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Tubular Surfaces of Weingarten Types in Minkowski 3-Space

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Abstract

In this paper, we study tubular surfaces in Minkowski 3*-space satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature. This paper is a completion of Weingarten and linear Weingarten tubular surfaces in Minkowski* 3*-space.*

Keywords: *Tubular surface, Minkowski 3-space, Gaussian curvature, Mean curvature, Second Gaussian curvature, Second mean curvature, Weingarten surfaces.*

1 Introduction

A surface M in Euclidean space E^3 or Minkowski space E_1^3 is called a Weingarten surface if there is a smooth relation $U(k_1, k_2) = 0$ between its two principal curvatures k_1 and k_2. If K and H denote respectively the Gauss curvature and the mean curvature of M, $U(k_1, k_2) = 0$ implies a relation $\Phi(K, H) = 0$. The existence of a non-trivial functional relation $\Phi(K, H) = 0$

on a surface M parameterized by a patch $x(s,t)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left|\frac{\partial(K,H)}{\partial(s,t)}\right| = 0$ [6].

The simplest case when $U = ak_1 + bk_2 - c$ or $\Phi = aH + bK - c$ (a,b and c are constants with $a^2 + b^2 \neq 0$), the surfaces are called linear Weingarten surfaces. When the constant $a = 0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $b = 0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [4].

The set of solutions of $U(k_1, k_2) = 0$ is called the curvature diagram of the surface [7]. If the curvature diagram degenerates to exactly one point then the surface has two constant principal curvatures which is possible for only a piece of a plane, a sphere or a circular cylinder. If the curvature diagram is contained in one of the coordinate axes through the origin then the surface is developable. If the curvature diagram is contained in the main diagonal $k_1 = k_2$ then the surface is a piece of a plane or a sphere because every point is umbilic. The curvature diagram is contained in a straight line parallel to the diagonal $k_1 = -k_2$ if and only if the mean curvature is zero. It is contained in a standard hyperbola $k_1 = \frac{c}{k_2}$ if and only if the Gaussian curvature is constant [6].

If the second fundamental form II of a surface M in E_1^3 is non-degenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the second Gaussian curvature K_{II} of non-degenerate second fundamental form II can be defined formally on the Riemannian or pseudo-Riemannian manifold (M, II) [4].

For a pair (X,Y), $X \neq Y$, of the curvatures K, H and K_{II} of M in E_1^3, if M satisfies $U(X,Y) = aX + bY - c = 0$, then it said to be a (X,Y)-Weingarten surface and (X,Y)-linear Weingarten surface, respectively [4].

Several geometers [3, 4, 8] have studied tubes in Euclidean 3-space and Minkowski 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature.Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K, the mean curvature H, the second Gaussian curvature K_{II} and the second mean curvature H_{II} an interesting geometric question is raised: Classify all surfaces in Euclidean 3-space satisfying the conditions

$$\Phi(X,Y) = 0 \tag{1.1}$$

$$aX + bY = c \tag{1.2}$$

where $X, Y \in \{K, H, K_{II}, H_{II}\}$, $X \neq Y$ and $(a,b,c) \neq (0,0,0)$.

In this paper, we studied Weingarten and linear Weingarten tubular surfaces and we obtained some conditions for that surfaces in E_1^3.We show that tubular surfaces are not umbilical and minimal by using their principal curvatures and they haven't curvature diagram.

2 Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the Lorentzian inner product

$$\langle X, Y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have the one of three Lorentzian causal characters: it is spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike) [6]. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ parametrized by a pseudo-arclength parameter such that $\langle \alpha'(s), \alpha'(s)\rangle = \pm 1$. If α is timelike curve, then Frenet formulae are

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N \tag{2.1}$$

where $\langle T, T\rangle = -1, \langle N, N\rangle = 1, \langle B, B\rangle = 1$. If α is a spacelike curve with a spacelike principal normal, then Frenet formulae are

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = \tau N \tag{2.2}$$

where $\langle T, T\rangle = \langle N, N\rangle = 1, \langle B, B\rangle = -1$. If α is a spacelike curve with a spacelike binormal, then Frenet formulae are

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N \tag{2.3}$$

where $\langle T, T\rangle = \langle B, B\rangle = 1, \langle N, N\rangle = -1$ [5].

