Gen. Math. Notes, Vol. 17, No. 1, July, 2013, pp. 82-87
ISSN 2219-7184; Copyright © ICSRS Publication, 2013
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A Novel Formula in Solving Tribonacci-like Sequence

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(Received: 17-4-13 / Accepted: 20-5-13)

Abstract

In mathematical terms, sequence is a list of numbers arranged in a specific order. Tribonacci sequence is a sequence of whole numbers with equations $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ in which $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. When the first three terms of the tribonacci sequence become arbitrary, it is known as tribonacci-like sequence. Tribonacci-like sequence can start at any desired number. This study aimed to derive and validate a formula in solving tribonacci-like sequence. Basic equations were derived on finding n^{th} term (S_n) of Tribonacci-like sequence given its arbitrary first three terms. Formulas of S_n for four terms ($n = 4$), five terms ($n = 5$), six terms ($n = 6$) and seven terms ($n = 7$) were solved. From these formulas, numerical coefficients of the first three terms were tabulated and patterns were recognized. In these patterns, a formula was obtained in solving n^{th} term of Tribonacci-like sequence using its first three terms and Tribonacci numbers. The novel formula is

$$S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3$$

Where
S_n = nth term of Tribonacci-like sequence
S_1 = first term

S_2 = second term
S_3 = third term
T_{n-1}, T_{n-2}, T_{n-3} = corresponding Tribonacci numbers.

Keywords: *Tribonacci sequence, Tribonacci-like sequence, Tribonacci numbers.*

1 Introduction

Sequence is a list of numbers arranged in a specific order. It can contain members similar to a set. However, sequence can have the same members repeated as much as possible at a divergent locations. Thus, pattern is a substantial element of a sequence.

There are lots of functional sequences existing today. Some sequences like of arithmetic, geometric, square, cube, Lucas, Fibonacci and Tribonacci are known to many for a very long time. Arithmetic, geometric, harmonic and fibonacci sequence have an established formula on finding the sum and the n^{th} term of the respective sequences. However, devising formula for the n^{th} term of fibonacci-like, tribonacci-like and other related sequences have received little attention from mathematicians.

Fibonacci sequence is a sequence starting from 0 and 1 where the succeeding terms are taken from two previous terms that are added. Moreover, fibonacci-like sequence is a derivative of the fibonacci sequence where the same pattern is applied. Their only difference is that Fibonacci-like sequence starts at any two given terms.

Recently, a formula derived by Natividad [1] in solving a Fibonacci-like sequence was established through amalgamating basic formula from important patterns with a Binet's formula. The resulting equation is

$$x = \frac{b - \left[\frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\right] a}{\frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\sqrt{5}}}, n \geq 1.$$

On the other hand, Tribonacci sequence is a sequence of whole numbers with equations $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ in which $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. This only means that the previous three terms are added to find the next term. It is also referred as series of Tribonacci numbers.

Noe, Piezas, and Weisstein [2] listed two explicit formulas for Tribonacci sequence. The implicit formula is

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

where α, β and γ are the three roots of the polynomial $x^3 - x^2 - x - 1$.

They gave also another explicit formula in the form of

$$T_n = \left| 3\frac{\left\{\frac{1}{3}(19+3\sqrt{33})^{\frac{1}{3}} + \frac{1}{3}(19-3\sqrt{33})^{\frac{1}{3}} + \frac{1}{3}\right\}^n (586+102\sqrt{33})^{1/3}}{(586+102\sqrt{33})^{2/3} + 4 - 2(586+102\sqrt{33})^{1/3}} \right|.$$

Just like fibonacci and fibonacci-like sequence, when the first three terms of the tribonacci sequence become arbitrary, it is known as tribonacci-like sequence. tribonacci-like sequence can start at any desired number. These sequences have many applications in environment, biology, chemistry, arts, mathematics, music and among others as it occurs naturally.

However, to the best knowledge of the researchers, there are no equations have been formulated on finding the nth term of the Tribonacci-like sequence. Thus, this study will present how to derive a formula on finding nth term of the Tribonacci-like sequence using recognition of patterns and by observing its first three terms.

2 Main Results

Tribonacci sequence, like Fibonacci, is a sequence where terms are obtained from adding previous three terms. The first three terms of this sequence are 0, 1, and 1. From this definition, we can define Tribonacci-like sequence as a Tribonacci sequence where the first three terms are arbitrary.

These first three terms of Tribonacci-like sequence can be designated as S_1, S_2, S_3. In this manner, a formula was obtained for S_n given the first three terms.

Definition 1: *The sequence $S_1, S_2, S_3, \ldots, S_n$ in which $S_n = S_{n-3} + S_{n-2} + S_{n-1}$ is a generalized tribonacci sequence (tribonacci-like sequence). This sequence follows the pattern of tribonacci sequence.*

Considering the Tribonacci-like sequence $S_1, S_2, S_3, \ldots, S_n$, a novel formula for nth term of tribonacci-like sequence was derived. Formula was obtained from the first, second and third terms designated as S_1, S_2 and S_3 only. Specific formula was observed and from these, general formula was deduced.

