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A Result on a Cyclic Polynomials

S.A. Wahid

Department of Mathematics & Statistics
U.W.I, Trinidad, West Indies
E-mail: shanazw@hotmail.com

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Abstract

This paper establishes a result on matching polynomials that is related to a conjecture by Gutman, see [4]. The principle of Inclusion and Exclusion is used to count matchings of certain reduced subgraphs. A function is then defined on each set of matchings to obtain a result on acyclic polynomials.

Keywords: *Matching, Acyclic polynomial, Weight, Path and Cover.*

1 Introduction

In the material which follows we consider finite undirected graphs without loops and multiple edges. Let *G* be such a graph with p nodes. By a matching in *G*, we mean a spanning subgraph whose components are nodes and edges only. *A k*-matching is a matching with *k* edges.

Let us assign weights w_1 and w_2 to each node and edge respectively in *G*. If a_k is the number of k-matchings in *G* then the total weight of the k-matchings is *G* is $a_k w_1^{p-2k} w_2^k$. The matching polynomial of *G*, see Farrell [1, 2] is defined as

$$M(G;\underline{w}) = \sum_{k=0}^{\left[\frac{p}{2}\right]} a_k\, w_1^{p-2k}\, w_2^{k}.,$$

The acyclic polynomial as defined by Gutman [5] is

$$\alpha(G;x) = \sum_{k=0}^{\left[\frac{p}{2}\right]} a_k\, x^{p-2k}\, (-1)^k.$$

This polynomial is easily obtained from matching polynomial by replacing w_1 by x and w_2 by -1. For further relations between the two polynomials, see Farrell [2].

Gutman's conjecture is as follows: Let G be a graph and A, B are two subgraphs of G such that $V(A) \cap V(B) = \phi$. Let $P_1, P_2, ..., P_s$ be the paths in *G* whose one endvertex belongs to *A* and the other endvertex belongs to *B* and no other node belongs to either *A* or *B*. Then

$$\alpha(G)\alpha(G - A - B) = \alpha(G - A)\alpha(G - B) - \sum_i \alpha(G - A - P_i)\alpha(G - B - P_i)$$
$$+ \sum_{i<j} \alpha(G - A - P_i - P_j)\alpha(G - B - P_i - P_j) + ...$$
$$+ (-1)^s \alpha(G - A - P_1 - P_2 - ... - P_s)\alpha(G - B - P_1 - P_2 - ... - P_s);$$

Where the convention is that whenever at least two among the paths $P_{i_1}, P_{i_2}, ..., P_{i_k}$ have at least one common vertex, then

$$\alpha(G - A - P_{i_1} - P_{i_2} - .... - P_{i_k}) = \alpha(G - B - P_{i_1} - P_{i_2} - .... - P_{i_k}) = 0.$$

Gutman's conjecture is closely related to the theory of Jacobi polynomials. Similar (but not equivalent) results were earlier published in Heilmann and Lieb (6) and Godsil [3].

In this paper a result is given which considers the case when the graphs *A* and *B* may have common nodes. As a consequence, Gutman's conjecture would follow. *A* path is a tree which has exactly two endnodes. The graph *G-A* is obtained by removing the nodes of *A i.e. V*(*A*) and all the incident edges from *G*. We sometimes write $\alpha(G - A; x)$ as $\alpha(G - A)$.

2 The Main Result

Let *G* be a labeled graph and *A*, *B* are two subgraphs. Let P_i be the paths as defined above. The length of P_i is k_i. It is convenient to write $M(G, \underline{w})$ as $M(G)$. Then

$$M(G-(A\cup B))M(G-(A\cap B)) = M(G-A)M(G-B) -$$
$$\sum_i M(G-A-P_i)M(G-B-P_i)(-w_2)^{k_i}$$
$$+\sum_{i<j} M(G-A-P_i-P_j)M(G-B-P_i-P_j)(-w_2)^{k_i+k_j} + \ldots$$
$$+(-1)^s M(G-A-P_1-\ldots-P_s)M(G-B-P_1-\ldots-P_s)(-w_2)^{k_1+\ldots+k_s} \;;$$

where by convention $M(\phi) = 1$.