We denote a surface M in E_1^3 by

$$M(s, t) = (m_1(s, t), m_2(s, t), m_3(s, t)).$$

Let U be the standard unit normal vector field on a surface M defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|}.$$

The first fundamental form I and the second fundamental form II of a surface M are

$$I = Eds^2 + 2Fdsdt + Gdt^2, \quad II = eds^2 + 2fdsdt + gdt^2$$

respectively, where

$$\begin{aligned} E &= \langle M_s, M_s\rangle, \quad F = \langle M_s, M_t\rangle, \quad G = \langle M_t, M_t\rangle \\ e &= \langle M_{ss}, U\rangle, \quad f = \langle M_{st}, U\rangle, \quad g = \langle M_{tt}, U\rangle \end{aligned}$$

[4]. The Gaussian curvature K and the mean curvature H are

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)},$$

respectively. From Brioschi's formula in a Minkowski 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g respectively [4]. Thus, the second Gaussian curvature K_{II} of a surface is

$$K_{II} = \frac{1}{\left(|eg| - f^2\right)^2}\left\{\begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}\right\}$$

and the second mean curvature H_{II} of a surface is defined by

$$H_{II} = H - \frac{1}{2\sqrt{|\det II|}}\sum_{i,j}\frac{\partial}{\partial u^i}\left(\sqrt{|\det II|}L^{ij}\frac{\partial}{\partial u^j}\left(\ln\sqrt{|K|}\right)\right) \tag{2.4}$$

where $(L^{ij}) = (L_{ij})^{-1}$ and L_{ij} are the coefficients of second fundamental forms [1,2]. The principal curvatures of the surface $M(s,t)$ can be found by the following equations

$$\left(EG - F^2\right)k_n^2 - (Eg + eG - 2fF)\,k_n + \left(eg - f^2\right) = 0$$

and then the principal curvatures are

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$

Definition 2.1 *Let k_1, k_2 be the principal curvatures of the surface $M(s,t)$ in E_1^3. The surface $M(s,t)$ has a curvature diagram if it satisfies*

$$f(k_1, k_2) = a_1k_1^2 + 2a_2k_1k_2 + a_3k_2^2 + a_4k_1 + a_5k_2 + a_6 = 0 \tag{2.5}$$

for some constant $a_i (i = 1, 2, ..., 6)$.

Remark 2.2 *It is well known that a minimal surface has vanishing second Gaussian curvature but a surface with vanishing second Gaussian curvature needs not be minimal* [4].

3 Weingarten Tubular Surfaces in Minkowski 3-Space

Definition 3.1 *Let* $\alpha : [a,b] \to E_1^3$ *be a unit-speed spacelike curve with timelike principal normal. A tubular surface of radius* $\lambda > 0$ *about* α *is the surface with parametrization*

$$M_1(s,\theta) = \alpha(s) + \lambda\left[N(s)\cosh\theta + B(s)\sinh\theta\right] \tag{3.1}$$

$a \leq s \leq b$ [5].

We have the natural frame $\{M_s, M_\theta\}$ of the surface Mgiven by

$$\begin{aligned} M_s &= (1+\lambda\kappa\cosh\theta)\,T + (\lambda\tau\sinh\theta)\,N + (\lambda\tau\cosh\theta)\,B, \\ M_\theta &= (\lambda\sinh\theta)\,N + (\lambda\cosh\theta)\,B. \end{aligned}$$

The components of the first fundamental form are

$$\begin{aligned} E &= \lambda^2\tau^2 + (1-\lambda\kappa\cosh\theta)^2 \\ F &= \lambda^2\tau \\ G &= \lambda^2. \end{aligned}$$

The unit normal vector field U is obtained by $U = N\cosh\theta + B\sinh\theta$. Since $\langle U,U\rangle = -1 < 0$, the surface M_1 is a spacelike surface. From this, the components of the second fundamental form of M_1 are given by

$$\begin{gathered} e = -\lambda\tau^2 - \kappa\cosh\theta\,(1+\lambda\kappa\cosh\theta) \\ f = -\lambda\tau \\ g = -\lambda. \end{gathered}$$

If the second fundamental form is non-degenerate, $eg - f^2 \neq 0$. Namely, κ, $(1+\lambda\kappa\cosh\theta)$ and $\cosh\theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature K_{II} and second mean curvature H_{II} on M_1. On the other hand, the curvatures K, H , K_{II} and H_{II} are

$$K = \frac{\kappa\cosh\theta\,(1+\lambda\kappa\cosh\theta)}{\lambda\,(-1+\lambda\kappa\cosh\theta)^2} \quad , \quad H = -\frac{(1-\lambda\kappa\cosh\theta) + 2\lambda^2\kappa^2\cosh^2\theta}{2\lambda\,(-1+\lambda\kappa\cosh\theta)^2}, \tag{3.2}$$

$$K_{II} = \frac{\kappa\left(1 + 6\lambda\kappa\cosh^3\theta + 4\lambda^2\kappa^2\cosh^4\theta + \cosh^2\theta\right)}{4\cosh\theta\,(1+\lambda\kappa\cosh\theta)}, \tag{3.3}$$

