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Derivations of Operators on Hilbert Modules

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Abstract

Let A be a C^*-algebra and X be a right Hilbert A-module. In this paper we study the relation between innerness of derivations on $\mathcal{K}(X)$, compact operators on X, and $\mathcal{L}(X)$, adjointable operators on X, also we show that with the certain conditions every derivation on $\mathcal{K}(X)$ and $\mathcal{L}(X)$ is zero.

Keywords: *C^*-algebra, Hilbert C^*-module, Derivation.*

1 Introduction

Hilbert C^*-modules were first introduced in the work of I. Kaplansky [5]. Hilbert C^*-modules are very useful in operator K-theory, operator algebra, Morita equivalence and others. Hilbert C^*-modules form a category in between Banach spaces and Hilbert spaces and obey the same axioms as a Hilbert space except that inner product takes values in a C^*-algebra rather than in the complex numbers. Let us recall some basic facts about the Hilbert C^*-modules.

Let A be a C^*-algebra. An right inner product A-module is a linear space X which is a right A-module (with compatible scalar multiplication: $\lambda(x.a) = (\lambda x).a = x.(\lambda a)$ for $x \in X$, $a \in A$, $\lambda \in \mathbf{C}$), together with a map $(x, y) \longmapsto \langle x, y\rangle_X : X \times X \longrightarrow A$ such that for all $x, y, z \in X$, $a \in A$, $\alpha, \beta \in \mathbf{C}$

(i) $\langle x, \alpha y + \beta z\rangle_X = \alpha\langle x, y\rangle_X + \beta\langle x, z\rangle_X$;
(ii) $\langle x, y.a\rangle_X = \langle x, y\rangle_X a$;
(iii) $\langle y, x\rangle_X = \langle x, y\rangle_X^*$;
(iv) $\langle x, x\rangle_X \geq 0$; if $\langle x, x\rangle_X = 0$ then $x = 0$.

A right pre-Hilbert A-module X is called a right Hilbert A-module if it is complete with respect to the norm $\| x \|=\| \langle x, x\rangle_X \|^{\frac{1}{2}}$. X is said to be full if the linear span of the set $\{\langle x, y\rangle_X : x, y \in X\}$ is dense in A. One interesting example of full right Hilbert C^*-modules is any C^*- algebra A as a right Hilbert A-module via $\langle a, b\rangle_A = a^*b \quad (a, b \in A)$.
Likewise, a left Hilbert A-module with an A-valued inner product ${}_X\langle ., .\rangle$ can be defined.

Let X be a right Hilbert A-module, we define $\mathcal{L}(X)$ to be the set of all maps $T : X \longrightarrow X$ for which there is a map $T^* : X \longrightarrow X$ such that $\langle Tx, y\rangle_X = \langle x, T^*y\rangle_X \ (x, y \in X)$. It is easy to see that T must be bounded A-linear and $\mathcal{L}(X)$ is a C^*-algebra. For $x, y \in X$, define the operator $\theta_{x,y}$ on X by $\theta_{x,y}(z) = x.\langle y, z\rangle_X \quad (z \in X)$. Denote by $\mathcal{K}(X)$ the closed linear span of $\{\theta_{x,y} : x, y \in X\}$, then $\mathcal{K}(X)$ is a closed two sided ideal in $\mathcal{L}(X)$. The reader is referred to [6] for more details on Hilbert C^*-modules.

In this paper a derivation of an algebra A is a linear mapping D from A into itself such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. For a fixed $b \in A$, the mapping $a \longmapsto ab - ba$ is clearly a derivation, which is called an inner derivation implemented by b.

2 Derivations of $\mathcal{K}(X)$ and $\mathcal{L}(X)$

The aim of present section is to study the derivations of operators on a Hilbert C^*-module. Throughout this section A is a C^*-algebra.