Proof: In this proof we first use the Principle of Inclusion and Exclusion to identify a number of matchings and then we apply a function f to convert the matchings into matching polynomials.

Let *A* and *B* be labeled subgraphs of *G* and $P_1, P_2, \ldots, P_s$ be the paths as defined above. Two graphs $G-A-P_i$ and $G-B-P_i$ are constructed from *G* with respect to a path P_i with no labels repeated. We let b_i be the property that a matching is obtained from the subgraphs $G-A-P_i$ and $G-B-P_i$ for a path P_i. In this way $N(b_i)$ is the number of matchings described with respect to property b_i. Similarly $N(b_i b_j)$ is the number of matchings described with respect to properties b_i and b_j. Using the principle of inclusion and exclusion, we get

$$N(b_1'b_2'\ldots b_s') = N - \sum_{i=1}^{s} N(b_i) + \sum_{i<j}^{s} N(b_i b_j) + \ldots + (-1)^s N(b_1 b_2 \ldots b_s). \qquad \ldots(1)$$

Firstly we examine the terms on the right side of equation 1.

We convert a matching to its matching polynomial by using the function f as follows:
f is defined as $f(N(b_i)) = M(G-A-P_i)M(G-B-P_i)(-w_2)^{k_i}$; where k_i is the length of path P_i.

N is the number of matchings of the graph $(G-A)\cup(G-B)$. In this case no paths are removed and thus in the unrestricted case only graphs *A* and *B* are removed once and separately from *G*. On applying *f* we get

$M(G-A;\underline{w})M(G-B;\underline{w})\left(-w_2\right)^0$ since no path is considered.

$N(b_ib_j)$ is the number of matchings of $G-A-P_i-P_j$ and $G-B-P_i-P_j$.
Applying f we get $M(G-A-P_i-P_j;\underline{w})\ M(G-B-P_i-P_j;\underline{w})\left(-w_2\right)^{k_i+k_j}$ since the paths P_i and P_j have k_i and k_j edges respectively.

All higher ordered terms on the right hand side of equation (1) are found as described. The term on the left hand side of (1) is now analyzed. We must find the two subgraphs that are to be removed from *G*. In examining the compliment property $b_i^{/}$, the following must be noted:

(a) In finding $N(b_i)$ graphs *A* and *B* are removed once and separately. Thus in $N(b_i^{/})$ these graphs are removed once but not separately.

(b) In finding $N(b_i)$ one endnode of path P_i is removed from *G – A* and the other endnode from *G – B*. Also the entire path P_i is removed from *G – A* and *G – B*. Thus in finding $N(b_i^{/})$, the entire path P_i is not to be removed. In addition, in finding $N(b_i^{/})$ both endnodes of path P_i are trivially removed since the graphs *A* and *B* are removed. Thus the whole of P_i is not to be removed. We now examine how the endnodes of P_i are to be removed.

(c) In finding $N(b_i)$ both endnodes of P_i are removed separately. Thus in finding $N(b_i^{/})$, the pair of endnodes are removed together. Also not removing one end node separate from the other is the same as removing zero endnodes together from some graph *X* and both endnodes together from another graph *Y*.

In considering $N(b_1^{/}b_2^{/}...b_s^{/})$, the points (a), (b) and (c) above are viewed with respect to the compliment of all properties. We need to identify the graphs *X* and *Y* as we are considering matchings of *G – X* and *G – Y*. The graphs $G-(A\cup B)$ and $G-(A\cap B)$ satisfy all the properties above. *A* and *B* are removed once in the term $G-(A\cup B)$ and not separately with respect to argument (a). Recall that a path P_i is selected such that there is one endnode in *A* and the other in *B*. There are no paths P_i that have an endnodein $A \cup B$ and next endnode in $A \cap B$. The removal of subgraph $A \cap B$ would ensure that no entire paths are removed as stated in argument (b). This is the same as removing zero endnodes trivially from some graph and removing both endnodes together from another graph.

