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Relative Type and Relative Weak Type Oriented Growth Analysis of Composite Entire and Meromorphic Functions

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Abstract

In this paper we deduce some comparative growth properties of composite entire and meromorphic functions on the basis of their relative type and relative weak type with respect to another entire function.

Keywords: *Meromorphic function, entire function, relative order, relative type, relative weak type.*

1 Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [9] and [14]. Therefore we do not explain those in details.

Now we just recall some definitions which will be needed in the sequel.

Definition 1 *The* order ρ_f *and* lower order λ_f *of an entire function* f *are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} .$$

When f *is meromorphic then*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

Definition 2 *The* type σ_f *and* lower type $\overline{\sigma}_f$ *of an entire function* f *are defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}},\ 0 < \rho_f < \infty .$$

If f *is meromorphic then*

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}},\quad 0 < \rho_f < \infty .$$

Datta and Jha [6] introduced the definition of *weak type* of an entire function of finite positive lower order in the following way:

Definition 3 *[6] The* weak type τ_f *and the growth indicator* $\overline{\tau}_f$ *of an entire function* f *of finite positive lower order* λ_f *are defined by*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty .$$

When f *is meromorphic then*

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty .$$

If an entire function g is non-constant then $M_g(r)$ and $T_g(r)$ are both strictly increasing and continuous function of r. Hence there exist inverse functions $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} M_g^{-1}(s) = \infty$ and $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ with $\lim_{s\to\infty} T_g^{-1}(s) = \infty$ respectively.

Bernal {[1], [2]} introduced the definition of *relative order* of an entire function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [13] if $g(z) = \exp z$.

Similarly, one can define the *relative lower order* of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Extending this notion, Lahiri and Banerjee [11] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [11] if $g(z) = \exp z$.

In the same way, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite *relative order* with respect to another entire function, Roy [12] introduced the notion of *relative type* of two entire functions in the following way:

Definition 4 *[12] Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the* relative type $\sigma_g(f)$ *of f with respect to g is defined as :*

$$\begin{aligned}&\sigma_g(f) \\ = \;& \inf\left\{k > 0 : M_f(r) < M_g\left(k r^{\rho_g(f)}\right) \textit{ for all sufficiently large values of } r\right\} \\ = \;& \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}} .\end{aligned}$$

Likewise, one can define the relative lower type *of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :*

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}}, \ 0 < \rho_g(f) < \infty .$$

Analogusly, to determine the relative growth of two entire functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [7] introduced the definition of *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way:

Definition 5 *[7] The* relative weak type $\tau_g(f)$ *of an entire function* f *with respect to another entire function* g *having finite positive* relative lower order $\lambda_g(f)$ *is defined as:*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} \ .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ *of an entire function* f *with respect to an entire function* g *in the following way :*

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}}, \ 0 < \lambda_g(f) < \infty \ .$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas also [7] gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 6 *[7] The* relative type $\sigma_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *are defined as*

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \textit{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the lower relative type $\overline{\sigma}_g(f)$ *in the following way:*

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \textit{where } 0 < \rho_g(f) < \infty.$$

Definition 7 *[7] The* relative weak type $\tau_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g(f)$ *is defined by*

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ *of a meromorphic function* f *with respect to an entire function* g *with finite positive relative lower order* $\lambda_g(f)$ *as*

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 4 , Definition 5, Definition 6 and Definition 7 coincide with the classical definitons of type (lower type) and weak type of entire ane meromorphic functions respectively. In this connection the following definition is relevant:

Definition 8 *[1] A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 < M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [1].*

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative type* as well as *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysists and they are not aware of the technical advantages of using the relative growth indicators of the functions. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order, relative type* and *relative weak type* some of which has been explored in this paper.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[3] Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) \ .$$

Lemma 2 *[4] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f(\exp(r^\mu)) \ .$$

Lemma 3 *[10] Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) > T_g(\exp(r^\mu)),$$

where $0 < \mu < \rho_g$.

