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Multi - Fuzzy Group and its Level Subgroups

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Abstract

In this paper, we define the algebraic structures of multi-fuzzy subgroup and some related properties are investigated. The purpose of this study is to implement the fuzzy set theory and group theory in multi-fuzzy subgroups. Characterizations of multi-level subsets of a multi-fuzzy subgroup of a group are given.

Keywords: *Fuzzy set, multi-fuzzy set, fuzzy subgroup, multi-fuzzy subgroup, anti fuzzy subgroup, multi-anti fuzzy subgroup.*

1 Introduction

S. Sabu and T.V. Ramakrishnan [5] proposed the theory of multi-fuzzy sets in terms of multi-dimensional membership functions and investigated some properties of multi-level fuzziness. L.A. Zadeh [4] introduced the theory of multi-fuzzy set is an extension of theories of fuzzy sets. In this paper we define a new algebraic structure of multi-fuzzy subgroups and study some of their related properties.

2 Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel.

2.1 Definition: *Let X be any non-empty set. A fuzzy subset μ of X is $\mu : X \to [0,1]$.*

2.2 Definition: *Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:*

$$A = \{(x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ...) : x \in X\}, \text{ where } \mu_i : X \to [0, 1] \text{ for all i.}$$

Remark:

i. If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A.
ii. The set of all multi-fuzzy sets in X of dimension k is denoted by $M^kFS(X)$.
iii. The multi-fuzzy membership function μ_A is a function from X to $[0,1]^k$ such that for all x in X, $\mu_A(x) = (\mu_1(x), \mu_2(x), ..., \mu_k(x))$.
iv. For the sake of simplicity, we denote the multi-fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), ..., \mu_k(x)) : x \in X\}$ as $A = (\mu_1, \mu_2, ..., \mu_k)$.

2.3 Definition: *Let k be a positive integer and let A and B in $M^kFS(X)$, where $A = (\mu_1, \mu_2, ..., \mu_k)$ and $B = (\nu_1, \nu_2..., \nu_k)$, then we have the following relations and operations:*

i. $A \subseteq B$ if and only if $\mu_i \le \nu_i$, for all i = 1, 2, ..., k;
ii. $A = B$ if and only if $\mu_i = \nu_i$, for all i = 1, 2, ..., k;
iii. $A \cup B = (\mu_1 \cup \nu_1, ..., \mu_k \cup \nu_k) = \{(x, \max(\mu_1(x), \nu_1(x)), ..., \max(\mu_k(x), \nu_k(x))) : x \in X\}$;
iv. $A \cap B = (\mu_1 \cap \nu_1, ..., \mu_k \cap \nu_k) = \{(x, \min(\mu_1(x), \nu_1(x)), ..., \min(\mu_k(x), \nu_k(x))) : x \in X\}$;
v. $A + B = (\mu_1 + \nu_1, ..., \mu_k + \nu_k) = \{(x, \mu_1(x) + \nu_1(x) - \mu_1(x)\nu_1(x), ..., \mu_k(x) + \nu_k(x) - \mu_k(x)\nu_k(x)) : x \in X\}$.

2.4 Definition: *Let $A = (\mu_1, \mu_2, ..., \mu_k)$ be a multi-fuzzy set of dimension k and let μ_i' be the fuzzy complement of the ordinary fuzzy set μ_i for i = 1, 2, ..., k. The Multi-fuzzy Complement of the multi-fuzzy set A is a multi-fuzzy set $(\mu_1', ..., \mu_k')$ and it is denoted by C(A) or A' or A^C.*

That is, $C(A) = \{(x, c(\mu_1(x)), ..., c(\mu_k(x))) : x \in X\} = \{(x, 1 - \mu_1(x), ..., 1 - \mu_k(x)) : x \in X\}$, where c is the fuzzy complement operation.

2.5 Definition: *Let A be a fuzzy set on a group G. Then A is said to be a fuzzy subgroup of G if for all* x, y $\in$ G,

i. $A(xy) \geq \min\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$.