To find the general formula for the nth term of the Tribonacci-like sequence, a pattern was recognized. All the equations derived were listed to find recognizable patterns easily.

$$S_4 = S_1 + S_2 + S_3$$
$$S_5 = S_1 + 2S_2 + 2S_3$$
$$S_6 = 2S_1 + 3S_2 + 4S_3$$
$$S_7 = 4S_1 + 6S_2 + 7S_3$$

After careful observation and investigation, all the numerical coefficients for S_1, S_2 and S_3 were tabulated as shown in Table 1.

Table 1: Coefficients of S_1, S_2, and S_3 of n^{th} term of Tribonacci-like Sequence

Number of Terms	n^{th} term of Tribonacci-like sequence	Coefficient		
		S_1	S_2	S_3
4	S_4	1	1	1
5	S_5	1	2	2
6	S_6	2	3	4
7	S_7	4	6	7
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
n	S_n	T_{n-2}	$T_{n-2} + T_{n-3}$	T_{n-1}

It can be noted that coefficients of S_1 and S_3 follows the Tribonacci sequence (Table 2).

Table 2: The first 18 Tribonacci Numbers

Nth Term	Tribonacci Number
1	0
2	1
3	1
4	2
5	4
6	7
7	13
8	24
9	44
10	81
11	149
12	274
13	504

14	927
15	1705
16	3136
17	5768
18	10609

From these observations, the following theorem is proposed.

Theorem 1: *For any real numbers S_1, S_2, and S_3, the formula for finding n^{th} term of tribonacci-like sequence is*

$$\boldsymbol{S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3}$$

where S_n is the nth term of tribonacci-like sequence, S_1 is the first term, S_2 is the second term, S_3 is the third term and T_{n-1}, T_{n-2} and T_{n-3} are the corresponding tribonacci numbers.

Proof: We will prove the formula for Tribonacci-like sequence for n≥4

$$S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3$$

The formula is true for n=3, n=4, n=5 etc. which is easy to verify.

Using n=4,

$$S_4 = T_2S_1 + (T_2 + T_1)S_2 + T_3S_3$$

From table 2,

$$S_4 = (1)S_1 + (1 + 0)S_2 + (1)S_3$$
$$S_4 = S_1 + S_2 + S_3$$

which is true from the definition of tribonacci-like sequence.

Next, we will show that the formula is valid for n is greater than 4. Assuming that P(k) is true and writing it as

$$S_k = T_{k-2}S_1 + (T_{k-2} + T_{k-3})S_2 + T_{k-1}S_3$$

Suppose we want to find the k+1 term of the sequence, so

$$S_{k+1} = T_{(k+1)-2}S_1 + [T_{(k+1)-2} + T_{(k+1)-3}]S_2 + T_{(k+1)-1}S_3$$
$$S_{k+1} = T_{k-1}S_1 + (T_{k-1} + T_{k-2})S_2 + T_kS_3$$

To verify, we will provide the assumption of P(k) implies the truth of P(k+1). To do so, we will add S_{k-1} and S_{k-2} to both sides of P(k). The left side of P(k) will become

$$S_{k-2} + S_{k-1} + S_k$$

which is equal to the left side of P(k+1) since $S_{k-2} + S_{k-1} + S_k = s_{k+1}$ from definition of Tribonacci-like sequence.

After adding S_{k-1} and S_{k-2}, the right side of P(k) will become

$$T_{k-2}S_1 + (T_{k-2} + T_{k-3})S_2 + T_{k-1}S_3 + S_{k-2} + S_{k-1}$$

But since

$S_{k-2} = T_{k-4}S_1 + (T_{k-4} + T_{k-5})S_2 + T_{k-3}S_3$ and $S_{k-1} = T_{k-3}S_1 + (T_{k-3} + T_{k-4})S_2 + T_{k-2}S_3$

The equation will become

$S_{k+1} = T_{k-2}S_1 + (T_{k-2} + T_{k-3})S_2 + T_{k-1}S_3 + T_{k-4}S_1 + (T_{k-4} + T_{k-5})S_2 + T_{k-3}S_3 + T_{k-3}S_1\ (T_{k-3} + T_{k-4})S_2 + T_{k-2}S_3$

$= (T_{k-2} + T_{k-3} + T_{k-4})\ S_1 + [(T_{k-2} + T_{k-3} + T_{k-4}) + (T_{k-3} + T_{k-4} + T_{k-5})]S_2 + (T_{k-1} + T_{k-2} + T_{k-3})\ S_3$

From the definition, the equation will become

$$T_{k-1}S_1 + (T_{k-1} + T_{k-2})S_2 + T_kS_3$$

which is the right side of P(k+1).

Through strong mathematical induction, conclusion follows.

Conclusions

It is concluded that patterns are essential in deriving formulas for sequences. Using these patterns, a formula was obtained in solving n^{th} term of tribonacci-like sequence using its first three terms and tribonacci numbers. The formula will be of great help for other people establishing new formula for other equations that receive little attention. The derived formula for solving the n^{th} term of a tribonacci-like sequence was verified in real numbers system by testing the formula in different examples and was verified through mathematical induction.

References

[1] L.R. Natividad, Deriving a formula in solving Fibonacci-like sequence, *International Journal of Mathematics and Scientific Computing*, 1(2010), 19-21.

[2] T. Noe, T. Piezas and E. Weisstein, Tribonacci Number, Retrieved from http://mathworld.wolfram.com/TribonacciNumber.html, September 30 (2012).