Lemma 4 *[5] Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f(\exp(r^\mu)) \ .$$

Lemma 5 *[5] Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_g\left(\exp\left(r^{\mu}\right)\right) \ .$$

Lemma 6 *[8] Let f be an entire function which satisfy the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f\left(r\right) < T_f\left(\alpha r^{\delta}\right) \ .$$

3 Main Results

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h\left(f\right) \leq \rho_h\left(f\right) < \infty$, $\sigma_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)} \leq \frac{\delta\cdot\sigma_g\cdot\rho_h\left(f\right)}{\lambda_h\left(f\right)} \ .$$

Proof. Let us suppose that $\alpha > 2$.
Since $T_h^{-1}\left(r\right)$ is an increasing function r, it follows from Lemma 1, Lemma 6 and the inequality $T_g\left(r\right) \leq \log M_g\left(r\right)$ $\{cf.$ [9]$\}$ for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}T_{f\circ g}\left(r\right) &\leqslant T_h^{-1}\left[\left\{1+o(1)\right\}T_f\left(M_g\left(r\right)\right)\right] \\ i.e.,\ T_h^{-1}T_{f\circ g}\left(r\right) &\leqslant \alpha\left[T_h^{-1}T_f\left(M_g\left(r\right)\right)\right]^{\delta} \\ i.e.,\ \log T_h^{-1}T_{f\circ g}\left(r\right) &\leqslant \delta\log T_h^{-1}T_f\left(M_g\left(r\right)\right)+O(1) \end{aligned} \quad (1)$$

$$i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)}$$

$$\leq \frac{\delta\log T_h^{-1}T_f\left(M_g\left(r\right)\right)+O(1)}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)} = \frac{\delta\log T_h^{-1}T_f\left(M_g\left(r\right)\right)+O(1)}{\log M_g\left(r\right)}\cdot \frac{\log M_g\left(r\right)}{r^{\rho_g}}\cdot\frac{\log\exp r^{\rho_g}}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)} \quad (2)$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)}$$

$$\leq \limsup_{r\to\infty}\frac{\delta\log T_h^{-1}T_f\left(M_g\left(r\right)\right)+O(1)}{\log M_g\left(r\right)}\cdot\limsup_{r\to\infty}\frac{\log M_g\left(r\right)}{r^{\rho_g}}\cdot \limsup_{r\to\infty}\frac{\log\exp r^{\rho_g}}{\log T_h^{-1}T_f\left(\exp r^{\rho_g}\right)}$$

$$i.e., \ \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(\exp r^{\rho_g})}\leq \delta\cdot\rho_h(f)\cdot\sigma_g\cdot\frac{1}{\lambda_h(f)}\ .$$

Thus the theorem is established.

In the line of Theorem 1 the following theorem can be proved :

Theorem 2 *Let f be a meromorphic function, g and h be any two entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\sigma_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(\exp r^{\rho_g})}\leq \frac{\delta\cdot\sigma_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Using the notion of l*ower type*, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 1 and Theorem 2 respectively.

Theorem 3 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(\exp r^{\rho_g})}\leq \frac{\delta\cdot\overline{\sigma}_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Theorem 4 *Let f be a meromorphic function, g and h be any two entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(\exp r^{\rho_g})}\leq \frac{\delta\cdot\overline{\sigma}_g\cdot\rho_h(f)}{\lambda_h(g)}\ .$$

Using the concept of the growth indicators τ_g and $\overline{\tau}_g$ of an entire function g, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 1, Theorem 2, Theorem 3 and Theorem 4 respectively.

Theorem 5 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\tau}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(\exp r^{\lambda_g})}\leq \frac{\delta\cdot\overline{\tau}_g\cdot\rho_h(f)}{\lambda_h(f)}\ .$$

Theorem 6 *Let f be a meromorphic function, g and h be any two entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\overline{\tau}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \overline{\tau}_g \cdot \rho_h(f)}{\lambda_h(g)} \ .$$

Theorem 7 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\tau_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(f)} \ .$$

Theorem 8 *Let f be a meromorphic function, g and h be any two entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\tau_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(g)} \ .$$

Theorem 9 *Let f be meromorphic and g, h be any two entire functions such that (i) $0 < \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \sigma_h(f) < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\sigma_h(f)} \ .$$

Proof. From (1) we get for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon) \log M_g(r) + O(1) \ . \tag{3}$$

Using the Definition 2, we obtain from (3) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_g} + O(1) \ . \tag{4}$$

Now in view of condition (ii), we obtain from (4) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon) \cdot r^{\rho_h(f)} + O(1) \ . \tag{5}$$

Again in view of Definition 6, we get for a sequence of values of r tending to infinity that

$$T_h^{-1} T_f(r) \geq (\sigma_h(f) - \varepsilon) r^{\rho_h(f)} \ . \tag{6}$$

Therefore from (5) and (6) , it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta\left(\rho_h(f)+\varepsilon\right)\left(\sigma_g+\varepsilon\right) \cdot r^{\rho_h(f)}+O(1)}{\left(\sigma_h(f)-\varepsilon\right) r^{\rho_h(f)}} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\sigma_h(f)} .$$

Hence the theorem follows.