Lemma 2.1 *[7] Every derivation of a C^*-algebra is bounded.*

Theorem 2.2 *Let X be a full right Hilbert A-module. If every derivation of $\mathcal{K}(X)$ is inner, then any derivation of $\mathcal{L}(X)$ is also inner.*

Proof: Let D be a derivation of $\mathcal{L}(X)$ and let $x, y \in X$. By Cohen's factorization Theorem [3] for x there exists $a \in A$, $z \in X$ such that $x = z.a$. Moreover since X is full, there exists (a_n) in $\langle X, X\rangle_X$ such that $a = \lim_n a_n$. Each a_n is of the form $a_n = \sum_{i=1}^{k_n} \langle x_{in}, y_{in}\rangle_X$ in which $x_{in}, y_{in} \in X$. Hence

$$\begin{aligned} D(\theta_{z.a_n,y}) &= D(\theta_{\sum_{i=1}^{k_n} z.\langle x_{in}, y_{in}\rangle_X, y}) \\ &= D(\sum_{i=1}^{k_n} \theta_{z.\langle x_{in}, y_{in}\rangle_X, y}) \\ &= \sum_{i=1}^{k_n} D(\theta_{z,x_{in}} \theta_{y_{in},y}) \\ &= \sum_{i=1}^{k_n} \theta_{z,x_{in}} D(\theta_{y_{in},y}) + \sum_{i=1}^{k_n} D(\theta_{z,x_{in}})\theta_{y_{in},y} \in \mathcal{K}(X). \end{aligned}$$

Since $\| \theta_{\sum_{i=1}^{k_n} z.\langle x_{in}, y_{in}\rangle_X, y} - \theta_{x,y} \| \leq \| \sum_{i=1}^{k_n} z.\langle x_{in}, y_{in}\rangle_X - x \| \| y \|$, we get $\theta_{\sum_{i=1}^{k_n} z.\langle x_{in}, y_{in}\rangle_X, y}$ converges to $\theta_{x,y}$ in norm topology, as n tends to ∞. It follows from Lemma (2.1) that D maps $\mathcal{K}(X)$ into itself. Now since every derivation of $\mathcal{K}(X)$ is inner, there exists $T \in \mathcal{K}(X)$ such that $D(K) = KT - TK$ for all $K \in \mathcal{K}(X)$. Now for $S \in \mathcal{L}(X)$ and $\theta_{x,y}$ we have $D(S\theta_{x,y}) = S\theta_{x,y}T - TS\theta_{x,y}$. On the other hand,

$$D(S\theta_{x,y}) = SD(\theta_{x,y}) + D(S)\theta_{x,y} = S\theta_{x,y}T - ST\theta_{x,y} + D(S)\theta_{x,y}$$

Consequently, we obtain $D(S)\theta_{x,y} = (ST - TS)\theta_{x,y}$. So for all $z \in X$, $D(S)\theta_{x,y}(z) = (ST - TS)\theta_{x,y}(z)$. Now since X is full, for every $u \in X$ there exist $x \in X$ and $(a_n) \subseteq A$ such that $u = \lim_n x.a_n$ and every a_n is of the form $a_n = \sum_{i=1}^{k_n} \langle x_{in}, y_{in}\rangle_X$ in which $x_{in}, y_{in} \in X$. Now since $D(S), (ST - TS) \in \mathcal{L}(X)$ we have

$$\begin{aligned}
D(S)(u) &= D(S)(\lim_n \sum_{i=1}^{k_n} x.\langle x_{in}, y_{in}\rangle_X) \\
&= \lim_n D(S)(\sum_{i=1}^{k_n} x.\langle x_{in}, y_{in}\rangle_X) \\
&= \lim_n \sum_{i=1}^{k_n} D(S)(x.\langle x_{in}, y_{in}\rangle_X) \\
&= \lim_n \sum_{i=1}^{k_n} (ST - TS)(x.\langle x_{in}, y_{in}\rangle_X) \\
&= (ST - TS)(\lim_n \sum_{i=1}^{k_n} x.\langle x_{in}, y_{in}\rangle_X) \\
&= (ST - TS)(u).
\end{aligned}$$

Hence $D(S) = ST - TS$ and this completes the proof.

The following definition of a Hilbert bimodule is orginally due to Brown, Mingo and Shen [2].