$$H_{II} = -\frac{1}{8\lambda\kappa^3\cosh^3\theta\,(1+\lambda\kappa\cosh\theta)^3\,(-1+\lambda\kappa\cosh\theta)^2}\left(\sum_{i=0}^{8} u_i\cosh^i\theta\right) \tag{3.4}$$

and where the coefficients u_i are

$$
\begin{aligned}
u_0 &= -3\lambda^2\kappa^2\tau^2 \\
u_1 &= +2\lambda\kappa\left(\kappa\tau' - \kappa'\tau\sinh\theta - 2\lambda\kappa^2\tau^2\right) - \kappa^3 \\
u_2 &= 3\lambda\left(\kappa'\right)^2 + \kappa\lambda\left(2\kappa\left(3\lambda\kappa\tau' + 2\kappa'\tau\right)\sinh\theta\right) \\
&\quad - \kappa\lambda\left(2\left(\kappa''\right)^2 + 3\kappa^3 + \kappa\tau^2 + 5\lambda^2\kappa^3\tau^2\right) \\
u_3 &= \kappa\left\{\begin{array}{c} 2\lambda^3\kappa^2\left(3\kappa'\tau - \kappa\tau'\right)\sinh\theta + 3\kappa^2 \\ +\lambda^2\left(2\kappa^2\tau^2\left(9\lambda^2\kappa^2 - 1\right) + 11\kappa^4 - 6\kappa\kappa'' + 4\left(\kappa'\right)^2\right) \end{array}\right\} \\
u_4 &= \lambda\kappa^2\left(6\lambda^3\kappa^2\left(4\kappa'\tau\sinh\theta - \kappa\tau'\right)\right) \\
&\quad + \lambda\kappa^2\left(\lambda^2\left(13\kappa^4 - 5\left(\kappa'\right)^2 - 2\kappa\kappa'' - 3\kappa^2\tau^2\right)\right) - 3\lambda\kappa^4 \\
u_5 &= \lambda^2\kappa^3\left(\lambda^2\left(6\kappa^4 + 6\kappa\kappa'' - 12\kappa^2\tau^2 - 18\left(\kappa'\right)^2\right) - 7\kappa^2\right) \\
u_6 &= 11\lambda^3\kappa^6 \\
u_7 &= 20\lambda^4\kappa^7 \\
u_8 &= 8\lambda^5\kappa^8.
\end{aligned}
$$

respectively. Differentiating K, H and K_{II} with respect to s and θ, we get

$$
K_s = -\frac{\kappa'\cosh\theta\left(1 + 3\lambda\kappa\cosh\theta\right)}{\lambda\left(-1 + \lambda\kappa\cosh\theta\right)^3}, \qquad K_\theta = -\frac{\kappa\sinh\theta\left(1 + 3\lambda\kappa\cosh\theta\right)}{\lambda\left(-1 + \lambda\kappa\cosh\theta\right)^3}, \tag{3.5}
$$

$$
H_s = -\frac{\kappa'\left(1 + 3\lambda\kappa\cosh\theta\right)}{2\left(-1 + \lambda\kappa\cosh\theta\right)^3}, \qquad H_\theta = -\frac{\kappa\sinh\theta\left(1 + 3\lambda\kappa\cosh\theta\right)}{2\left(-1 + \lambda\kappa\cosh\theta\right)^3},, \tag{3.6}
$$

$$
\begin{aligned}
\left(K_{II}\right)_s &= \frac{\kappa'\left(\cosh^2\theta + 12\lambda\kappa\cosh^3\theta + 18\lambda^2\kappa^2\cosh^4\theta + 8\lambda^3\kappa^3\cosh^5\theta + 1\right)}{4\cosh\theta\left(1 + \lambda\kappa\cosh\theta\right)^2}, \\
\left(K_{II}\right)_\theta &= \frac{\kappa\sinh\theta\left(\cosh^2\theta + 12\lambda\kappa\cosh^3\theta + 18\lambda^2\kappa^2\cosh^4\theta\right)}{4\cosh^2\theta\left(1 + \lambda\kappa\cosh\theta\right)^2} \\
&\quad + \frac{\kappa\sinh\theta\left(8\lambda^3\kappa^3\cosh^5\theta - 1 - 2\lambda\kappa\cosh\theta\right)}{4\cosh^2\theta\left(1 + \lambda\kappa\cosh\theta\right)^2}
\end{aligned} \tag{3.7}
$$