Using the notion of *lower type* and *relative lower type*, we may state the following theorem without its proof as it can be carried out in the line of Theorem 9 :

Theorem 10 *Let f be meromorphic and g, h be any two entire functions with (i) $0<\rho_h(f)<\infty$, (ii) $\rho_h(f)=\rho_g$, (iii) $\overline{\sigma}_g<\infty$, (iv) $0<\overline{\sigma}_h(f)<\infty$ and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \overline{\sigma}_g}{\overline{\sigma}_h(f)} .$$

Similarly using the notion of *type* and *relative lower type,* one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 9 :

Theorem 11 *Let f be meromorphic and g, h be any two entire functions such that (i) $0<\lambda_h(f) \leq \rho_h(f)<\infty$, (ii) $\rho_h(f)=\rho_g$, (iii) $\sigma_g<\infty$, (iv) $0<\overline{\sigma}_h(f)<\infty$ and h satisfies the Property (A). Then for any $\delta>1$,*

$$\liminf_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \lambda_h(f) \cdot \sigma_g}{\overline{\sigma}_h(f)} .$$

Theorem 12 *Let f be meromorphic and g, h be any two entire functions with (i) $0<\rho_h(f)<\infty$, (ii) $\rho_h(f)=\rho_g$, (iii) $\sigma_g<\infty$, (iv) $0<\overline{\sigma}_h(f)<\infty$ and h satisfies the Property (A). Then for any $\delta>1$,*

$$\limsup_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\overline{\sigma}_h(f)} .$$

Similarly, using the concept of *weak type* and *relative weak type*, we may state next four theorems without their proofs as those can be carried out in the line of Theorem 9, Theorem 10, Theorem 11 and Theorem 12 respectively.

Theorem 13 *Let f be meromorphic and g, h be any two entire functions such that (i) $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, (ii) $\lambda_h(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \overline{\tau}_h(f) < \infty$ and h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \overline{\tau}_g}{\overline{\tau}_h(f)} .$$

Theorem 14 *Let f be meromorphic and g, h be any two entire functions with (i) $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, (ii) $\lambda_h(f) = \lambda_g$, (iii) $\tau_g < \infty$, (iv) $0 < \tau_h(f) < \infty$ and h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \tau_g}{\tau_h(f)} .$$

Theorem 15 *Let f be meromorphic and g, h be any two entire functions such that (i) $0 < \lambda_h(f) < \infty$, (ii) $\lambda_h(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \tau_h(f) < \infty$ and h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \lambda_h(f) \cdot \overline{\tau}_g}{\tau_h(f)} .$$

Theorem 16 *Let f be meromorphic and g, h be any two entire functions with (i) $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, (ii) $\lambda_h(f) = \lambda_g$, (iii) $\overline{\tau}_g < \infty$, (iv) $0 < \tau_h(f) < \infty$ and h satisfies the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\delta \cdot \rho_h(f) \cdot \overline{\tau}_g}{\tau_h(f)} .$$

Theorem 17 *Let f be meromorphic g, h and l be any three entire functions such that $0 < \overline{\sigma}_h(f\circ g) \leq \sigma_h(f\circ g) < \infty$, $0 < \overline{\sigma}_l(f) \leq \sigma_l(f) < \infty$ and $\rho_h(f\circ g) = \rho_l(f)$. Then*

$$\frac{\overline{\sigma}_h(f\circ g)}{\sigma_l(f)} \leq \liminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_l(f)} \leq \limsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\sigma_h(f\circ g)}{\overline{\sigma}_l(f)} .$$

