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γ-Operation in M-Topological Space

S.A. El-Sheikh[1], R.A.K. Omar[2] and M. Raafat[3]

[1,2,3]Mathematics Department, Faculty of Education
Ain Shams University, Cairo, Egypt
[1]E-mail: elsheikh33@hotmail.com
[2]E-mail: ben_omar1955@yahoo.com
[3]E-mail: dr_mahmoudraafat2011@yahoo.com

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Abstract

In this paper we extend the notions of γ-operation, pre-open msets , α-open msets, semi open msets, b-open msets and β-open msets to M-topological spaces. These types of msets are new classes of multisets. We study the relations between these different types of submsets of M-topological spaces. Also, we study some of their properties and show that these types generalize the notion of open (closed) msets.

Keywords: *Mset, M-topological space, γ-operation, Pre-open mset, α-open mset, Semi-open mset, β-open mset, b-open mset.*

1 Introduction

The notion of a multiset is well established both in mathematics and computer science [6, 7, 9, 13, 21, 23, 25, 26]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [8, 14, 21, 22, 24]. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by A(x). One of the most natural

and simplest examples is the multiset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the multiset $M = \{3/2, 2/3, 1/7\}$ where $C_M(2) = 3$, $C_M(3) = 2$, $C_M(7) = 1$.
Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

Generalized open sets play a very important role in general topology and they are now the research topies of many topologists worldwide. Andrijevic [2, 5] introduced a class of generalized open sets in a topological space as b-open sets and α-sets. The class of b-open sets is contained in the class of β-open sets and contains both semi-open sets and pre-open sets. Levine [16], Mashhour et. al.[18] , Njastad [20]and Abd El-Monsef et. al.[1] introduced semi-open sets, pre-open sets, α-sets and β-open sets respectively. Andrijevic [4]called β-open sets as semi-preopen sets. The complement of a semi-open (resp. preopen, α-open, semi-preopen, b-open) set is called a semi-closed (resp. preclosed, α-closed, semi-preclosed, b-closed) set. Andrijevic [3] introduced a new class of topology generated by preopen sets and the corresponding closure and interior operators. Also, a γ-operation and decompositions of some forms of soft continuity in soft topological spaces given by Kandil et al. in [15].
In this paper we introduce the notions of γ-operation, pre-open msets , α-open msets, semi open msets, b-open msets and β-open msets in M-topological space. We study their properties and relations between them.

2 Preliminaries

Definition 2.1 *[11] A mset X drawn from the set U is represented by a Count function X or C_X defined as $C_X : U \to N$ where N represents the set of non negative integers .*

Here C_X (x) is the number of occurrences of the element x in the mset X. We present the mset X drawn from the set $U = \{x_1, x_2, x_3, ..., x_n\}$ as $X =$

$\{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}$ where m_i is the number of occurreneces of the element x_i , $i = 1, 2, 3, ..., n$ in the mset X .

Definition 2.2 *[11] A domain U, is defined as a set of elements from which msets are constructed. The mset space $[U]^w$ is the set of all msets whose elements are in U such that no element in the mset occurs more than w times. The mset space $[U]^\infty$ is the set of all msets over a domain U such that there is no limit on the number of occurrences of an element in a mset. If $U = \{x_1, x_2, ..., x_k\}$ then $[U]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : m_i \in \{0, 1, 2, ..., w\}$ for $i = 1, 2, ..., k\}$.*

Definition 2.3 *[11] Let X and Y be two msets drawn from a set U. Then :*

1. *$X = Y$ if $C_X(x) = C_Y(x)$ for all $x \in U$.*
2. *$X \subseteq Y$ if $C_X(x) \leq C_Y(x)$ for all $x \in U$.*
3. *$P = X \cup Y$ if $C_P(x) = Max\ \{C_X(x), C_Y(x)\}$ for all $x \in U$.*
4. *$P = X \cap Y$ if $C_P(x) = Min\ \{C_X(x), C_Y(x)\}$ for all $x \in U$.*
5. *$P = X \oplus Y$ if $C_P(x) = Min\ \{C_X(x) + C_Y(x)\ ,\ w\}$ for all $x \in U$.*
6. *$P = X \ominus Y$ if $C_P(x) = Max\ \{C_X(x) - C_Y(x)\ ,\ 0\}$ for all $x \in U$ where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.*

Definition 2.4 *[11] Let X be a mset drawn from the set U and if $C_X(x)=0$ $\forall x \in U$ then, X is called empty mset and denoted by ϕ i.e. $\phi(x)=0\ \forall x$.*