Now, we investigate a tubular surface M in E_1^3 satisfying the Jacobi equation $\Phi(K, H_{II}) = 0$. By using (3.5), we obtained $\Phi(K, H_{II})$ in the form

$$
\Phi(K, H_{II}) = \frac{1}{4\lambda^2\kappa^3\cosh^3\theta\left(1 + \lambda\kappa\cosh\theta\right)^3\left(-1 + \lambda\kappa\cosh\theta\right)^5}\sum_{i=0}^{7} a_i\cosh^i\theta \tag{3.8}
$$

with respect to the Gaussian curvature K and the second mean curvature H_{II} where

$$a_0 = 3\lambda\kappa^3\tau\tau'\sinh\theta + 3\lambda\kappa^2\kappa'\tau^2\sinh\theta$$
$$a_1 = \kappa^2\left\{\kappa\kappa'\sinh\theta + \lambda\kappa\tau'' + 13\lambda^2\kappa\tau\left(\kappa\tau' + \kappa'\tau\right) - \lambda\kappa''\tau\right\}$$
$$a_2 = \lambda\left\{\left\{\begin{array}{c} 7\lambda^2\kappa^4\tau\left(\kappa'\tau + \kappa\tau'\right) - 4\kappa\kappa'\kappa'' \\ +\kappa^2\left(\kappa''' - \kappa\tau\tau'\right) + 3(\kappa')^3 \end{array}\right\}\sinh\theta + \lambda\kappa^3\left(5\kappa'\tau' - \kappa''\tau + 6\kappa\tau''\right)\right\}$$
$$a_3 = \lambda\kappa\left\{\begin{array}{c} \lambda\left\{\begin{array}{c} 6\kappa^2\kappa''' - \kappa^3\tau\tau'\left(33\lambda^2\kappa^2 + \kappa'\right) - 20\kappa^4\kappa' \\ +\lambda\kappa^3\kappa'\left(17\tau' - 33\lambda\kappa\tau^2\right) - 19\kappa\kappa'\kappa'' + 13\left(\kappa'\right)^3 \end{array}\right\}\sinh\theta \\ +\left(\kappa\kappa''\tau\left(1 + 9\lambda^2\kappa^2\right) - 3\left(\kappa'\right)^2\tau - \kappa\kappa'\tau' + \kappa^2\tau''\left(8\lambda^2\kappa^2 - 1\right)\right) \end{array}\right\}$$
$$a_4 = \lambda^2\kappa^2\left\{\begin{array}{c} \lambda\left\{\begin{array}{c} \kappa^3\tau\tau'\left(9 - 54\lambda^2\kappa^2\right) + \left(8\kappa^2\kappa''' + 7\left(\kappa'\right)^3\right) \\ -\kappa\kappa'\left(46\kappa^3 + 54\lambda^2\kappa^3\tau^2 + 15\kappa''\right) \end{array}\right\}\sinh\theta \\ +\left\{\begin{array}{c} \lambda^2\kappa^3\left(21\kappa''\tau + 15\kappa'\tau' - 6\kappa\tau''\right) \\ +\kappa\kappa''\tau + \kappa\kappa'\tau' - \left(\kappa'\right)^2\tau - 6\kappa^2\tau'' \end{array}\right\} \end{array}\right\}$$
$$a_5 = \lambda^3\kappa^3\left\{\begin{array}{c} \lambda\left\{\begin{array}{c} 3\kappa'\left(13\kappa\kappa'' - 11\left(\kappa'\right)^2\right) - 9\kappa^4\left(5\kappa' + \lambda\tau''\right) \\ +21\kappa^3\tau\tau' - 6\kappa^2\kappa''' \end{array}\right\}\sinh\theta \\ +\left(9\kappa\kappa''\tau\left(4\lambda^2\kappa^2 - 1\right) + 9\kappa\kappa'\tau'\left(3\lambda^2\kappa^2 - 1\right) + 9\left(\kappa'\right)^2\tau - 8\kappa^2\tau''\right) \end{array}\right\}$$
$$a_6 = \lambda^4\kappa^4\left\{\begin{array}{c} \lambda\left(\kappa'\left(63\kappa\kappa'' - 18\kappa^4 - 54\left(\kappa'\right)^2\right) - 9\kappa^2\kappa''' + 36\kappa^3\tau\tau'\right)\sinh\theta \\ +3\left(2\kappa^2\tau'' + 7\left(\left(\kappa'\right)^2\tau - \kappa\kappa''\tau - \kappa\kappa'\tau'\right)\right) \end{array}\right\}$$
$$a_7 = 9\lambda^5\kappa^5\left(\kappa^2\tau'' + 4\left(\kappa'\right)^2\tau - 4\kappa\kappa''\tau - 4\kappa\kappa'\tau'\right).$$

Then, by (3.5), equation (3.8) becomes

$$\sum_{i=0}^{7} a_i\cosh^i\theta = 0 \tag{3.9}$$

Hence we have the following theorem.

