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Numerical Solution of SIR Model using Differential Transformation Method and Variational Iteration Method

F.S. Akinboro[1], S. Alao[2] and F.O. Akinpelu[3]

[1,2,3]Department of Pure & Applied Mathematics
LAUTECH Ogbomoso, Nigeria
[1]E-mail: folexboro@gmail.com
[2]E-mail: saheedalao4k@yahoo.com
[3]E-mail: foakinpelu.co.uk

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Abstract

This study investigates the application of differential transformation method and variational iteration method in finding the approximate solution of Epidemiology (SIR) model. SIR models are nonlinear system of ordinary differential equation that has no analytic solution. VIM uses the general Lagrange multiplier to construct the correction functional for the problem while DTM uses the transformed function of the original nonlinear system. The result revealed that both methods are in complete agreement, accurate and efficient for solving systems of ODE.

Keywords: *Differential Transformation method, Epidemiology model, Lagrange multipliers, Variational Iteration method, Transformed function.*

1 Introduction

Epidemiology studies the spread of diseases in population and primarily the human population. Often the work of a mathematical epidemiologist consists of model building, estimation of parameters and investigation of the sensitivity of models to changes in the parameters and numerical simulations.

All these activities are expected to tell us something about the spread of the disease in the population, the possibility to control this spread and maybe how to make the disease disappear from the population [3]. These diseases modeled most often are the so called infectious diseases, i.e. diseases that are contagious and can be transferred from one individual to another through contact. Examples of such diseases are the childhood diseases: measles, rubella, chicken pox and mumps the sexually transmitted diseases: HIV/AIDS, gonorrhea, syphilis and others.

The S-I-R model was introduced by W.O. Kermack and has played a major role in mathematical epidemiology. In the model, a population is divided into three groups: the susceptible (**s**), the infective (**i**), and the recovered (**r**), with numbers s, i, and r respectively. The total population (**n**) is

$$\mathbf{n = s + i + r}$$

The susceptible are those who are not infected and not immune, the infective are those who are infected and can transmit the disease, and the recovered are those who have been infected, have recovered and are permanently immune.

Consider an epidemic that occurs on a timescale that is much shorter than that of the population, in other words regard the population as having a constant size and ignore births and deaths then we have the following system of nonlinear ordinary differential equation

$$\begin{aligned} \frac{ds}{dt} &= -c\beta s(t)\frac{i(t)}{n} \\ \frac{di}{dt} &= c\beta s(t)\frac{i}{n} - \gamma i(t) \\ \frac{dr}{dt} &= \gamma i(t) \end{aligned} \tag{1}$$

This model is appropriate to viral diseases such as measles, mumps and rubella. The SIR model has no analytical solution, but we can conduct numerical simulation for the approximate solution. Including births and deaths in the standard S-I-R model (1) for epidemics the resulting model will allow us to look at events of longer duration**.** In this case, we consider a model with a constant birth rate and a constant per-capita death rate.

We assume that all births are into the susceptible and the death rate is equal for members of all three classes. The birth and death rates are equal so that population is stationary then we have the system

$$\frac{ds}{dt} = \mu(s+i+r) - \alpha s(t)i(t) - \mu s(t)$$
$$\frac{di}{dt} = \alpha s(t)i(t) - (\gamma + \mu)i(t) \qquad (2)$$
$$\frac{dr}{dt} = \gamma i(t) - \mu r(t)$$

Where $s(t), i(t), r(t)$ represent the number of susceptible, infective and recovered individuals at time t respectively, c is number of contacts per unit time, β is Probability of disease transmission per contact, γ is Per-capita recovery rate, μ is Per-capita removal rate and α is the transitivity.

In this paper, equation (1) and (2) are solved using variational iteration method and the differential transformation method for numerical comparison.

Differential transformation method is one of the well-known techniques to solve linear and nonlinear equations. It was first introduced by Zhou [15] for solving linear and nonlinear initial value problems in electrical circuit analysis. This method has been used to solve differential algebraic equation [7], Schrödinger equations [22], fractional differential equation [1], Lane-Emden type equation, free vibration analysis of rotating beams, unsteady rolling motion of spheres equation in inclined tubes. The main advantage of this method is that it can be applied directly to linear and nonlinear ODEs without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate.

Also, the basic idea of the variational iteration method [9, 10] is to construct an iteration method based on correction functional that include a generalized Lagrange multiplier [18]. The VIM was proposed by J.H He (1999) where the value of the multiplier was chosen using variational theory so that the each improves the accuracy of the solution. The initial approximation i.e. trial function usually includes unknown coefficient which can be determined to satisfy any boundary and initial conditions. VIM does not require specific transformation for nonlinear terms as required by other techniques and is now widely used by many researchers to study autonomous ordinary differential equation[13], Integro-differential systems[20],Linear Helmholtz partial differential equation [21] and other fields[11,19,5,12,14]. In this method the solution is given in an infinite series usually convergent to an accurate solution.