Definition 2.5 *[11](Whole submset) A submset Y of X is a whole submset of X with each element in Y having full multiplicity as in X. i.e., $C_Y(x) = C_X(x)$ for every x in Y^*.*

Definition 2.6 *[11](Partial Whole submset) A submset Y of X is a partial whole submset of X with at least one element in Y having full multiplicity as in X. i.e., $C_Y(x) = C_X(x)$ for some x in Y^*.*

Definition 2.7 *[11](Full submset) A submset Y of X is a full submset of X if each element in X is an element in Y with the same or lesser multiplicity as in X. i.e., $X^* = Y^*$ with $C_Y(x) \leq C_X(x)$ for every x in Y^*.*

Remark 2.8 *[11] Empty mset ϕ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset X.*

Example 2.9 *[11] Let $X = \{2/x, 3/y, 5/z\}$ be a mset. Following are the some of the submsets of X which are whole submsets, partial whole submsets and full submsets :*

1. *A submset {2/x,3/y} is a whole submset and partial whole submset of X but it is not full submset of X.*
2. *A submset {1/x,3/y,2/z} is partial whole submset and full submset of X but it is not a whole submset of X.*
3. *A submset {1/x,3/y} is partial whole submset of X which is neither whole submset nor full submset of X.*

Definition 2.10 *[11](Power Whole Mset) Let $X \in [U]^w$ be a mset. The power whole mset of X denoted by PW(X) is defined as the set of all whole submsets of X. i.e., for constructing power whole submsets of X, every element of X with its full multiplicity behaves like an element in a classical set. The cardinality of PW(X) is 2^n where n is the cardinality of the support set (root set) of X.*

Definition 2.11 *[11](Power Full Mset) Let $X \in [U]^w$ be a mset. The power full mset of X denoted by PF(X) is defined as the set of all full submsets of X. The cardinality of PF(X) is the product of the counts of the elements in X.*

Remark 2.12 *[11] PW(X) and PF(X) are ordinary sets whose elements are msets.*

If X is an ordinary set with n distinct elements, then the power set P(X) of X contains exactly 2^n elements. If X is a multiset with n elements (repetitions counted), then the power set P(X) contains strictly less than 2^n elements because singleton submsets do not repeat in P(X). In classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of X for finite mset X that preserves Cantor's power set theorem.

Definition 2.13 *[11] (Power Mset) Let $X \in [U]^w$ be a mset. The power mset P(X) of X is the set of all submsets of X. We have $Y \in P(X)$ if and only if $Y \subseteq X$. If $Y = \phi$, then $Y \in^1 P(X)$; and if $Y \neq \phi$, then $Y \in^k P(X)$ where $k = \prod_z \begin{pmatrix} \mid [X]_z \mid \\ \mid [Y]_z \mid \end{pmatrix}$, the product $\prod_z$ is taken over by distinct elements of z of the mset Y and $\mid [X]_z \mid = m$ iff $z \in^m X$, $\mid [Y]_z \mid = n$ iff $z \in^n Y$, then $\begin{pmatrix} \mid [X]_z \mid \\ \mid [Y]_z \mid \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$*
The power set of a mset is the support set of the power mset and is denoted by $P^*(X)$. The following theorem shows the cardinality of the power set of a mset.

Theorem 2.14 *[24] Let P(X) be a power mset drawn from the mset $X = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ and $P^*(X)$ be the power set of a mset X. Then Card($P^*(X)$) = $\Pi_{i=1}^n$ (1+m_i).*

Definition 2.15 *[11] Let $X \in [U]^w$ and $\tau \subseteq P^*(X)$. Then τ is called a multiset topology (for short, M-topology) of X if τ satisfies the following properties :*

1. *The mset X and the empty mset ϕ are in τ.*
2. *The mset union of the elements of any subcollection of τ is in τ.*
3. *The mset intersection of the elements of any finite subcollection of τ is in τ.*

Hence, (X,τ) is called M-topological space. Each element in τ is called open mset. Also, OM(X) is the set of all open submsets of X.