Theorem 3.2 *Let M_1 be a tubular surface defined by (3.1) with non-degenerate second fundamental form. M_1 is a (K, H_{II})-Weingarten surface if and only if M_1 is a tubular surface around a circle or a helix in Minkowski 3-space.*

Proof. Let assume that M_1 is a (K, H_{II})-Weingarten surface then the Jacobi equation (3.9) satisfied. Since the polynomial in (3.9) is equal to zero for every θ, all its coefficients must be zero. Therefore, the solutions of $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$ are $\kappa', \tau = 0$ and $\kappa', \tau' = 0$ that is M_1 is a tubular surface around a circle or a helix in Minkowski 3-space, respectively.

Conversely, suppose that M_1 is a tubular surface around a circle or a helix in Minkowski 3-space then it is easily to see that $\Phi(K, H_{II}) = 0$ is satisfied for the cases both κ', $\tau = 0$ and κ', $\tau' = 0$. Thus M_1 is a (K, H_{II})-Weingarten surface in Minkowski 3-space.

We suppose that a tubular surface $M_1(s, \theta)$ with non-degenerate second fundamental form in E_1^3 is (H, H_{II})-Weingarten surface. From (3.6), $\Phi(H, H_{II})$ is

$$\Phi(H, H_{II}) = \frac{1}{8\lambda\kappa^3\cosh^3\theta\,(1+\lambda\kappa\cosh\theta)^3\,(-1+\lambda\kappa\cosh\theta)^5}\sum_{i=0}^{7} m_i \cosh^i\theta \tag{3.10}$$

with respect to the variable $\cosh\theta$, where

$$m_0 = -3\lambda\kappa^2\kappa'\tau\,(\tau+\kappa\tau')\sinh\theta \tag{3.11}$$

$$m_1 = -\kappa^3\left(13\lambda^2\kappa\tau\tau' + 13\lambda^2\kappa'\tau^2 + \kappa'\right)\sinh\theta + \lambda\kappa^2\left(\kappa''\tau - \kappa\tau''\right)$$

$$\begin{aligned} m_2 \;=\;& \lambda\sinh\theta\kappa^3\tau\tau'\left(1-7\lambda^2\kappa^2\right) - \kappa^2\kappa''' + \kappa'\left(4\kappa\kappa'' - 7\lambda^2\kappa^4\tau^2 - 3\left(\kappa'\right)^2\right) \\ & +\lambda\sinh\theta\left(\lambda\kappa^3\left(\kappa''\tau - 5\kappa'\tau' - 6\kappa\tau''\right)\right) \end{aligned}$$

$$\begin{aligned} m_3 \;=\;& \lambda^2\kappa\sinh\theta\left(\kappa^3\kappa'\tau\left(\tau' + 33\lambda^2\kappa\tau\right) + 33\lambda^2\kappa^5\tau\tau'\right) \\ & +\lambda^2\kappa\sinh\theta\left(\kappa'\left(20\kappa^4 + 19\kappa\kappa'' - 13\left(\kappa'\right)^2\right) - 6\kappa^2\kappa'''\right) \\ & +\lambda^2\kappa\left(\kappa^2\tau''\left(1-8\lambda^2\kappa^2\right) - \kappa\kappa'\tau'\left(1+17\lambda^2\kappa^2\right)\right) \\ & -\lambda^2\kappa\left(\kappa\kappa''\tau\left(9\lambda^2\kappa^2+1\right) + (\kappa')^2\tau\right) \end{aligned}$$

$$\begin{aligned} m_4 \;=\;& \lambda^2\kappa^2\lambda\sinh\theta\kappa\kappa'\left(15\kappa'' + 46\kappa^3 + 54\lambda^2\kappa^3\tau^2\right) \\ & +\lambda^2\kappa^2\lambda\sinh\theta\left(9\kappa^3\tau\tau'\left(6\lambda^2\kappa^2-1\right) - 8\kappa^2\kappa''' - 7\left(\kappa'\right)^3\tau\right) \\ & +\lambda^2\kappa^2\left(\left(\kappa'\right)^2\tau - \kappa\kappa'\tau'\left(15\lambda^2\kappa^2+1\right)\right) \\ & +\lambda^2\kappa^2\left(6\kappa^2\tau''\left(\lambda^2\kappa^2+1\right) - \kappa\kappa''\tau\left(21\lambda^2\kappa^2+1\right)\right) \end{aligned}$$