2.6 Definition: *A multi-fuzzy set A of a group G is called a multi-fuzzy subgroup of G if for all* x, y $\in$ G,

i. $A(xy) \geq \min\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$

2.7 Definition: *Let A be a fuzzy set on a group G. Then A is called an anti fuzzy subgroup of G if for all x, y $\in$ G,*

i. $A(xy) \leq \max\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$.

2.8 Definition: *A multi-fuzzy set A of a group G is called a multi-anti fuzzy subgroup of G if for all x, y $\in$ G,*

i. $A(xy) \leq \max\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$

2.9 Definition: *Let A and B be any two multi-fuzzy sets of a non-empty set X. Then for all x$\in$ X,*

i. $A \subseteq B$ iff $A(x) \leq B(x)$,
ii. $A = B$ iff $A(x) = B(x)$,
iii. $A \cup B(x) = \max\{A(x), B(x)\}$,
iv. $A \cap B(x) = \min\{A(x), B(x)\}$.

2.10 Definition: *Let A and B be any two multi-fuzzy sets of a non-empty set X. Then*

i. $A \cup A = A$, $A \cap A = A$,
ii. $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$,
iii. $A \subseteq B$ iff $A \cup B = B$,
iv. $A \subseteq B$ iff $A \cap B = A$.

3 Properties of Multi-Fuzzy Subgroups

In this section, we discuss some of the properties of multi-fuzzy subgroups.

3.1 Theorem: *Let 'A' be a multi-fuzzy subgroup of a group G and 'e' is the identity element of G. Then*

i. $A(x) \leq A(e)$ for all $x \in G$.
ii. The subset $H = \{x \in G \,/\, A(x) = A(e)\}$ is a subgroup of G.

Proof:

i. Let $x \in G$.

$$\begin{aligned} A(x) &= \min\{A(x), A(x)\} \\ &= \min\{A(x), A(x^{-1})\} \\ &\leq A(xx^{-1}) \\ &= A(e). \end{aligned}$$

Therefore, $A(x) \leq A(e)$, for all $x \in G$.

ii. Let $H = \{x \in G \,/\, A(x) = A(e)\}$
Clearly H is non-empty as $e \in H$.

Let $x, y \in H$. Then, $A(x) = A(y) = A(e)$

$$\begin{aligned} A(xy^{-1}) &\geq \min\{A(x), A(y^{-1})\} \\ &= \min\{A(x), A(y)\} \\ &= \min\{A(e), A(e)\} \\ &= A(e) \end{aligned}$$

That is, $A(xy^{-1}) \geq A(e)$ and obviously $A(xy^{-1}) \leq A(e)$ by i.

Hence, $A(xy^{-1}) = A(e)$ and $xy^{-1} \in H$.

Clearly, H is a subgroup of G.

3.2 Theorem: *A is a multi-fuzzy subgroup of G iff AC is a multi-anti fuzzy subgroup of G.*

Proof: Suppose A is a multi-fuzzy subgroup of G. Then for all $x, y \in G$,

$$\begin{aligned} & A(xy) \geq \min\{A(x), A(y)\} \\ \Leftrightarrow\ & 1 - A^c(xy) \geq \min\{(1 - A^c(x)), (1 - A^c(y))\} \\ \Leftrightarrow\ & A^c(xy) \leq 1 - \min\{(1 - A^c(x)), (1 - A^c(y))\} \\ \Leftrightarrow\ & A^c(xy) \leq \max\{A^c(x), A^c(y)\}. \end{aligned}$$

We have, $A(x) = A(x^{-1})$ for all x in G

$$\Leftrightarrow 1 - A^c(x) = 1 - A^c(x^{-1})$$

Therefore, $A^c(x) = A^c(x^{-1})$.

Hence A^c is a multi-anti fuzzy subgroup of G.

3.3 Theorem: *Let 'A' be any multi-fuzzy subgroup of a group G with identity 'e'. Then* $A(xy^{-1}) = A(e) \Rightarrow A(x) = A(y)$ for all $x, y \in G$.

Proof: Given A is a multi-fuzzy subgroup of G and $A(xy^{-1}) = A(e)$.
Then for all $x, y \in G$,

$$\begin{aligned} A(x) &= A(x(y^{-1}y)) \\ &= A((xy^{-1})y) \\ &\geq \min\{A(xy^{-1}), A(y)\} \\ &= \min\{A(e), A(y)\} \\ &= A(y). \end{aligned}$$

That is, $A(x) \geq A(y)$.