2.1 The Differential Transformation Method

An arbitrary function $f(x)$ can be expanded in Taylor series about a point $x=0$ as

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0}$$

The differential transformation of $f(x)$ is defined as

$$F(x)=\frac{1}{k!}\left[\frac{d^k f}{dx^k}\right]_{x=0}$$

Then the inverse differential transform is

$$f(x)=\sum_{k=0}^{\infty}x^k F(k)$$

The fundamental mathematical operations performed by differential transform method are listed in Table 1.

Table 1: The fundamental operations of differential transformation method (DTM)

Original Function	**Transformed Function**
$y(x)=g(x)\pm h(x)$	$Y(k)=G(k)\pm H(k)$
$y(x)=\alpha\, g(x)$	$Y(k)=\alpha\, G(k)$
$y(x)=\dfrac{dg(x)}{dx}$	$Y(k)=(k+1)G(k+1)$
$y(x)=\dfrac{d^2 g(x)}{dx^2}$	$Y(k)=(k+1)(k+2)G(k+2)$
$y(x)=\dfrac{d^m g(x)}{dx^m}$	$Y(k)=(k+1)(k+2)...(k+m)G(k+m)$
$y(x)=1$	$Y(k)=\delta(k)$
$y(x)=x$	$Y(k)=\delta(k-1)$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$
$y(x)=g(x)h(x)$	$Y(k)=\sum_{m=0}^{k}H(m)G(k-m)$
$y(x)=e^{(\lambda x)}$	$Y(k)=\dfrac{\lambda^k}{k!}$

$y(x)=(1+x)^m$	$Y(k)=\dfrac{m(m-1)...(m-k+1)}{k!}$

2.2 Variational Iteration Method

According to the variational iteration method of [9, 10, 20] we consider the following general differential equation

$$Lu+Nu=g(x) \tag{3}$$

where, L is a linear operator, N is a nonlinear operator and g(x) is an inhomogeneous term. We can construct a correctional function as follows

$$u_{n+1}=u_n(x)+\int_0^x \lambda\{Lu_n(s)+N\tilde{u}_n(s)-g(s)\}ds \tag{4}$$

Where λ is a Lagrangian multiplier [20] which can be identified optimally via variational theory. The subscript n denotes the nth approximation and $\tilde{u}_n$ is considered as a restricted variation i.e $\delta\tilde{u}_n=0$ .Consider the stationary condition of the above correction functional, then the Lagrange multiplier can be expressed as

$$\lambda_i(w)=\frac{(-1)^m}{(m-1)!}(w-x)^{m-1} \tag{5}$$

Where m is the highest order of the differential equation

3.1 Application of Differential Transformation Method

Using the transformed function of the original function in table 1, we obtained the recurrence relation of equation (1) as

$$\begin{aligned}
S(k+1)&=\frac{1}{k+1}\left[\frac{-c\beta}{n}\sum_{m=0}^{k}S(m)\,I(k-m)\right]\\
I(k+1)&=\frac{1}{k+1}\left[\frac{c\beta}{n}\sum_{m=0}^{k}S(m)I(k-m)-\gamma I(k)\right]\\
R(k+1)&=\frac{1}{k+1}\left[\gamma I(k)\right]
\end{aligned} \tag{6}$$

With initial condition $s(0)=2000, i(0)=300, r(0)=200$ and parameter $c=6, \beta=0.05, \gamma=0.1,\ n=2500$, then $S(0)=2000, I(0)=300, R(0)=200$

Applying the condition in (6) we obtain

$S(1) = -72,\quad S(2) = -3.744,\quad S(3) = 0.034368,\quad S(4) = 0.013309,\ \ldots,$

$I(1) = 42,\quad I(2) = 1.644,\quad I(3) = -0.089168,\quad I(4) = -0.011080,\ \ldots,$

$R(1) = 30,\quad R(2) = 2.1,\quad R(3) = 0.0548,\quad R(4) = -0.002229,\ \ldots,$

Then, the closed form of the solution where k=4 can be written as

$$
\begin{aligned}
s(t) &= \sum_{m=0}^{k} S(k)t^k = 2000 - 72t - 3.744\,t^2 + 0.034368\,t^3 + 0.013309\,t^4 + \ldots,\\
i(t) &= \sum_{m=0}^{k} I(k)t^k = 300 + 42t + 1.644\,t^2 - 0.089168\,t^3 - 0.011080\,t^4 + \ldots,\\
r(t) &= \sum_{m=0}^{k} R(k)t^k = 200 + 30t + 2.1\,t^2 + 0.0548\,t^3 - 0.002229\,t^4 + \ldots,
\end{aligned}
\tag{7}
$$