Definition 2.3 *Let X be an A-bimodule. X is said to be a Hilbert A-bimodule, when X is a left and right Hilbert A-module and satisfies the relation ${}_X\langle x, y\rangle.z = x.\langle y, z\rangle_X$.*

Example 2.4 *Let A be a C^*-algebra. Then A is a Hilbert A-bimodule with left and right inner products given by ${}_A\langle a, b\rangle = ab^*$ and $\langle a, b\rangle_A = a^*b \quad (a, b \in A)$.*

Proposition 2.5 *Let X be a Hilbert A-bimodule. If A is commutative then $\mathcal{K}(X)$ is commutative.*

Proof: Since $\mathcal{K}(E)$ is the closed linear span of $\{\theta_{x,y} : x, y \in X\}$, we show that $\theta_{x,y}\theta_{u,v}(z) = \theta_{u,v}\theta_{x,y}(z)$ for every $x, y, u, v, z \in X$.

$$\begin{aligned}\theta_{x,y}\theta_{u,v}(z) = \theta_{x.\langle y,u\rangle_X ,v}(z) = x.\langle y,u\rangle_X \langle v,z\rangle_X &= {}_X\langle x.\langle y,u\rangle_X , v\rangle.z \\ &= {}_X\langle {}_X\langle x,y\rangle.u, v\rangle.z \\ &= {}_X\langle x,y\rangle {}_X\langle u,v\rangle.z.\end{aligned}$$

$$\begin{aligned}\theta_{u,v}\theta_{x,y}(z) = \theta_{u.\langle v,x\rangle_X ,y}(z) = u.\langle v,x\rangle_X \langle y,z\rangle_X &= {}_X\langle u.\langle v,x\rangle_X , y\rangle.z \\ &= {}_X\langle {}_X\langle u,v\rangle.x, y\rangle.z \\ &= {}_X\langle u,v\rangle {}_X\langle x,y\rangle.z. \\ &= {}_X\langle x,y\rangle {}_X\langle u,v\rangle.z.\end{aligned}$$

Therefore $\theta_{x,y}\theta_{u,v} = \theta_{u,v}\theta_{x,y}$, as claimed.

Remark 2.6 *Let A be a C^*-algebra. In [4, theorem 2] R. V. Kadison showed that Each derivation of A annihilates its center.*

Corollary 2.7 *Let X be a Hilbert A-bimodule. If A is commutative then every derivation on $\mathcal{K}(X)$ is zero.*

Proof: Since A is commutative, the C^*-algebra $\mathcal{K}(X)$ is commutative. So by remark (2.6) every derivation on $\mathcal{K}(X)$ is zero.

Let X be a Hilbert A-bimodule and $T \in \mathcal{L}(X)$. Then there exists an operator $T^* \in \mathcal{L}(X)$ such that $\langle Tx, y\rangle_X = \langle x, T^*y\rangle_X$ $(x, y \in X)$. Here there exist one interesting point about T and T^*, in fact we can't conclude that ${}_X\langle Tx, y\rangle = {}_X\langle x, T^*y\rangle$. For example let A be a noncommutative C^*-algebra and $Z(A)$ be the center of A. Then for Hilbert A-bimodule A, the operator T_c $(c \notin Z(A))$ defined by $T_c(a) = ca$ on A is a adjointable operator and ${T_c}^* = T_{c^*}$, because

$$\langle T_c(a), b\rangle_A = \langle ca, b\rangle_A = (ca)^*b = a^*c^*b = \langle a, c^*b\rangle_A = \langle a, T_{c^*}b\rangle_A,$$

But since ${}_A\langle T_c(a), b\rangle = {}_A\langle ca, b\rangle = cab^*$ and ${}_A\langle a, T_{c^*}(b)\rangle = {}_A\langle a, c^*b\rangle = a(c^*b)^* = ab^*c$, we have ${}_A\langle T_c(a), b\rangle \neq {}_A\langle a, T_{c^*}(b)\rangle$.

Remark 2.8 *Let A be a commutative C^*-algebra and X a Hilbert C^*-bimodule over A. In [1, Proposition 1.4] B. Abadie and R. Exel proved that ${}_X\langle x, y\rangle.z = {}_X\langle z, y\rangle.x$ for all $x, y, z \in X$. By this Proposition, for all $x, y, z, t \in X$ we have:*

$$\begin{aligned}{}_X\langle {}_X\langle x,y\rangle.z, t\rangle = {}_X\langle {}_X\langle z,y\rangle.x, t\rangle = {}_X\langle z,y\rangle {}_X\langle x,t\rangle = &= {}_X\langle x,t\rangle {}_X\langle z,y\rangle \\ &= {}_X\langle {}_X\langle x,t\rangle.z, y\rangle \\ &= {}_X\langle x.\langle t,z\rangle_X , y\rangle.\end{aligned}$$

Proposition 2.9 *Let A be commutative and X be a Hilbert A-bimodule. If $T \in \mathcal{L}(X)$ Then for all $x, y \in X$, ${}_X\langle Tx, y\rangle = {}_X\langle x, T^*y\rangle$.*