Definition 2.16 *[12] Let (X,τ) be a M-topological space and Y is a submset of X. The collection $\tau_Y = \{G' = Y \cap G; G \in \tau\}$ is a M-topology on Y, called the subspace M-topology.*

Remark 2.17 *[12] The complement of any submset Y in a mset topological space (X,τ) is defined by : $Y^c = X \ominus Y$.*

Definition 2.18 *[12] A submset Y of a M-topological space X in $[U]^w$ is said to be closed if the mset $X \ominus Y$ is open.*

Example 2.19 *Let $X = \{2/x, 3/y, 1/z\}$ be a mset and let $\tau = \{\phi, X, \{2/x\}, \{1/y\}, \{2/x, 1/y\}\}$ be a M-topological space. Then, the complement of any submset Y in a M-topological space (X,τ) is shown as:*

1. *If $Y = \{2/x, 1/y\}$ then, $Y^c = \{2/y, 1/z\}$.*
2. *If $Y = \{3/y\}$ then, $Y^c = \{2/x, 1/z\}$.*
3. *If $Y = \{3/y, 1/z\}$ then, $Y^c = \{2/x\}$.*
4. *If $Y = \{1/x, 1/y, 1/z\}$ then, $Y^c = \{1/x, 2/y\}$.*
5. *If $Y = X$ then, $Y^c = \phi$.*

Note that De Morgan laws are satisfied in multisets by Wildberger [26].

Definition 2.20 *[12] Given a submset A of a M-topological space (X, τ), the interior of A is defined as the mset union of all open msets contained in A and is denoted by Int(A).*
i.e., $Int(A) = \cup\{G \subseteq X : G \text{ is an open mset and } G \subseteq A\}$
and $C_{Int(A)}(x) = max\{C_G(x) : G \subseteq A, G \in \tau\}$.

Definition 2.21 *[12] Given a submset A of a M-topological space (X, τ), the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by Cl(A), i.e., $Cl(A) = \cap\{K \subseteq X : K$ is a closed mset and $A \subseteq K\}$
and $C_{Cl(A)}(x) = min\{C_K(x) : A \subseteq K, K \in \tau^c\}$.*

If (X,τ) is a M-topological space and A is submset of X. Then, these laws are satisfied in [12] :

- $int(A^c) = (cl(A))^c$.
- $cl(A^c) = (int(A))^c$.

Definition 2.22 *[23] A mset X is called a M-singleton and denoted by $\{m/x\}$ if $C_X : U \to N$ such that $C_X(x) = m$ and $C_X(x') = 0 \;\forall\; x' \in U - \{x\}$. Note that, $x \in^k X$, means $C_X(x) = k$,so $\{k/x\}$ is called M-singleton submset of X and $\{m/x\}$ is called simple M-singleton where $0 < m < k$.*

Definition 2.23 *[22] Two msets A and B are said to be similar msets if for all x ($x \in A \Leftrightarrow x \in B$), where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.*

Definition 2.24 *[6] Let X be a mset and if $x \in^m X$, $x \in^n X$. Then $m = n$.*

3 Submsets of M-Topological Spaces

Definition 3.1 *Let (X, τ) be a multiset topological space (for short, M-topological space). A mapping $\gamma : P^*(X) \to P^*(X)$ is said to be an operation on $OM(X)$, if $N \subseteq \gamma(N) \;\forall\; N \in OM(X)$. The collection of all γ-open msets is denoted by, $OM(\gamma) = \{N : N \subseteq \gamma(N), N \in P^*(X)\}$. Also, the complement of γ-open mset is called γ-closed mset and the set of all γ-closed msets denoted by, CM(γ).*

Definition 3.2 *Let (X, τ) be a M-topological space. Different cases of γ-operations on $P^*(X)$ are as follows:*

(1) *If $\gamma = int(cl)$, then γ is called pre-open M-operator. We denote the set of all pre-open msets by $POM(X, \tau)$, or $POM(X)$ and the set of all pre-closed msets by $PCM(X, \tau)$, or $PCM(X)$.*

(2) *If $\gamma = int(cl(int))$, then γ is called α-open M-operator. We denote the set of all α-open msets by $\alpha OM(X, \tau)$, or $\alpha OM(X)$ and the set of all α-closed msets by $\alpha CM(X, \tau)$, or $\alpha CM(X)$.*

(3) *If* $\gamma = cl(int)$*, then* γ *is called semi open M-operator. We denote the set of all semi open msets by* $SOM(X, \tau)$*, or* $SOM(X)$ *and the set of all semi closed msets by* $SCM(X, \tau)$*, or* $SCM(X)$*.*

(4) *If* $\gamma = cl(int(cl))$*, then* γ *is called* β*-open M-operator. We denote the set of all* β*-open msets by* $\beta OM(X, \tau)$*, or* $\beta OM(X)$ *and the set of all* β*-closed msets by* $\beta CM(X, \tau)$*, or* $\beta CM(X)$*.*

