{B}||\mathrm{T}}{(e^{||\mathrm{B}||\mathrm{T}}-||\mathrm{E}||)}$,

$$N_3=\frac{||\mathrm{C}||\mathrm{T}}{(e^{||\mathrm{C}||\mathrm{T}}-||\mathrm{E}||)}$$

Proof: From the properties of the functions $x^0(t, x_0, y_0, z_0), y^0(t, x_0, y_0, z_0)$ and $z^0(t, x_0, y_0, z_0)$ are fixative intheorem 1.

Then the functions $\Delta_1(x_0, y_0, z_0), \Delta_2(x_0, y_0, z_0)$ and $\Delta_3(x_0, y_0, z_0)$ are continuous, bounded in the domain (1.)

By using (34), we get

$$\|\Delta_1(x_0, y_0, z_0)\| = \|\frac{\mathrm{A}^2}{(e^{\mathrm{AT}} - \mathrm{TAE} - \mathrm{E})}\left[\frac{x_0}{\mathrm{A}}(e^{\mathrm{AT}} - \mathrm{E}) - d_1 + \right.$$

$$\left. + \int_0^T Lf(t, x^0, y^0, z^0)dt\right] + \frac{\mathrm{A}}{(e^{\mathrm{AT}} - \mathrm{E})}\int_0^T e^{A(T-s)} f(s, x^0(s, x_0, y_0, z_0),$$

$$, y^0(s, x_0, y_0, z_0), z^0(s, x_0, y_0, z_0))ds\|$$

$$\therefore \|\Delta_1(x_0, y_0, z_0)\| \le N_1 \frac{\gamma_1}{\lambda_1} M_1 + b(x_0) \qquad \cdots (54)$$

Similarly by using (35), (36) we get

$$\|\Delta_2(x_0, y_0, z_0)\| \le N_2 \frac{\gamma_2}{\lambda_2} M_2 + c(y_0) \qquad \cdots (55)$$

$$\|\Delta_3(x_0, y_0, z_0)\| \le N_3 \frac{\gamma_3}{\lambda_3} M_3 + e(z_0) \qquad \cdots (56)$$

Then we rewrite (54), (55) and (56) by the vector form we get (53). □

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