Proof: Suppose that $u = {}_X\langle Tx, y\rangle - {}_X\langle x, T^*y\rangle$, we prove that $uu^* = 0$.

$$\begin{aligned} uu^* &= ({}_X\langle Tx, y\rangle - {}_X\langle x, T^*y\rangle)({}_X\langle y, Tx\rangle - {}_X\langle T^*y, x\rangle) \\ &= {}_X\langle Tx, y\rangle {}_X\langle y, Tx\rangle - {}_X\langle Tx, y\rangle {}_X\langle T^*y, x\rangle \\ &- {}_X\langle x, T^*y\rangle {}_X\langle y, Tx\rangle + {}_X\langle x, T^*y\rangle {}_X\langle T^*y, x\rangle \\ &= {}_X\langle {}_X\langle Tx, y\rangle .y, Tx\rangle - {}_X\langle {}_X\langle Tx, y\rangle .T^*y, x\rangle \\ &- {}_X\langle {}_X\langle x, T^*y\rangle .y, Tx\rangle + {}_X\langle {}_X\langle x, T^*y\rangle .T^*y, x\rangle \end{aligned}$$

Now by Remark (2.8), we have

$$\begin{aligned} uu^* &= {}_X\langle Tx.\langle Tx, y\rangle_X, y\rangle - {}_X\langle Tx.\langle x, T^*y\rangle_X, y\rangle \\ &- {}_X\langle x.\langle Tx, y\rangle_X, T^*y\rangle + {}_X\langle x.\langle x, T^*y\rangle_X, T^*y\rangle \\ &= {}_X\langle {}_X\langle Tx, Tx\rangle .y, y\rangle - {}_X\langle Tx.\langle Tx, y\rangle_X, y\rangle \\ &- {}_X\langle x.\langle x, T^*y\rangle_X, T^*y\rangle + {}_X\langle {}_X\langle x, x\rangle .T^*y, T^*y\rangle \\ &= {}_X\langle Tx, Tx\rangle {}_X\langle y, y\rangle - {}_X\langle Tx, Tx\rangle {}_X\langle y, y\rangle \\ &- {}_X\langle x, x\rangle {}_X\langle T^*y, T^*y\rangle + {}_X\langle x, x\rangle {}_X\langle T^*y, T^*y\rangle \\ &= 0. \end{aligned}$$

Thus we conclude that ${}_X\langle Tx, y\rangle - {}_X\langle x, T^*y\rangle = 0$ as claimed.

Theorem 2.10 *Let X be a Hilbert A-bimodule. If A is commutative then every derivation on $\mathcal{L}(X)$ is zero.*

Proof: Let D be a derivation of $\mathcal{L}(X)$. First notice that for every x, y in X, the operator $\theta_{x,y}$ belongs to the center of $\mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$, then $\theta_{x,y}T(z) = \theta_{x,T^*y}(z) = x.\langle T^*y, z\rangle_X = {}_X\langle x, T^*y\rangle .z$. Now by Proposition (2.9) we have

$$\theta_{x,y}T(z) = {}_X\langle Tx, y\rangle .z = Tx.\langle y, z\rangle_X = \theta_{Tx,y}(z) = T\theta_{x,y}(z).$$

So Remark (2.6) implies that for every x, y in X, $D(\theta_{x,y}) = 0$. Now we prove that for every operator $T \in \mathcal{L}(X)$, $D(T) = 0$. For this goal, let $x \in X$. Thus $D(T)\theta_{x,D(T)(x)} = D(T\theta_{x,D(T)(x)}) - TD(\theta_{x,D(T)(x)}) = 0$.
Hence for every $z \in X$ we conclude that

$$D(T)\theta_{x,D(T)(x)}(z) = D(T)(x.\langle D(T)(x), z\rangle_X) = D(T)(x).\langle D(T)(x), z\rangle_X = 0$$

Now by setting $z = D(T)(x)$ we have $D(T)(x).\langle D(T)(x), D(T)(x)\rangle_X = 0$ and so $\langle D(T)(x), D(T)(x)\rangle_X \langle D(T)(x), D(T)(x)\rangle_X = 0$. This implies that $\langle D(T)(x), D(T)(x)\rangle_X = 0$, consequently we obtain $D(T)(x) = 0$ and the proof is complete.

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