Proof. From the definition of $\sigma_l(f)$ and $\overline{\sigma}_h(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1} T_{f\circ g}(r) \geqslant (\overline{\sigma}_h(f\circ g) - \varepsilon)\, r^{\rho_h(f\circ g)} \tag{7}$$

and

$$T_l^{-1} T_f(r) \leq (\sigma_l(f) + \varepsilon)\, r^{\rho_l(f)} . \tag{8}$$

Now from (7), (8) and in view of the condition $\rho_h(f\circ g) = \rho_l(f)$, it follows for all sufficiently large values of r that

$$\frac{T_h^{-1} T_{f\circ g}(r)}{T_l^{-1} T_f(r)} \geqslant \frac{(\overline{\sigma}_h(f\circ g) - \varepsilon)}{(\sigma_l(f) + \varepsilon)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\geqslant\frac{\overline{\sigma}_h(f\circ g)}{\sigma_l(f)}\ . \tag{9}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r)\leq(\overline{\sigma}_h(f\circ g)+\varepsilon)\,r^{\rho_h(f\circ g)} \tag{10}$$

and for all sufficiently large values of r,

$$T_l^{-1}T_f(r)\geqslant(\overline{\sigma}_l(f)-\varepsilon)\,r^{\rho_l(f)}\ . \tag{11}$$

Combining (10), (11) and in view of the condition $\rho_h(f\circ g)=\rho_l(f)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{(\overline{\sigma}_h(f\circ g)+\varepsilon)}{(\overline{\sigma}_l(f)-\varepsilon)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_l(f)}\ . \tag{12}$$

Also for a sequence of values of r tending to infinity that

$$T_l^{-1}T_f(r)\leq(\overline{\sigma}_l(f)+\varepsilon)\,r^{\rho_l(f)}\ . \tag{13}$$

Now from (7), (13) and using the condition $\rho_h(f\circ g)=\rho_l(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\geq\frac{(\overline{\sigma}_h(f\circ g)-\varepsilon)}{(\overline{\sigma}_l(f)+\varepsilon)}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\geq\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_l(f)}\ . \tag{14}$$

Also we obtain for all sufficiently large values of r that

$$T_h^{-1}T_{f\circ g}(r)\leq(\sigma_h(f\circ g)+\varepsilon)\,r^{\rho_h(f\circ g)}\ . \tag{15}$$

In view of the condition $\rho_h(f\circ g)=\rho_l(f)$, it follows from (11) and (15) for all sufficiently large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{(\sigma_h(f\circ g)+\varepsilon)}{(\overline{\sigma}_l(f)-\varepsilon)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{\sigma_h(f\circ g)}{\overline{\sigma}_l(f)}\,. \tag{16}$$

Thus the theorem follows from $(9)\,,(12)\,,(14)$ and $(16)\,.$

The following theorem can be proved in the line of Theorem 17 and so its proof is omitted.

Theorem 18 *Let f be meromorphic, g,h and k be any three entire functions with $0<\overline{\sigma}_h(f\circ g)\leq\sigma_h(f\circ g)<\infty$, $0<\overline{\sigma}_k(g)\leq\sigma_k(g)<\infty$ and $\rho_h(f\circ g)=\rho_k(g)$. Then*

$$\frac{\overline{\sigma}_h(f\circ g)}{\sigma_k(g)}\leq\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)}\leq\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_k(g)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)}\leq\frac{\sigma_h(f\circ g)}{\overline{\sigma}_k(g)}\,.$$

Theorem 19 *Let f be meromorphic g, h and l be any three entire functions such that $0<\sigma_h(f\circ g)<\infty$, $0<\sigma_l(f)<\infty$ and $\rho_h(f\circ g)=\rho_l(f)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{\sigma_h(f\circ g)}{\sigma_l(f)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\,.$$

Proof. From the definition of $\sigma_l(f)\,,$ we get for a sequence of values of r tending to infinity that

$$T_l^{-1}T_f(r)\geqslant(\sigma_k(f)-\varepsilon)\,r^{\rho_l(f)}\,. \tag{17}$$

Now from (15), (17) and in view of the condition $\rho_h(f\circ g)=\rho_l(f)\,,$ it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{(\sigma_h(f\circ g)+\varepsilon)}{(\sigma_l(f)-\varepsilon)}\,.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\leq\frac{\sigma_h(f\circ g)}{\sigma_l(f)}\,. \tag{18}$$

Again for a sequence of values of r tending to infinity ,

$$T_h^{-1}T_{f\circ g}(r)\geqslant(\sigma_h(f\circ g)-\varepsilon)\,r^{\rho_h(f\circ g)}\,. \tag{19}$$

So combining (8), (19) and using the condition $\rho_h(f\circ g)=\rho_l(f)\,,$ we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\geqslant\frac{(\sigma_h(f\circ g)-\varepsilon)}{(\sigma_l(f)+\varepsilon)}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\geqslant\frac{\sigma_h(f\circ g)}{\sigma_l(f)}\ . \tag{20}$$

Thus the theorem follows from (18) and (20).

