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m-Systems and n-Systems in Ordered Ternary Semigroups

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Abstract

In this note, the concepts of m-systems and n-systems in ordered ternary semigroups will be introduced and studied.

Keywords: *Ordered ternary semigroup, m-system, n-system, weakly prime and weakly semiprime ideal.*

1 Preliminaries

In [4], Niovi Kehayopulu introduced the concepts of m-systems and n-systems in ordered semigroups and showed that these concepts being related to the concepts of weakly prime and weakly semiprime ideals. In this note, we introduce the concepts of m-systems and n-systems in ordered ternary semigroups.

Ternary algebraic systems have been introduced by Lehmer [3] in 1932. The author investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. Ternary semigroups were introduced by Banach. He showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup. The following definitions can be founded in [1, 2].

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Definition 1.1 *Let S be a nonempty set. Then S is called a ternary semigroup if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that*

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

In [1], the author showed that $S = \{-i, 0, i\}$ is a ternary semigroup under the usual multiplication of complex numbers. However, S is not an ordinary semigroup under the usual multiplication of complex numbers because $(-i)i = 1 \notin S$.

For nonempty subsets A_1, A_2 and A_3 of a ternary semigroup S, let

$$[A_1A_2A_3] = \{[x_1x_2x_3] \mid x_i \in A_i, 1 \leq i \leq 3\}.$$

For $x, y \in S$, let $[xA_1A_2] = [\{x\}A_1A_2]$ and $[xA_1y] = [\{x\}A_1\{y\}]$. For any other cases can be defined analogously.

Definition 1.2 *A ternary semigroup S is called an ordered ternary semigroup if there is an ordered relation $\leq$ on S such that*

$$x \leq y \Rightarrow [xx_1x_2] \leq [yx_1x_2], [x_1xx_2] \leq [x_1yx_2], [x_1x_2x] \leq [x_1x_2y]$$

for all $x, y, x_1, x_2 \in S$.

Let S be an ordered ternary semigroup. For $A \subseteq S$, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Definition 1.3 *Let S be an ordered ternary semigroup. A nonempty subset A of S is called an ideal of S if (i) $[SSA] \subseteq A$, $[SAS] \subseteq A$ and $[ASS] \subseteq A$; (ii) for $x \in A, y \in S$, $y \leq x$ implies $y \in A$. An ideal A is said to be proper if $A \neq S$.*

A nonempty subset A of a ternary semigroup S is called an ideal of S if $[SSA] \subseteq A$, $[SAS] \subseteq A$ and $[ASS] \subseteq A$.

Note that if A is an ideal of an ordered ternary semigroup S, then $([aSb]] \subseteq A$ if and only if $[aSb] \subseteq A$.

Definition 1.4 *Let S be an ordered ternary semigroup (or ternary semigroup) and A an ideal of S. Then A is said to be weakly prime if for all $a, b \in S$, $[aSb] \subseteq A$ implies $a \in A$ or $b \in A$. A is said to be weakly semiprime if for all $a \in S$, $[aSa] \subseteq A$ implies $a \in A$.*

Definition 1.5 *Let S be an ordered ternary semigroup (or ternary semigroup) and $\emptyset \neq A \subseteq S$. Then A is called an m-system of S if for each $a, b \in A$ there exist $c \in A$ and $x \in S$ such that $c \leq [axb]$.*

Equivalent definition: For any $a, b \in A$ there exists $c \in A$ such that $c \in ([aSb]]$.

Definition 1.6 *Let S be an ordered ternary semigroup (or ternary semigroup) and $\emptyset \neq A \subseteq S$. Then A is called an n-system of S if for each $a \in A$ there exist $c \in A$ and $x \in S$ such that $c \leq [axa]$.*

Equivalent definition: For each $a \in A$ there exists $c \in A$ such that $c \in ([aSa]]$.

Note that Every m-systems is an n-system.

2 Main Results

Theorem 2.1 *Let S be an ordered ternary semigroup and A an ideal of S.*

(i) *If A is weakly prime and $S \setminus A \neq \emptyset$, then $S \setminus A$ is an m-system.*

(ii) *If $S \setminus A$ is an m-system, then A is weakly prime.*

Proof. This is a modification of the proof in [4, Proposition 1, p. 56].

(i) Assume that A is weakly prime and $S \setminus A \neq \emptyset$. Clearly, $S \setminus A \subseteq S$. Let $a, b \in S \setminus A$. To prove that there exists $c \in S \setminus A$ such that $c \in ([aSb]]$, we suppose not. Then $c \notin ([aSb]]$ for every $c \in S \setminus A$. Then $[aSb] \subseteq A$. This implies $a \in A$ or $b \in A$. A contradiction.

(ii) Assume that $S \setminus A$ is an m-system. Let $a, b \in S$ be such that $[aSb] \subseteq A$. To show that $a \in A$ or $b \in A$, suppose not. Then $a, b \in S \setminus A$. By assumption, there exists $c \in S \setminus A$ such that $c \in ([aSb]]$. Let $c \leq [axb]$ for some $x \in S$. Since $[axb] \in A$, we have $c \in A$. A contradiction.

Similarly, we have the following.

Theorem 2.2 *Let S be an ordered ternary semigroup and A an ideal of S.*

(i) *If A is weakly semiprime and $S \setminus A \neq \emptyset$, then $S \setminus A$ is an n-system.*

(ii) *If $S \setminus A$ is an n-system, then A is weakly semiprime.*

Proof. (i) Assume that A is weakly semiprime and $S \setminus A \neq \emptyset$. Then $\emptyset \neq S \setminus A \subseteq S$. Let $a \in S \setminus A$. Suppose that $c \notin ([aSc]]$ for every $c \in S \setminus A$. Then $[aSa] \subseteq A$. This implies $a \in A$. A contradiction.

(ii) Assume that $S \setminus A$ is an n-system. Let $a \in S$ be such that $[aSa] \subseteq A$. Suppose that $a \in S \setminus A$. By assumption, there exists $c \in S \setminus A$ such that $c \in ([aSa]]$. Let $c \leq [axa]$ for some $x \in S$. Since $[axa] \in A$, we obtain $c \in A$. A contradiction.

Let S be a ternary semigroup. Define a relation on S by $x \leq y$ if and only if $x = y$. Then S forms an ordered ternary semigroup. Therefore, by Theorem 2.1 and 2.2, we have the following.

Corollary 2.3 *Let S be a ternary semigroup and A an ideal of S.*

(i) *If A is weakly prime and $S \setminus A \neq \emptyset$, then $S \setminus A$ is an m-system.*

(ii) *If $S \setminus A$ is an m-system, then A is weakly prime.*

Corollary 2.4 *Let S be a ternary semigroup and A an ideal of S.*

(i) *If A is weakly semiprime and $S \setminus A \neq \emptyset$, then $S \setminus A$ is an n-system.*

(ii) *If $S \setminus A$ is an n-system, then A is weakly semiprime.*

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