Now, $A(y) = A(y^{-1})$, since A is a multi-fuzzy subgroup of G.

$$\begin{aligned} &= A(ey^{-1}) \\ &= A((x^{-1}x)y^{-1}) \\ &= A(x^{-1}(xy^{-1})) \\ &\geq \min\{A(x^{-1}), A(xy^{-1})\} \\ &= \min\{A(x), A(e)\} \\ &= A(x). \end{aligned}$$

That is, $A(y) \geq A(x)$.

Hence, $A(x) = A(y)$.

3.4 Theorem: *A is a multi-fuzzy subgroup of a group G if and only if* $A(xy^{-1}) \geq \min\{A(x), A(y)\}$, for all $x, y \in G$.

Proof: Let A be a multi-fuzzy subgroup of a group G. Then for all x ,y in G,

$$A(xy) \geq \min\{A(x), A(y)\}$$

And $A(x) = A(x^{-1})$.

Now,

$$\begin{aligned} A(xy^{-1}) &\geq \min\{A(x), A(y^{-1})\}. \\ &= \min\{A(x), A(y)\} \end{aligned}$$

$$\Leftrightarrow \quad A(xy^{-1}) \geq \min\{A(x), A(y)\}.$$

4 Properties of Multi-Level Subsets of a Multi-Fuzzy Subgroup

In this section, we introduce the concept of multi-level subset of a multi-fuzzy subgroup and discuss some of its properties.

4.1 Definition: *Let A be a multi-fuzzy subgroup of a group G. For any* $t = (t_1, t_2, \ldots, t_k, \ldots)$ *where* $t_i \in [0,1]$, *for all i, we define the multi-level subset of A is the set,*

$$L(A\,;\,t) = \{x \in G \,/\, A(x) \geq t\}.$$

4.1 Theorem: *Let A be a multi-fuzzy subgroup of a group G. Then for any* $t = (t_1, t_2, \ldots, t_k, \ldots)$, *where* $t_i \in [0,1]$ *for all i such that* $t \leq A(e)$, *where 'e' is the identity element of G, L(A; t) is a subgroup of G.*

Proof: For all x, y ∈ L (A ; t) , we have,

$$A(x) \geq t \; ; \; A(y) \geq t.$$

$$\text{Now,} \quad A(x\,y^{-1}) \geq \min\{A(x), A(y)\}.$$

$$\geq \min\{t, t\} = t$$

$$\text{That is,} \quad A(x\,y^{-1}) \geq t.$$

Therefore, $x\,y^{-1} \in L(A;t)$. Hence L (A ; t) is a subgroup of G.

4.2 Theorem: *Let G be a group and A be a multi-fuzzy subset of G such that L(A; t) is a subgroup of G. Then for* $t = (t_1, t_2, \ldots, t_k, \ldots)$, *where* $t_i \in [0,1]$ *for all i such that* $t \leq A(e)$ *where 'e' is the identity element of G, A is a multi-fuzzy subgroup of G.*

Proof: Let x, y ∈ G and A(x) = r and A(y) = s , where $r = (r_1, r_2, \ldots, r_k, \ldots)$, $s = (s_1, s_2, \ldots, s_k, \ldots)$, for $r_i, s_i \in [0,1]$ for all i.

Suppose $r < s$.
Now A(x) = r which implies x ∈ L(A ; r).
And now A(y) = s > r which implies y ∈ L(A ; r).
Therefore x, y ∈ L (A ; r).

As L(A ; r) is a subgroup of G, $xy^{-1} \in L(A;r)$.

$$\text{Hence,} \quad A(xy^{-1}) \geq r = \min\{r, s\}$$

$$\geq \min\{A(x), A(y)\}$$

$$\text{That is,} \quad A(xy^{-1}) \geq \min\{A(x), A(y)\}.$$

Hence A is a multi-fuzzy subgroup of G.