(5) *If* $\gamma = cl(int) \cup int(cl)$*, then* γ *is called b-open M-operator. We denote the set of all b-open msets by* $BOM(X, \tau)$*, or* $BOM(X)$ *and the set of all b-closed msets by* $BCM(X, \tau)$*, or* $BCM(X)$*.*

Theorem 3.3 *Let* (X, τ) *be a M-topological space and* $\gamma : P^*(X) \rightarrow P^*(X)$ *be an operation on* $OM(X)$*.*
If $\gamma \in \{int(cl), int(cl(int)), cl(int), cl(int(cl)), cl(int) \cup int(cl)\}$*. Then,*

(1) *Arbitrary union of* γ*-open msets is* γ*-open mset.*

(2) *Arbitrary intersection of* γ*-closed msets is* γ*-closed mset.*

Proof.

(1) We give the proof for the case of pre-open M-operator i.e $\gamma = int(cl)$, and the other cases are similar. Let $\{N_i : i \in I\} \subseteq POM(X)$. Then, $\forall\ i \in I,\ N_i \subseteq int(cl(N_i))$. It follows that, $\bigcup_i N_i \subseteq \bigcup_i int(cl(N_i)) \subseteq int(\bigcup_i cl(N_i)) = int(cl(\bigcup_i N_i))$. Hence, $\bigcup_i N_i \in POM(X)\ \forall\ i \in I$.

(2) Immediate.

Remark 3.4 *A finite intersection of pre-open (resp. semi open,*β*-open, b-open) msets need not to be pre-open (resp. semi open,*β*-open, b-open) mset, as shown in the following examples. But, the finite intersection of* α*-open msets is* α*-open mset. This means that,* $\alpha OM(X)$ *is a M-topology.*

Examples 3.5 **(1)** *Let* $X = \{2/x, 2/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x, 2/y\}\}$ *be a M-topological space on X. Then the msets* $A = \{2/x, 1/z\}$ *,* $B = \{1/x, 1/y\}$ *are pre-open msets of* (X, τ)*, but* $A \cap B = \{1/x\}$ *is not pre-open mset.*

(2) *Let* $X = \{2/x, 3/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x\}, \{1/y\}, \{1/x, 1/y\}\}$ *be a M-topological space on X. Then the msets* $A = \{1/x, 1/z\}$ *,* $B = \{1/y, 1/z\}$ *are semi-open msets of* (X, τ)*, but* $A \cap B = \{1/z\}$ *is not semi-open mset.*

(3) *Let* $X = \{2/a, 2/b, 1/c\}$ *be a mset and* $\tau = \{X, \phi, \{1/a, 2/b\}\}$ *be a M-topological space on X. Then the msets* $A = \{2/a, 1/c\}$ *,* $B = \{1/a, 1/b\}$ *are* β*-open msets of* (X, τ)*, but* $A \cap B = \{1/a\}$ *is not* β*-open mset.*

(4) *Let* $X = \{2/x, 2/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x, 2/y\}\}$ *be a M-topological space on X. Then the msets* $A = \{2/x, 1/z\}$, $B = \{1/x, 1/y\}$ *are b-open msets of* (X, τ), *but* $A \cap B = \{1/x\}$ *is not b-open mset.*

Definition 3.6 *Let* (X, τ) *be a M-topological space,* $N \in P^*(X)$ *and* $k/x \in X$. *Then:*

(1) k/x *is called* γ*- interior M-point of N if* $\exists\, G \in OM(\gamma)$ *such that* $k/x \in G \subseteq N$, *the set of all* γ*-interior M-points of N is called the* γ*-interior of N and is denoted by* $\gamma M(int(N))$ *consequently,* $\gamma M(int(N)) = \bigcup\{G : G \subseteq N,\ G \in OM(\gamma)\}$.

(2) k/x *is called* γ*-cluster M-point of N if* $N \cap G \neq \phi\ \forall\, G \in OM(\gamma)$. *The set of all* γ*-cluster M-points of N is called* γ*-closure of N and is denoted by* $\gamma M(cl(N))$ *consequently,* $\gamma M(cl(N)) = \bigcap\{G : G \in CM(\gamma),\ N \subseteq G\}$.

Theorem 3.7 *Let* (X, τ) *be a M-topological space,* $\gamma : P^*(X) \rightarrow P^*(X)$ *be one of the operations in Definition 3.2 and* $F, G \in P^*(X)$. *Then the following properties are satisfied for the* γ*M-interior operators, denoted by* $\gamma M(int)$.