Removing $A \cup B$ from G ensures that both endnodes of each path P_i are removed together. Thus *X* is $A \cup B$ and *Y* is $A \cap B$.

On collecting all terms we get,

$$M(G-(A\cup B))M(G-(A\cap B)) = M(G-A)M(G-B)$$

$$-\sum_{i} M(G-A-P_i)M(G-B-P_i)(-w_2)^{k_i}$$

$$+\sum_{i<j} M(G-A-P_i-P_j)M(G-B-P_i-P_J)(-w_2)^{k_i+k_j} + \ldots$$

$$+(-1)^S M(G-A-P_1-\ldots-P_S)M(G-B-P_1-\ldots-P_S)(-w_2)^{k_1+\ldots k_s}$$

where by convention $M(\phi)=1$. …(2)

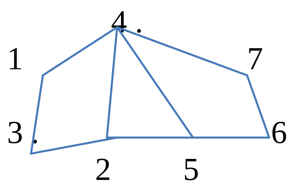
In order to convert (2) into a result on acyclic polynomials we use the conversion as stated in the introduction.

For disjoint subgraphs A and B, equation(2) is Gutman’s conjecture.

We illustrate with an example where $A \cap B \neq \phi$.

3 Example

Let *V*(*A*) = {1, 2, 4, 6} and *V*(*B*) = {2, 3, 5} and *G* is the following :



The paths are $P_1 = \{5,6\}$, $P_2 = \{5,4\}$ and $P_3 = \{1,3\}$. The following calculations can be easily confirmed.

$M(G - A) = w_1^3$; $M(G - B) = w_1^4 + 3w_2 w_1^2 + w_2^2$;

$M(G - (A \cup B)) = w_1$, $M(G - (A \cap B)) = w_1^6 + 6w_2 w_1^4 + 8w_2^2 + 2w_2^3$.

$M(G - A - P_1) = w_1^2$; $M(G - B - P_1) = w_1^3 + 2w_2 w_1$; $M(G - A - P_2) = w_1^2$;

$M(G - B - P_2) = w_1^3 + w_2 w_1$; $M(G - A - P_3) = w_1^2$; $M(G - B - P_3) = w_1^3 + 2w_2 w_1$

$M(G - A - P_1 - P_3) = w_1$; $M(G - B - P_1 - P_3) = w_1^2 + w_2$; $M(G - A - P_2 - P_3) = w_1$

$M(G - B - P_2 - P_3) = w_1^2 + w_2$.

$M(G - (A \cup B))M(G - (A \cap B)) = w_1(w_1^6 + 6w_2 w_1^4 + 8w_2^2 w_1^2 + 2w_2^3)$.

On R.H.S. we have $w_1^3(w_1^4 + 3w_2 w_1^2 + w_2^2) - w_1^2(w_1^3 + 2w_2 w_1)(-w_2)^1$

$-w_1^2(w_1^3 + w_2 w_1)(-w_2)^1 - w_1^2(w_1^3 + 2w_2 w_1)(-w_2)^1 + w_1(w_1^2 + w_2)(-w_2)^2 +$
$w_1(w_1^2 + w_2)(-w_2)^2$.

$= w_1(w_1^6 + 6w_2 w_1^4 + 8w_2^2 w_1^2 + 2w_2^3)$.

This is converted to acyclic polynomials as

$x(x^6 - 6x^4 + 8x^2 - 2) = x^3(x^4 - 3x^2 + 1) - x^2(x^3 - 2x) - x^2(x^3 - x)$
$- x^2(x^3 - 2x) + x(x^2 - 1) + x(x^2 - 1)$.
$= x^7 - 6x^5 + 8x^3 - 2x$.

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