4.2 Definition: *Let A be a multi-fuzzy subgroup of a group G. The subgroups L(A ; t) for* $t = (t_1, t_2, \ldots, t_k, \ldots)$ *where* $t_i \in [0,1]$ *for all i and* $t \leq A(e)$ *where 'e' is the identity element of G, are called multi-level subgroups of A.*

4.3 Theorem: *Let A be a multi-fuzzy subgroup of a group G and 'e' is the identity element of G. If two multi-level subgroups L(A ; r), L(A ; s), for* $r = (r_1, r_2, \ldots, r_k, \ldots)$, $s = (s_1, s_2, \ldots, s_k, \ldots)$, *where* $r_i, s_i \in [0,1]$ *for all i and* $r, s \leq A(e)$ *with* $r < s$ *of A are equal, then there is no x in G such that* $r \leq A(x) < s$.

Proof:

Let L(A ; r) = L(A ; s).
Suppose there exists a x ∈ G such that $r \leq A(x) < s$.
Then L(A ; s) ⊆ L(A ; r).
That is, x ∈ L(A; r) , but x ∉ L(A; s), which contradicts the assumption that, L(A ; r) = L(A ; s).

Hence there is no x in G such that $r \leq A(x) < s$.
Conversely, Suppose that there is no x in G such that $r \leq A(x) < s$.
Then, by the definition, $L(A ; s) \subseteq L(A ; r)$.
Let $x \in L(A ; r)$ and there is no x in G such that $r \leq A(x) < s$.
Hence $x \in L(A ; s)$ and therefore, $L(A ; r) \subseteq L(A ; s)$.
Hence $L(A ; r) = L(A ; s)$.

4.4 Theorem: *A multi-fuzzy subset A of G is a multi-fuzzy subgroup of a group G if and only if the multi-level subsets L(A ; t), for* $t = (t_1, t_2, \ldots, t_k, \ldots)$ *where* $t_i \in [0,1]$ *for all i and* $t \leq A(e)$, *are subgroups of G.*

Proof: It is clear.

4.5 Theorem: *Any subgroup H of a group G can be realized as a multi-level subgroup of some multi-fuzzy subgroup of G.*

Proof:

Let A be a multi-fuzzy subset and $x \in G$.

Define,

$$A(x) = \begin{cases} 0 & \text{if } x \notin H \\ t & \text{if } x \in H, \text{ for } t = (t_1, t_2, \ldots, t_k, \ldots) \text{ where } t_i \in [0,1] \text{ for all i and} \end{cases}$$

$t \leq A(e)$.

We shall prove that A is a multi-fuzzy subgroup of G. Let $x, y \in G$.

i. Suppose $x, y \in H$. Then $xy \in H$ and $xy^{-1} \in H$.

   $A(x) = t$, $A(y) = t$, $A(xy) = t$ and $A(xy^{-1}) = t$.

   Hence $A(xy^{-1}) \geq \min\{ A(x), A(y) \}$.

ii. Suppose $x \in H$ and $y \notin H$. Then $xy \notin H$ and $xy^{-1} \notin H$.

   $A(x) = t$, $A(y) = 0$ and $A(xy^{-1}) = 0$.

   Hence $A(xy^{-1}) \geq \min\{ A(x), A(y) \}$.

iii. Suppose $x, y \notin H$. Then $xy^{-1} \in H$ or $xy^{-1} \notin H$.

   $A(x) = 0$, $A(y) = 0$ and $A(xy^{-1}) = t$ or 0.

Hence $A(xy^{-1}) \geq \min\{ A(x), A(y) \}$.

Thus in all cases, A is a multi-fuzzy subgroup of G.

For this multi-fuzzy subgroup A, $L(A; t) = H$.

Remark: As a consequence of the Theorem 4.3, the multi-level subgroups of a multi-fuzzy subgroup A of a group G form a chain. Since $A(e) \geq A(x)$ for all x in G where 'e' is the identity element of G , therefore $L(A; t_0)$, where $A(e) = t_0$ is the smallest and we have the chain:

$\{e\} \subseteq L(A; t_0) \subset L(A; t_1) \subset L(A; t_2) \subset \ldots \subset L(A; t_n) = G$, where $t_0 > t_1 > t_2 > \ldots\ldots > t_n$.

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