(1) $\gamma M(int(X)) = X$ *and* $\gamma M(int(\phi)) = \phi$.

(2) $\gamma M(int(F)) \subseteq F$.

(3) $\gamma M(int(F))$ *is the largest* γ*-open mset contained in F.*

(4) *If* $F \subseteq G$, *then* $\gamma M(int(F)) \subseteq \gamma M(int(G))$.

(5) $\gamma M(int(\gamma M(int(F)))) = \gamma M(int(F))$.

(6) $\gamma M(int(F)) \cup \gamma M(int(G)) \subseteq \gamma M(int(F \cup G))$.

(7) $\gamma M(int(F \cap G)) \subseteq \gamma M(int(F)) \cap \gamma M(int(G))$.

Theorem 3.8 *Let* (X, τ) *be a M-topological space,* $\gamma : P^*(X) \rightarrow P^*(X)$ *be one of the operations in Definition 3.2 and* $F, G \in P^*(X)$. *Then the following properties are hold for the* γ*M-closure operators, denoted by* $\gamma M(cl)$.

(1) $\gamma M(cl(X)) = X$ *and* $\gamma M(cl(\phi)) = \phi$.

(2) $F \subseteq \gamma M(cl(F))$.

(3) $\gamma M(cl(F))$ *is the smallest* γ*-closed mset containing F.*

(4) *If* $F \subseteq G$, *then* $\gamma M(cl(F)) \subseteq \gamma M(cl(G))$.

(5) $\gamma M(cl(\gamma M(cl(F)))) = \gamma M(cl(F))$.

(6) $\gamma M(cl(F)) \cup \gamma M(cl(G)) \subseteq \gamma M(cl(F \cup G))$.

(7) $\gamma M(cl(F \cap G)) \subseteq \gamma M(cl(F)) \cap \gamma M(cl(G))$.

Remark 3.9 *Note that the family of all γ-open msets on a M-topological space (X, τ) forms a supra M-topology, i.e. the family contains X, ϕ and closed under arbitrary union [19].*

4 Relations between Submsets of M-Topological Space

In this section we introduce the relations between some special submsets of a M-topological space (X, τ).

Theorem 4.1 *In a M-topological space (X, τ), the following statements hold :*

(1) *Every open mset is pre-open (resp. semi-open, α-open, β-open, b-open) mset.*

(2) *Every closed mset is pre-closed (resp. semi-closed, α-closed, β-closed, b-closed) mset.*

Proof. We give the proof for the case of pre-open M-operator i.e., $\gamma = int(cl)$, the other cases are similar.

(1) Let $N \in OM(X)$. Then, $int(N) = N$. Since $N \subseteq cl(N)$, then $N \subseteq int(cl(N))$. Therefore, $N \in POM(X)$.

(2) By a similar way.

Remark 4.2 *The converse of Theorem 4.1 is not true in general as shown in the following examples.*

Examples 4.3 **(1)** *Let $X = \{2/x, 2/y, 1/z\}$ be a mset and $\tau = \{X, \phi, \{1/x, 2/y\}\}$ be a M-topological space on X. Then, the mset $A = \{2/x, 1/z\}$ is pre-open mset of (X, τ), but it is not open mset.*

(2) *Let $X = \{2/x, 3/y, 1/z\}$ be a mset and $\tau = \{X, \phi, \{1/x\}, \{1/y\}, \{1/x, 1/y\}\}$ be a M-topological space on X. Then, the mset $A = \{1/x, 1/z\}$ is semi-open mset of (X, τ), but it is not open mset.*

(3) *Let $X = \{2/x, 2/y, 1/z\}$ be a mset and $\tau = \{X, \phi, \{2/x, 1/y\}\}$ be a M-topological space on X. Then, the mset $A = \{2/x, 2/y\}$ is α-open mset of (X, τ), but it is not open mset.*

(4) *Let $X = \{3/x, 2/y, 1/z\}$ be a mset and $\tau = \{X, \phi, \{2/x, 2/y\}\}$ be a M-topological space on X. Then, the mset $A = \{3/x, 1/z\}$ is β-open mset of (X, τ), but it is not open mset.*

(5) *Let $X = \{2/x, 2/y, 1/z\}$ be a mset and $\tau = \{X, \phi, \{1/x, 2/y\}\}$ be a M-topological space on X. Then, the mset $A = \{2/x, 1/z\}$ is b-open mset of (X, τ), but it is not open mset.*

Theorem 4.4 *Let (X, τ) be a M-topological space, then the following statements are hold :*

(1) *Every α-open (resp. α-closed) mset is semi-open (resp. semi-closed).*

(2) *Every α-open (resp. α-closed) mset is pre-open (resp. pre-closed).*

(3) *Every pre-open (resp. pre-closed) mset is b-open (resp. b-closed).*

(4) *Every semi-open (resp. semi-closed) mset is b-open (resp. b-closed).*

(5) *Every b-open (resp. b-closed) mset is β-open (resp. β-closed).*

Proof. We prove the assertion in the case of open mset, the other case is similar.