$$m_5 = \lambda^3\kappa^3\left\{\begin{array}{c} \lambda\left\{\begin{array}{c} 6\kappa^2\kappa''' + \kappa\kappa'\left(45\kappa^3 - 39\kappa''\right) \\ -3\kappa^3\tau\left(12\lambda\kappa'' + 7\tau'\right) + 33\left(\kappa'\right)^3 \end{array}\right\}\sinh\theta \\ +9\kappa\kappa''\tau + 8\kappa^2\tau'' + 9\kappa\kappa'\tau' + 9\lambda^2\kappa^4\tau'' - 9\left(\kappa'\right)^2\tau - 27\lambda^2\kappa^3\kappa'\tau' \end{array}\right\}$$

$$m_6 = \lambda^4\kappa^4\left\{\begin{array}{c} \lambda\left\{54\left(\kappa'\right)^3 + 9\kappa^2\kappa''' - 36\kappa^3\tau\tau' - 63\kappa\kappa'\kappa'' + 18\kappa^4\kappa'\right\}\sinh\theta \\ +21\kappa\kappa''\tau - 21\left(\kappa'\right)^2\tau - 6\kappa^2\tau'' + 21\kappa\kappa'\tau' \end{array}\right\}$$

$$m_7 = 9\lambda^5\kappa^5\left\{4\kappa\kappa''\tau\sinh\theta - 4\left(\kappa'\right)^2\tau + 4\kappa\kappa'\tau' - \kappa^2\tau''\right\}.$$

Then, by $\Phi(H, H_{II}) = 0$, the equation (3.10) can be written in the following form.

$$\sum_{i=0}^{7} m_i \cosh^i \theta = 0. \tag{3.12}$$

Thus we state the following theorem.

Theorem 3.3 *Let M_1 be a tubular surface defined by (3.1) with non-degenerate second fundamental form. M_1 is a (H, H_{II})-Weingarten surface if and only if M_1 is a tubular surface around a circle or a helix in Minkowski 3-space.*

Proof. Considering $\Phi(H, H_{II}) = 0$ and using (3.11) one can obtain the solutions κ', $\tau = 0$ and κ', $\tau' = 0$ of the equation $m_0 = m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = 0$ for all θ. Thus it is easly proved that M_1 is a (H, H_{II})-Weingarten surface if and only if M_1 is a tubular surface around a circle or a helix in Minkowski 3-space.

We consider a tubular surface M_1 as (K_{II}, H_{II})-Weingarten surface with non-degenerate second fundamental form in E_1^3. By using (3.7),

$$\Phi(K_{II}, H_{II}) = \frac{1}{16\lambda\kappa^3 \cosh^5\theta \left(1 + \lambda\kappa\cosh\theta\right)^5 \left(-1 + \lambda\kappa\cosh\theta\right)^3} \sum_{i=0}^{12} n_i \cosh^i \theta. \tag{3.13}$$

for some n_i.

Since $\Phi(K_{II}, H_{II}) = 0$, then the equation (3.13) becomes in following form.

$$\sum_{i=0}^{12} n_i \cosh^i \theta = 0. \tag{3.14}$$

Hence we have the following theorem.

Theorem 3.4 *Let M_1 be a tubular surface defined by (3.1) with non-degenerate second fundamental form. M_1 is a (K_{II}, H_{II})-Weingarten surface if and only if M_1 is a tubular surface around a circle or a helix in Minkowski 3-space.*

Proof. It can be easly proved similar to theorems 1 and 2.

Consequently, we can give the following main theorem for the end of this part.

Theorem 3.5 *Let $(X, Y) \in \{(K, H_{II}), (H, H_{II}), (K_{II}, H_{II})\}$ and let M_1 be a tubular surface defined by (3.1) with non-degenerate second fundamental form. M_1 is a (X, Y)-Weingarten surface if and only if M_1 is a tubular surface around a circle or a helix in Minkowski 3-space.*

Theorem 3.6 *The spacelike tubular surface $M_1(s,\theta)$ is neither umblical nor minimal.*

Proof. Let k_1, k_2 be the principal curvatures of $M_1(s,\theta)$ in E_1^3 . The principal curvatures are obtained as follows;

$$k_1 = -\frac{\kappa\cosh\theta}{1+\lambda\kappa\cosh\theta}\ , \quad k_2 = -\frac{1}{\lambda}. \tag{3.15}$$

Since M_1 has not a curvature diagram such that $k_1 - k_2 = 0$ and $k_1 + k_2 = 0$ then M_1 is neither umblical nor minimal.