The following theorem can be carried out in the line of Theorem 19 and therefore we omit its proof.

Theorem 20 *Let f be meromorphic, g,h and k be any three entire functions with $0<\sigma_h(f\circ g)<\infty$, $0<\sigma_k(g)<\infty$ and $\rho_h(f\circ g)=\rho_k(g)$. Then*

$$\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\leq\frac{\sigma_h(f\circ g)}{\sigma_k(g)}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .$$

The following theorem is a natural consequence of Theorem 17 and Theorem 19.

Theorem 21 *Let f be meromorphic g, h and l be any three entire functions such that $0<\overline{\sigma}_h(f\circ g)\leq\sigma_h(f\circ g)<\infty$, $0<\overline{\sigma}_l(f)\leq\sigma_l(f)<\infty$ and $\rho_h(f\circ g)=\rho_l(f)$. Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}&\leq\min\left\{\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_l(f)},\frac{\sigma_h(f\circ g)}{\sigma_l(f)}\right\}\\&\leq\max\left\{\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_l(f)},\frac{\sigma_h(f\circ g)}{\sigma_l(f)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)}\ .\end{aligned}$$

The proof is omitted.

Analogously, one may state the following theorem without its proof as it is also a natural consequence of Theorem 18 and Theorem 20.

Theorem 22 *Let f be meromorphic, g,h and k be any three entire functions with $0<\overline{\sigma}_h(f\circ g)\leq\sigma_h(f\circ g)<\infty$, $0<\overline{\sigma}_k(g)\leq\sigma_k(g)<\infty$ and $\rho_h(f\circ g)=\rho_k(g)$. Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}&\leq\min\left\{\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_k(g)},\frac{\sigma_h(f\circ g)}{\sigma_k(g)}\right\}\\&\leq\max\left\{\frac{\overline{\sigma}_h(f\circ g)}{\overline{\sigma}_k(g)},\frac{\sigma_h(f\circ g)}{\sigma_k(g)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .\end{aligned}$$

In the same way using the concept of *relative weak type*, we may state the next two theorems without their proofs as those can be carried out in the line of Theorem 17 and Theorem 19 respectively.

Theorem 23 *Let f be meromorphic g, h and l be any three entire functions such that $0 < \tau_h(f \circ g) \leq \overline{\tau}_h(f \circ g) < \infty$, $0 < \tau_l(f) \leq \overline{\tau}_l(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_l(f)$. Then*

$$\frac{\tau_h(f \circ g)}{\overline{\tau}_l(f)} \leq \liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)} \leq \frac{\tau_h(f \circ g)}{\tau_l(f)} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)} \leq \frac{\overline{\tau}_h(f \circ g)}{\tau_l(f)} .$$

Theorem 24 *Let f be meromorphic g, h and l be any three entire functions with $0 < \overline{\tau}_h(f \circ g) < \infty$, $0 < \overline{\tau}_l(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_l(f)$. Then*

$$\liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)} \leq \frac{\overline{\tau}_h(f \circ g)}{\overline{\tau}_l(f)} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)} .$$

The following theorem is a natural consequence of Theorem 23 and Theorem 24:

Theorem 25 *Let f be meromorphic g, h and l be any three entire functions such that $0 < \tau_h(f \circ g) \leq \overline{\tau}_h(f \circ g) < \infty$, $0 < \tau_l(f) \leq \overline{\tau}_l(f) < \infty$ and $\lambda_h(f \circ g) = \lambda_l(f)$. Then*

$$\begin{aligned} \liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)} &\leq \min\left\{\frac{\overline{\tau}_h(f \circ g)}{\overline{\tau}_l(f)}, \frac{\tau_h(f \circ g)}{\tau_l(f)}\right\} \\ &\leq \max\left\{\frac{\overline{\tau}_h(f \circ g)}{\overline{\tau}_l(f)}, \frac{\tau_h(f \circ g)}{\tau_l(f)}\right\} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_l^{-1}T_f(r)} . \end{aligned}$$

The following two theorems can be proved in the line of Theorem 23 and Theorem 24 respectively and therefore their proofs are omitted.