(1) Let $N \in \alpha OM(X)$. Then, $N \subseteq int(cl(int(N))) \subseteq cl(int(N))$. Hence, $N \in SOM(X)$.

(2) Let $N \in \alpha OM(X)$. Then, $N \subseteq int(cl(int(N)))$. Since, $int(N) \subseteq cl(N)$. Then, $cl(int(N)) \subseteq cl(N)$. Hence, $N \subseteq int(cl(int(N))) \subseteq int(cl(N))$. Thus, $N \subseteq int(cl(N))$. It follows that, $N \in POM(X)$.

(3) Let $N \in POM(X)$. Then, $N \subseteq int(cl(N))$. Thus, $N \subseteq int(cl(N)) \cup cl(int(N))$. Hence, $N \in BOM(X)$.

(4) Let $N \in SOM(X)$. Then, $N \subseteq cl(int(N))$. Thus, $N \subseteq int(cl(N)) \cup cl(int(N))$. Hence, $N \in BOM(X)$.

(5) Let $N \in BOM(X)$. Then,
$N \subseteq int(cl(N)) \cup cl(int(N))$
$\subseteq cl(int(N)) \cup cl(int(cl(N)))$
$= cl[int(N) \cup int(cl(N))]$
$\subseteq cl(int[N \cup cl(N)])$
$= cl(int(cl(N)))$. Hence, $N \in \beta OM(X)$.

Remark 4.5 *The converse of Theorem 4.4 is not true in general as shown by the following examples.*

Examples 4.6 **(1)** *Let* $X = \{2/x, 3/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x\}, \{1/y\}, \{1/x, 1/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{1/x, 1/z\}$ *is semi-open mset of* (X, τ)*, but it is not* α*-open mset.*

(2) *Let* $X = \{3/x, 2/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{2/x, 2/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{3/x, 1/z\}$ *is* β*-open mset of* (X, τ)*, but it is not semi-open mset.*

(3) *Let* $X = \{2/x, 3/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x\}, \{1/y\}, \{1/x, 1/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{1/x, 1/z\}$ *is* β*-open mset of* (X, τ)*, but it is not pre-open mset.*

(4) *Let* $X = \{3/x, 2/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{2/x, 2/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{3/x, 1/z\}$ *is pre-open mset of* (X, τ)*, but it is not* α*-open mset.*

(5) *Let* $X = \{2/x, 3/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{1/x\}, \{1/y\}, \{1/x, 1/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{1/x, 1/z\}$ *is b-open mset of* (X, τ)*, but it is not pre-open mset.*

(6) *Let* $X = \{3/x, 2/y, 1/z\}$ *be a mset and* $\tau = \{X, \phi, \{2/x, 2/y\}\}$ *be a M-topological space on X. Then, the mset* $A = \{3/x, 1/z\}$ *is b-open mset of* (X, τ)*, but it is not semi-open mset.*

(7) *Let* $X = \{2/a, 3/b, 1/c, 1/d\}$ *be a mset and* $\tau = \{X, \phi, \{1/d\}, \{1/a, 2/b\}, \{1/a, 2/b, 1/d\}\}$ *be a M-topological space on X. Then, the mset* $A = \{2/a\}$ *is* β*-open mset of* (X, τ)*, but it is not b-open mset.*

Corollary 4.7 *On account of Theorem 4.1, Example 4.3, Theorem 4.4 and Example 4.6 we have the following diagram for a M-topological space* (X, τ) *:*

$$\begin{tikzcd}
OM(X) \arrow[r] \arrow[d] & \alpha OM(X) \arrow[r] \arrow[dl] & SOM(X) \arrow[dl] \\
POM(X) \arrow[r] & BOM(X) \arrow[r] & \beta OM(X)
\end{tikzcd}$$

Theorem 4.8 *[17] Let* (X, τ) *be a M-topological space and* $N \in P^*(X)$*. Then :*