Also, using the transformation in table 1, equation (2) gives

$$
\begin{aligned}
S(k+1) &= \frac{1}{k+1}\left[\mu(S(k)+I(k)+R(k)) - \alpha\sum_{m=0}^{k} S(m)I(k-m) - \mu S(k)\right]\\
I(k+1) &= \frac{1}{k+1}\left[\alpha\sum_{m=0}^{k} S(m)I(k-m) - \gamma I(k) - \mu I(k)\right]\\
R(k+1) &= \frac{1}{k+1}\left[\gamma I(k) - \mu R(k)\right]
\end{aligned}
\tag{8}
$$

with initial condition $s(0) = 990, i(0) = 10, r(0) = 0$ and parameter

$\alpha = 0.003, \gamma = 1, \mu = 0.05,\ n = 1000$.

From the initial condition, $S(0) = 990, I(0) = 10, R(0) = 0$ then equation (8) gives

$S(1) = -29.20,\quad S(2) = -27.344,\quad S(3) = -16.524247,\quad S(4) = -6.813137,\ \ldots,$

$I(1) = 19.20,\quad I(2) = 17.994,\quad I(3) = 10.682080,\quad I(4) = 4.215644,\ \ldots,$

$R(1) = 10,\quad R(2) = 9.35,\quad R(3) = 5.842166,\quad R(4) = 2.559749,\ \ldots,$

Therefore, the closed form of the solution where k=4 can be written as

$$s(t)=\sum_{m=0}^{k}S(k)t^k=990-29.20\,t-27.344t^2-16.524247\,t^3-6.813137\,t^4+\ldots,$$
$$i(t)=\sum_{m=0}^{k}I(k)t^k=10+19.20t+17.994t^2+10.682080\,t^3+4.215644\,t^4+\ldots, \quad (9)$$
$$r(t)=\sum_{m=0}^{k}R(k)t^k=10t+9.35\,t^2+5.842167t^3-2.559749\,t^4+\ldots,$$

3.2 Application of the Variational Iteration Method

Applying the variational iteration method in (1), we derive the correctional functional as follows

$$s_{n+1}(t)=s_n(t)+\int_0^t \lambda_1(w)\left\{s_n^{'}(w)+\frac{c\beta}{n}\tilde{s}_n(w)\,\tilde{i}_n(w)\right\}dw$$
$$i_{n+1}(t)=i_n(t)+\int_0^t \lambda_2(w)\left\{i_n^{'}(w)-\frac{c\beta}{n}\tilde{s}_n(w)\,\tilde{i}_n(w)+\gamma\tilde{i}_n(w)\right\}dw \quad (10)$$
$$r_{n+1}(t)=r_n(t)+\int_0^t \lambda_3(w)\left\{r_n^{'}(w)-\gamma\tilde{i}_n(w)\right\}dw$$

Where $\lambda_1,\lambda_2,\lambda_3$ are general Lagrange multipliers, $\tilde{s}_n(w),\tilde{i}_n(w),\tilde{r}_n(w)$ are considered restricted variation which means that $\delta\tilde{s}_n(w)=\delta\tilde{i}_n(w)=\delta\tilde{r}_n(w)=0$.

From equation (5) $\lambda_1(w)=-1,\lambda_2(w)=-1,\lambda_3(w)=-1$ then equation (10) becomes

$$s_{n+1}(t)=s_n(t)-\int_0^t \left\{s_n^{'}(w)+\frac{c\beta}{N}\tilde{s}_n(w)\tilde{i}_n(w)\right\}dw$$
$$i_{n+1}(t)=i_n(t)-\int_0^t \left\{i_n^{'}(w)-\frac{c\beta}{N}\tilde{s}_n(w)\tilde{i}_n(w)+\gamma\tilde{i}_n(w)\right\}dw \quad (11)$$
$$r_{n+1}(t)=r_n(t)-\int_0^t \left\{r_n^{'}(w)-\gamma\tilde{i}_n(w)\right\}dw$$