Theorem 26 *Let f be meromorphic, g, h and k be any three entire functions with $0 < \tau_h(f \circ g) \leq \overline{\tau}_h(f \circ g) < \infty$, $0 < \tau_k(g) \leq \overline{\tau}_k(g) < \infty$ and $\lambda_h(f \circ g) = \lambda_k(g)$. Then*

$$\frac{\tau_h(f \circ g)}{\overline{\tau}_k(g)} \leq \liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)} \leq \frac{\tau_h(f \circ g)}{\tau_k(g)} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)} \leq \frac{\overline{\tau}_h(f \circ g)}{\tau_k(g)} .$$

Theorem 27 *Let f be meromorphic, g, h and k be any three entire functions such that $0 < \overline{\tau}_h(f \circ g) < \infty$, $0 < \overline{\tau}_k(g) < \infty$ and $\lambda_h(f \circ g) = \lambda_k(g)$. Then*

$$\liminf_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} \leq \frac{\overline{\tau}_h(f \circ g)}{\overline{\tau}_k(g)} \leq \limsup_{r\to\infty} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} .$$

The following theorem is a natural consequence of Theorem 26 and Theorem 27:

Theorem 28 *Let f be meromorphic, g, h and k be any three entire functions with $0<\tau_h(f\circ g)\leq\overline{\tau}_h(f\circ g)<\infty$, $0<\tau_k(g)\leq\overline{\tau}_k(g)<\infty$ and $\lambda_h(f\circ g)=\lambda_k(g)$. Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} &\leq \min\left\{\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_k(g)},\frac{\tau_h(f\circ g)}{\tau_k(g)}\right\}\\ &\leq \max\left\{\frac{\overline{\tau}_h(f\circ g)}{\overline{\tau}_k(g)},\frac{\tau_h(f\circ g)}{\tau_k(g)}\right\}\leq\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .\end{aligned}$$

Theorem 29 *Let f be meromorphic, g and h be any two entire functions such that $0<\lambda_h(f)\leq\rho_h(f)<\rho_g\leq\infty$ and $\sigma_h(f)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)}\geq\frac{\lambda_h(f)}{\sigma_h(f)}\ .$$

Proof. Since $\rho_h(f)<\rho_g$ and $T_h^{-1}(r)$ is a increasing function of r, we get from Lemma 2 for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r) &\geq \log T_h^{-1}T_f(\exp(r^{\mu}))\\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\geq (\lambda_h(f)-\varepsilon)\cdot r^{\mu}\\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\geq (\lambda_h(f)-\varepsilon)\cdot r^{\rho_h(f)}\ . \end{aligned}\tag{21}$$

Again in view of Definition 6, we get for all sufficiently large values of r that

$$T_h^{-1}T_f(r)\leq(\sigma_h(f)+\varepsilon)\,r^{\rho_h(f)}\ .\tag{22}$$

Now from (21) and (22), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)}\geq\frac{(\lambda_h(f)-\varepsilon)\,r^{\rho_h(f)}}{(\sigma_h(f)+\varepsilon)\,r^{\rho_h(f)}}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)}\geq\frac{\lambda_h(f)}{\sigma_h(f)}\ .$$

Thus the theorem follows.

In the line of Theorem 29, the following theorem can be proved and therefore its proof is omitted:

Theorem 30 *Let f be meromorphic, g and h be any two entire functions with $0<\lambda_h(f)$, $0<\rho_h(g)<\rho_g\leq\infty$ and $\sigma_h(g)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)}\geq\frac{\lambda_h(f)}{\sigma_h(g)}\ .$$

The following two theorems can also be proved in the line of Theorem 29 and Theorem 30 respectively and with help of Lemma 3. Hence their proofs are omitted.

Theorem 31 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(g)$, $0 < \lambda_f$, $0 < \rho_h(f) < \rho_g < \infty$ and $\sigma_h(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \geq \frac{\lambda_h(g)}{\sigma_h(f)} .$$

Theorem 32 *Let f be meromorphic, g and h be any two entire functions with $0 < \lambda_h(g)$, $0 < \lambda_f$, $0 < \rho_h(g) < \rho_g < \infty$ and $\sigma_h(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \geq \frac{\lambda_h(g)}{\sigma_h(g)} .$$

Now we state the following four theorems without their proofs as those can be carried out in the line of Theorem 29, Theorem 30, Theorem 31 and Theorem 32 and with the help of Definition 7:

Theorem 33 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(f) < \rho_g \leq \infty$ and $\overline{\tau}_h(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \geq \frac{\lambda_h(f)}{\overline{\tau}_h(f)} .$$

Theorem 34 *Let f be meromorphic, g and h be any two entire functions with $0 < \lambda_h(f)$, $0 < \lambda_h(g) < \rho_g \leq \infty$ and $\overline{\tau}_h(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \geq \frac{\lambda_h(f)}{\overline{\tau}_h(g)} .$$

Theorem 35 *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ and $\overline{\tau}_h(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \geq \frac{\lambda_h(g)}{\overline{\tau}_h(f)} .$$

Theorem 36 *Let f be meromorphic, g and h be any two entire functions with $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ and $\overline{\tau}_h(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \geq \frac{\lambda_h(g)}{\overline{\tau}_h(g)} .$$

Theorem 37 *Let f be meromorphic with non zero finite order and lower order. Also let g and h be any two entire functions such that $0 < \lambda_g < \rho_h(f) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(f)} .$$

Proof. As $\lambda_g < \rho_h(f)$ and $T_h^{-1}(r)$ is a increasing function of r, it follows from Lemma 4 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &< \log T_h^{-1} T_f(\exp(r^{\mu})) \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r) &< (\rho_h(f) + \varepsilon) \cdot r^{\mu} \\ i.e., \ \log T_h^{-1} T_{f\circ g}(r) &< (\rho_h(f) + \varepsilon) \cdot r^{\rho_h(f)} . \end{aligned} \tag{23}$$

Further in view of Definition 6, we obtain for all sufficiently large values of r that

$$T_h^{-1} T_f(r) \geq (\overline{\sigma}_h(f) - \varepsilon) r^{\rho_h(f)} . \tag{24}$$

Since $\varepsilon (> 0)$ is arbitrary, therefore from (23) and (24) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} &\leq \frac{(\rho_h(f) + \varepsilon) \cdot r^{\rho_h(f)}}{(\overline{\sigma}_h(f) - \varepsilon) r^{\rho_h(f)}} \\ i.e., \ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} &\leq \frac{\rho_h(f)}{\overline{\sigma}_h(f)} . \end{aligned}$$

Hence the theorem is established.

In the line of Theorem 37 the following theorem can be proved and therefore its proof is omitted:

Theorem 38 *Let f be meromorphic with non zero finite order and lower order, g and h be any two entire functions with $\rho_h(f) < \infty$, $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \leq \frac{\rho_h(f)}{\overline{\sigma}_h(g)} .$$

Moreover, the following two theorems can also be deduced in the line of Theorem 29 and Theorem 30 respectively and with help of Lemma 5 and therefore their proofs are omitted.

Theorem 39 *Let f be a meromorphic function of finite order, g and h be any two entire functions such that $\rho_h(g) < \infty$, $0 < \lambda_g < \rho_h(f) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\rho_h(g)}{\overline{\sigma}_h(f)} .$$

Theorem 40 *Let f be a meromorphic function of finite order, g and h be any two entire functions with $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \leq \frac{\rho_h(g)}{\overline{\sigma}_h(g)} .$$

Finally we state the following four theorems without their proofs as those can be carried out in the line of Theorem 37, Theorem 38, Theorem 39 and Theorem 40 using the concept of *relative weak type:*

Theorem 41 *Let f be meromorphic with non zero finite order and lower order, g and h be any two entire functions such that $0 < \lambda_g < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\tau_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\rho_h(f)}{\tau_h(f)} .$$

Theorem 42 *Let f be meromorphic with non zero finite order and lower order, g and h be any two entire functions with $\rho_h(f) < \infty$, $0 < \lambda_g < \lambda_h(g) < \infty$ and $\tau_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \leq \frac{\rho_h(f)}{\tau_h(g)} .$$

Theorem 43 *Let f be a meromorphic function of finite order, g and h be any two entire functions such that $\rho_h(g) < \infty$, $0 < \lambda_g < \lambda_h(f) < \infty$ and $\tau_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_f(r)} \leq \frac{\rho_h(g)}{\tau_h(f)} .$$

Theorem 44 *Let f be a meromorphic function of finite order, g and h be any two entire functions with $0 < \lambda_g < \lambda_h(f) \leq \rho_h(g) < \infty$ and $\tau_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_h^{-1} T_g(r)} \leq \frac{\rho_h(g)}{\tau_h(g)} .$$

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