(i) $N \in SOM(X)$ *if and only if* $cl(N) = cl(int(N))$*.*

(ii) $N \in SOM(X)$ *if and only if there exists* $G \in OM(X)$ *such that* $G \subseteq N \subseteq cl(G)$*.*

(iii) *If $N \in SOM(X)$ and $N \subseteq G \subseteq cl(N)$, then $G \in SOM(X)$.*

Theorem 4.9 *Let (X, τ) be a M-topological space, $\gamma : P^*(X) \rightarrow P^*(X)$ be one of the operations in Definition 3.2 and $N \in P^*(X)$. Then, the following hold:*

(1) $\gamma M(int(N^c)) = [\gamma M(cl(N))]^c$.

(2) $\gamma M(cl(N^c)) = [\gamma M(int(N))]^c$.

Proof. We give the proof for the case of pre-open M-operator i.e $\gamma = int(cl)$, the other cases are similar.

(1) $pcl(N) = \cap\{G : N \subseteq G, G \in PCM(X)\}$
$[pcl(N)]^c = \cup\{G^c : G^c \subseteq N^c, G^c \in POM(X)\} = pint(N^c)$.

(2) By a similarly way.

Theorem 4.10 *Let (X, τ) be a M-topological space and $N,\ G \in P^*(X)$. Then,*

(1) *$N \in \alpha OM(X)$ if and only if $\exists\ H \in OM(X)$ such that $H \subseteq N \subseteq int(cl(H))$.*

(2) *If $N \in \alpha OM(X)$ and $N \subseteq G \subseteq int(cl(N))$, then $G \in \alpha OM(X)$.*

Proof.

(1) $\Rightarrow$: *Suppose that* $int(N) = H \in OM(X). Then,\ H \subseteq N \subseteq int(cl(H))$.

$\Leftarrow$: *Let* $H \subseteq N \subseteq int(cl(H)),\ H \in OM(X). Then, int(H) = H \subseteq int(N). It\ follows\ that,\ N \subseteq int(cl(int(H))) \subseteq int(cl(int(N))). Thus, N \in \alpha OM(X)$.

(2) *Let* $N \in \alpha OM(X),\ thenN \subseteq int(cl(int(N))). Hence,\ N \subseteq G \subseteq int(cl(int(cl(int(N))))) \subseteq int(cl(int(N))) \subseteq int(cl(int(G))). Thus,\ G \in \alpha OM(X)$.

Theorem 4.11 *Let (X, τ) be a M-topological space and $N \in P^*(X)$. Then,*

(1) *$N \in \alpha OM(X)$ if and only if $N \in POM(X) \cap SOM(X)$.*

(2) *$N \in \alpha CM(X)$ if and only if $N \in PCM(X) \cap SCM(X)$.*

Proof.

(1) $\Rightarrow$: *Let* $N \in \alpha OM(X)$. *Then,* $N \subseteq int(cl(int(N)))$. *Hence,* $N \subseteq cl(int(N))$ *and* $N \subseteq int(cl(N))$. *Thus,* $N \in POM(X) \cap SOM(X)$.
$\Leftarrow$: *Let* $N \in POM(X) \cap SOM(X)$. *Then,* $N \subseteq cl(int(N))$ *and* $N \subseteq int(cl(N))$. *Thus,* $N \subseteq int(cl(cl(int(N)))) = int(cl(int(N)))$.
It follows that, $N \in \alpha OM(X)$.

(2) By a similar way.

Theorem 4.12 *Let* (X, τ) *be a M-topological space. If* $N \in \alpha OM(X)$ *and* $N^c \in POM(X)$. *Then,* $N \in OM(X)$.

Proof. Let $N \in \alpha OM(X)$ and $N^c \in POM(X)$. Then, $N \in PCM(X)$. Hence, $cl(int(N)) \subseteq N \subseteq int(cl(int(N))) \subseteq cl(int(N))$. This means that, $cl(int(N) = N$. Thus, $N \subseteq int(cl(int(N))) = int(N)$. Therefore, $N \in OM(X)$.

The proof of the following proposition is straight forward, so it is omitted.

Proposition 4.13 *Let* (X, τ) *be a M-topological space and* $N \in P^*(X)$. *Then :*

(1) $N \in PCM(X)$ *if and only if* $cl(int(N)) \subseteq N$.

(2) $N \in \alpha CM(X)$ *if and only if* $cl(int(cl(N))) \subseteq N$.