3.1 Linear Weingarten Tubular Surfaces in Minkowski 3-Space

In this part of this paper, we study on linear Weingarten tubular surfaces (K, H_{II}), (H, H_{II}), (K_{II}, H_{II}) and (K, H, H_{II}), (K, H, K_{II}), (H, K_{II}, H_{II}), (K, K_{II}, H_{II}) and (K, H, K_{II}, H_{II}) in E_1^3, linear Weingarten tubes (K, H), (K, K_{II}), (H, K_{II}) are studied in [3,4].

Let a_1, a_2, a_3, a_4 and p be constants. In general, a linear combination of K, H, K_{II} and H_{II} can be constructed as

$$a_1K + a_2H + a_3K_{II} + a_4H_{II} = p. \tag{4.1}$$

By the straightforward calculations, we obtained the reduced form of the equation (4.1) as

$$\begin{aligned} & a_1K + a_2H + a_3K_{II} + a_4H_{II} - p \\ = & \frac{-1}{8\lambda\kappa^3\left(1+\lambda\kappa\cosh\theta\right)^3\left(-1+\lambda\kappa\cosh\theta\right)^2\cosh^3\theta}\sum_{i=0}^{10} w_i(\cosh\theta)^i \end{aligned}$$

where the coefficients are

$$w_0 = -3a_4\lambda\kappa^2\tau^2$$

$$w_1 = \kappa\left\{2a_4\lambda\left(\kappa\tau' - \kappa'\tau\right)\sinh\theta - a_4\kappa^2\left(4\lambda^2\tau^2+1\right)\right\}$$

$$w_2 = \lambda\left\{\begin{array}{c} 2a_4\lambda\kappa^2\left(3\kappa\tau' + 2\kappa'\tau\right)\sinh\theta \\ +a_4\left(\kappa^2\tau^2\left(5\lambda^2\kappa^2+1\right) - 2\kappa\kappa'' + 3(\kappa')^2 + 3\kappa^4\right) - 2a_3\kappa^4 \end{array}\right\}$$

$$w_3 = \kappa\left\{\begin{array}{c} 2a_4\lambda^3\kappa^2\left(3\kappa'\tau - \kappa\tau'\right)\sinh\theta + 4a_2\kappa^2 + 8p\lambda\kappa^2 \\ +a_4\left(3\kappa^2 - 6\lambda^2\kappa\kappa'' + \lambda^2\left(11\kappa^4 - 2\kappa^2\tau^2 + 18\lambda^2\kappa^4\tau^2 + 4(\kappa')^2\right)\right) \end{array}\right\}$$

$$w_4 = \begin{array}{c} \kappa^2\left(6a_4\lambda^4\kappa^2\left(4\kappa'\tau - \kappa\tau'\right)\sinh\theta - 8a_1\kappa^2 + 8a_2\lambda\kappa^2 - 2a_3\lambda\kappa^2\left(1-2\lambda^2\kappa^2\right)\right) \\ +\kappa^2\left(a_4\lambda\left(13\lambda^2\kappa^4 - 3\kappa^2 + 2\lambda^2\kappa\kappa'' - 5\lambda^2(\kappa')^2 - 3\lambda^2\kappa^2\tau^2\right) + 8p\lambda^2\kappa^2\right) \end{array}$$

$$w_5 = \lambda\kappa^3\left\{\begin{array}{c} -32a_1\kappa^2 + 8a_2\lambda\kappa^2 - 12a_3\lambda\kappa^2 - 16p\lambda^2\kappa^2 \\ +a_4\lambda\left(\kappa^2\left(6\lambda^2\kappa^2 - 12\lambda^2\tau^2 - 7\right) - 18\lambda^2(\kappa')^2 + 6\lambda^2\kappa\kappa''\right) \end{array}\right\}$$

$$w_6 = \lambda^2\kappa^6\left\{-48a_1 + 16a_2\lambda - 4a_3\lambda - 2a_3\lambda^3\kappa^2 + 11a_4\lambda - 16p\lambda^2\right\}$$
$$w_7 = 4\lambda^3\kappa^7\left\{-8a_1 + 5a_2\lambda + 6a_3\lambda + 5a_4\lambda + 2p\lambda^2\right\}$$
$$w_8 = 8a_4\lambda^5\kappa^8 - 8a_1\lambda^4\kappa^8 + 8p\lambda^6\kappa^8 + 8a_2\lambda^5\kappa^8 + 14a_3\lambda^5\kappa^8$$
$$w_9 = -12a_3\lambda^6\kappa^9$$
$$w_{10} = -8a_3\lambda^7\kappa^{10}.$$

Since "$(1+\lambda\kappa\cosh\theta)^3\cosh^3\theta$" is not constant, p has to be zero. From $w_0 = 0$, one has $\kappa = 0$. Hence for all the cases $a_2 = a_3 = 0$, $a_1 = a_3 = 0$ and $a_1 = a_2 = 0$, we can give the following theorems.