With initial approximation $s_0(t)=s(0)=2000$, $i_0(t)=i(0)=300$
$r_0(t)=r(0)=200$

which satisfy the initial conditions, to give

$$s_1(t) = 2000 - 72\,t$$
$$i_1(t) = 300 + 42\,t \tag{12}$$
$$r_1(t) = 200 + 30\,t$$

$$s_2(t) = 2000 - 72\,t - 3.744t^2 + 0.12096t^3$$
$$i_2(t) = 300 + 42\,t + 1.644t^2 - 0.12096t^3 \tag{13}$$
$$r_2(t) = 200 + 30\,t + 2.1\,t^2$$

$$s_3(t) = 2000 - 72\,t - 3.744\,t^2 + 0.034368t^3 + 0.014437t^4 - 0.000183t^5 - 0.000013t^6 + 2.51\,(10^{-7})\,t^7$$
$$i_3(t) = 300 + 42\,t + 1.644t^2 - 0.089168t^3 - 0.014413t^4 + 0.000183t^5 + 0.000013t^6 + 2.51\,(10^{-7})\,t^7 \tag{14}$$
$$r_3(t) = 200 + 30\,t + 2.1\,t^2 + 0.0548\,t^3 - 0.003024\,t^4$$

It can be observed that the result of the epidemic system of equation (1) is in complete agreement with the result obtained by the differential transformation method.

Also, in the same approach applying VIM to equation (2) the correction functional is as follows

$$s_{n+1}(t) = s_n(t) + \int_0^t \lambda_1(w)\left\{s_n'(w) - \mu(\tilde{s}_n(w) + \tilde{i}_n(w) + \tilde{r}_n(w)) + \alpha\tilde{s}_n(w)\,\tilde{i}_n(w) + \mu\tilde{s}_n(w)\right\}dw$$
$$i_{n+1}(t) = i_n(t) + \int_0^t \lambda_2(w)\left\{i_n'(w) - \alpha\tilde{s}_n(w)\,\tilde{i}_n(w) + \gamma\tilde{i}_n(w) + \mu\tilde{i}_n(w)\right\}dw \tag{15}$$
$$r_{n+1}(t) = r_n(t) + \int_0^t \lambda_3(w)\left\{r_n'(w) - \gamma\tilde{i}_n(w) - \mu\tilde{r}_n(w)\right\}dw$$

The Lagrange multipliers, $\lambda_1(w) = -1, \lambda_2(w) = -1, \lambda_3(w) = -1$, substituting the multipliers into the correction functional above, we have the following iteration formula

$$s_{n+1}(t) = s_n(t) - \int_0^t \left\{s_n'(w) - \mu(\tilde{s}_n(w) + \tilde{i}_n(w) + \tilde{r}_n(w)) + \alpha\tilde{s}_n(w)\,\tilde{i}_n(w) + \mu\tilde{s}_n(w)\right\}dw$$
$$i_{n+1}(t) = i_n(t) - \int_0^t \left\{i_n'(w) - \alpha\tilde{s}_n(w)\,\tilde{i}_n(w) + \gamma\tilde{i}_n(w) + \mu\tilde{i}_n(w)\right\}dw \tag{16}$$
$$r_{n+1}(t) = r_n(t) - \int_0^t \left\{r_n'(w) - \gamma\tilde{i}_n(w) - \mu\tilde{r}_n(w)\right\}dw$$

Given the initial condition and parameter s(0)= 990, i(0)=10, r(0)=0, α =0.003, γ=1, μ=0.05, n=1000, with initial approximations $s_0(t) = 990, i_0(t) = 10, r_0(t) = 0$ which satisfy the initial conditions, we obtain

$$
\begin{aligned}
s_1(t) &= 990 - 29.20\,t \\
i_1(t) &= 10 + 19.20\,t \\
r_1(t) &= 10t
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
s_2(t) &= 990 - 29.20\,t - 27.344t^2 + 0.560640\,t^3 \\
i_2(t) &= 10 + 19.20t + 17.994t^2 - 0.560640\,t^3 \\
r_2(t) &= 10t + 9.35\,t^2
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
s_3(t) &= 990 - 29.20\,t - 27.344\,t^2 - 16.524247t^3 + 1.192885t^4 + 0.278936t^5 \\
&\quad - 0.012709t^6 + 0.000135t^7 \\
i_3(t) &= 10 + 19.20t + 17.994t^2 + 10.68208t^3 - 1.052724t^4 - 0.278936t^5 \\
&\quad + 0.012709t^6 - 0.000135t^7 \\
r_3(t) &= 10t + 9.35\,t^2 + 5.842167t^3 - 0.140160t^4
\end{aligned}
\tag{19}
$$

The result above indicates that both methods are in complete agreement which implies that both are powerful tools for solving system of differential equations.

4 Conclusion

In this paper, differential transformation method (DTM) and variational iteration method (VIM) has been successfully employed to obtain the approximate solution of SIR model with initial condition. The presented methods have been applied in a direct way without linearization, discretization or perturbation. Result obtained by this methods shows that both are in excellent agreement which indicates their effectiveness and reliability. These two methods can be considered as an alternative method for solving a wide class of linear and non-linear problems which arise in various field of study.

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