(3) $N \in SCM(X)$ *if and only if* $int(cl(N)) \subseteq N$.

(4) $N \in \beta CM(X)$ *if and only if* $int(cl(int(N))) \subseteq N$.

(5) $N \in BCM(X)$ *if and only if* $cl(int(N)) \cap int(cl(N)) \subseteq N$.

5 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the mset theory and easily applied to many problems having uncertainties from social life. In this paper, we firstly gave the definition of "γ-operation" and then presented some examples of it such as pre-open M-operator, α-open M-operator, semi open M-operator, b-open M-operator and β-open M-operator. We study the relations between these different types of submsets of M-topological spaces. We notice that the family η of all γ-open msets on a M-topological space (X, τ) forms a supra M-topology, i.e $X, \phi \in \eta$ and η is closed under arbitrary union. So that, our next study is to introduce supra M-topological spaces and extending some M-topological properties to such spaces.

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References

[1] M.E.A. El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.*, 12(1983), 77-90.

[2] D. Andrijevic, On b-open sets, *Math. Vesnik*, 48(1996), 59-64.

[3] D. Andrijevic, On the topology generated by pre-open sets, *Math. Vesnik*, 39(1987), 367-376.

[4] D. Andrijevic, Semi-preopen sets, *Math. Vesnik*, 38(1986), 24-32.

[5] D. Andrijevic, Some properties of the topology of α-sets, *Math. Vesnik*, 36(1984), 1-10.

[6] W.D. Blizard, Multiset theory, *Notre Dame Journal of Formal Logic*, 30(1989 a), 36-66.

[7] W.D. Blizard, Real-valued multisets and fuzzy sets, *Fuzzy Sets and Systems*, 33(1989 b), 77-97.

[8] W.D. Blizard, The development of multiset theory, *Modern Logic*, 1(1991), 319-352.

[9] G.F. Clements, On multiset k-families, *Discrete Mathematics*, 69(1988), 153-164.

[10] M. Conder, S. Marshall and A.M. Slinko, Orders on multisets and discrete cones, *A Journal on the Theory of Ordered Sets and its Applications*, 24(2007), 277-296.

[11] K.P. Girish and S.J. John, Multiset topologies induced by multiset relations, *Information Sciences*, 188(2012), 298-313.

[12] K.P. Girish and S.J. John, On multiset topologies, *Theory and Applications of Mathematics & Computer Science*, 2(2012), 37-52.

[13] K.P. Girish and S.J. John, Relations and functions in multiset context, *Inf. Sci.*, 179(2009), 758-768.

[14] K.P. Girish and S.J. John, Rough multisets and information multisystems, *Advances in Decision Sciences*, Article ID 495392 (2011), 1-17.

[15] A. Kandil, O.A.E. Tantawy, S.A. El-Sheikh and A.M.A. El-Latif, γ-operation and decompositions of some forms of soft continuity in soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 7(2014), 181-196.

[16] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.

[17] J. Mahanta and D. Das, Semi-compactness in multiset topology, *arXiv preprint*, 1403:5642, (2014).

[18] A.S. Mashhour, M.E.A. El-Monsef and S.N. El-Deep, On pre-continuous and weak pre-continuous mappings, *Proc. Math. and Phys. Soc. Egypt*, 53(1982), 47-53.

[19] A.S. Mashhour, A.A. Allam, F.S. Mahmoud and F.H. Khedr, On supratopological space, *Indian J. Pure Appl. Math.*, 14(1983), 502-510.

[20] P. Njastad, On some classes of nearly open sets, *Pacifi J. Math.*, 15(1965), 961-970.

[21] D. Singh, A note on the development of multiset theory, *Modern Logic*, 4(1994), 405-406.

[22] D. Singh, A.M. Ibrahim, T. Yohana and J.N. Singh, An overview of the applications of multisets, *Novi Sad J. Math*, 37(2007), 73-92.

[23] D. Singh, A.M. Ibrahim, T. Yohana and J.N. Singh, Complementation in multiset theory, *International Mathematical Forum*, 6(2011), 1877-1884.

[24] D. Singh and J.N. Singh, Some combinatorics of multisets, *International Journal of Mathematical Education in Science and Technology*, 34(2003), 489-499.

[25] J. Slapal, Relations and topologies, *Czechoslovak Mathematical Journal*, 43(1993), 141-150.

[26] N.J. Wildberger, *A New Look at Multisets*, Preprint, University of New South Wales, Sydney, Australia, (2003), 1-21.