Theorem 3.7 *Let (X, Y) be one of $(K, H_{II}), (H, H_{II})$, (K_{II}, H_{II}). Then there are no (X, Y)-linear Weingarten tubular surfaces M_1 in Minkowski 3-space defined by (3.1) with non-degenerate second fundamental form.*

Theorem 3.8 *Let (X, Y, Z) be one of $(K, H, K_{II}), (K, H, H_{II}), (K, K_{II}, H_{II})$ and (H, K_{II}, H_{II}). Then there are no (X, Y, Z)-linear Weingarten tubular surfaces in Minkowski 3-space defined by (3.1) with non-degenerate second fundamental form.*

Theorem 3.9 *Let M_1 be a tubular surface defined by (3.1) with non-degenerate second fundamental form. Then (K, H, K_{II}, H_{II}) are not linear Weingarten surface in Minkowski 3-space.*

Definition 3.10 *Let $\alpha : [a, b] \to E_1^3$ be a unit-speed spacelike curve with spacelike principal normal. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization*

$$M_2(s,\theta) = \alpha(s) + \lambda\left[N(s)\cosh\theta - B(s)\sinh\theta\right]$$

$a \le s \le b$. If the curve α is unit-speed timelike, then the tubular surface defined by

$$M_3(s,\theta) = \alpha(s) + \lambda\left[N(s)\cos\theta + B(s)\sin\theta\right].$$

For the timelike tubular surface M_2, the components of the first and second fundamental forms are

$$E = -\lambda^2\tau^2 + (1-\lambda\kappa\cosh\theta)^2$$
$$F = \lambda^2\tau, G = -\lambda^2$$
$$e = \left(\lambda\tau^2 + (\kappa\cosh\theta)(1-\lambda\kappa\cosh\theta)\right)$$
$$f = -\lambda\tau$$
$$g = \lambda$$

and the unit normal vector field is $U_2 = N\cosh\theta - B\sinh\theta$. Since $\langle U_2, U_2\rangle = 1$, the surface M_2 is a timelike. The curvatures K, H and K_{II} are

$$K = -\frac{\kappa\cosh\theta}{\lambda\left(1-\lambda\kappa\cosh\theta\right)}$$
$$H = \frac{-1+2\lambda\kappa\cosh\theta}{2\lambda\left(1-\lambda\kappa\cosh\theta\right)}$$
$$K_{II} = \frac{-4\lambda^2\kappa^2\cosh^2\theta + 6\lambda\kappa\cosh\theta + \tanh^2\theta - 2}{4\lambda\left(-1+\lambda\kappa\cosh\theta\right)^2}$$

respectively and the principal curvatures are obtained as follows

$$k_1 = -\frac{1}{\lambda}$$
$$k_2 = \frac{\kappa\cosh\theta}{1-\lambda\kappa\cosh\theta}.$$

For the timelike tubular surface M_3, from which the components of the first and second fundamental forms are

$$E = \lambda^2\tau^2 - \left(1+\lambda\kappa\cos\theta\right)^2$$
$$F = \lambda^2\tau$$
$$G = \lambda^2$$
$$e = -\lambda\tau^2 + \left(\kappa\cos\theta\right)\left(1+\lambda\kappa\cos\theta\right)$$
$$f = -\lambda\tau$$
$$g = -\lambda$$

and the unit normal vector field is $U_3 = N\cos\theta + B\sin\theta$. Since $\langle U_3, U_3\rangle = 1$, the surface M_3 is a timelike surface. The curvatures K, H and K_{II} are

$$K = \frac{\kappa\cos\theta}{\lambda\left(1+\lambda\kappa\cos\theta\right)}$$
$$H = -\frac{1+2\lambda\kappa\cos\theta}{2\lambda\left(1+\lambda\kappa\cos\theta\right)}$$
$$K_{II} = -\frac{4\lambda^2\kappa^2\cos^2\theta + 6\lambda\kappa\cos\theta + \tanh^2\theta + 2}{4\lambda\left(1+\lambda\kappa\cos\theta\right)^2}$$

respectively and the principal curvatures are obtained as follows

$$k_1 = -\frac{1}{\lambda}$$
$$k_2 = -\frac{\kappa\cos\theta}{1+\lambda\kappa\cos\theta}.$$

We can obtain similar theorems for the timelike tubular surfaces M_2 and M_3. They can be easly proved in the similar ways for the spacelike tubular surface M_1.

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