Gen. Math. Notes, Vol. 18, No. 2, October, 2013, pp. 92-103
ISSN 2219-7184; Copyright © ICSRS Publication, 2013
www.i-csrs.org
Available free online at http://www.geman.in

On Fractional Calculus Operators of a Class of Meromorphic Multivalent Functions

Waggas Galib Atshan[1], Laila Ali Alzopee[2] and Mohammad Mostafa Alcheikh[3]

[1] Department of Mathematics
College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq
E-mail: waggashnd@gmail.com; waggas_hnd@yahoo.com

[2, 3] Department of Mathematics
College of Science, Damascus University
Damascus, Syria
[2] E-mail: lailaalzopee@gmail.com
[3] E-mail: mohammadalcheikh@mail2world.com

(Received: 15-7-13 / Accepted: 24-8-13)

Abstract

In the present paper, a class of meromorphic multivalent functions is defined by using fractional differ-integral operators. Coefficients estimates, radii of starlikeness and convexity are obtained. Also distortion and closure theorems for the class $\sum_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ *are also established.*

Keywords: *Meromorphic Functions, Fractional Calculus, Radius of starlikeness.*

1 Introduction:

Let $\sum_p$ denote the class of meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \, , p \in N, \tag{1}$$

which are analytic and p-valent in the puncture unit disk

$$U^* = \{z \in C: 0 < |z| < 1\}.$$

A function $f \in \sum_p$ is said to be in the class $\sum_p^*(\alpha)$ of meromorphic p-valently starlike function of order α if:

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad , \ (z \in U^*, 0 \le \alpha < p \, , p \in N). \tag{2}$$

A function $f \in \sum_p$ is said to be in the class $\sum_p^k(\alpha)$ of meromorphic p-valently convex function of order α if:

$$-Re\{\, 1 + \frac{zf''(z)}{f'(z)}\} > \alpha \, , (z \in U^*, 0 \le \alpha < p \, , p \in N). \tag{3}$$

In this paper, we discuss and study a new class of meromorphic p-valently convex functions by making use of the fractional differ-integral operator contained in:

Definition 1:

$$W_{0,z}^{\lambda,\mu,v,\eta} f(z) = \begin{cases} \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p+\eta+1} J_{0,z}^{\lambda,\mu,v,\eta}[z^{\mu+p} f(z)\,] \ (0 \le \lambda < 1), \\ \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)} z^{-p-\eta+1} I_{0,z}^{-\lambda,\mu,v,\eta}[z^{\mu+p} f(z)] \ \ (-\infty \le \lambda < 0) \end{cases} \tag{4}$$

where $J_{0,z}^{\lambda,\mu,v,\eta}$is the generalized fractional derivative operator of order α defined by

$$J_{0,z}^{\lambda,\mu,v,\eta} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz}\left\{z^{\lambda-\mu} \int_0^z t^{\eta-1} (z-t)^{-\lambda} {}_2F_1(\mu-\lambda, 1-v; 1-\lambda; 1-\frac{t}{2}) f(t) dt\right\} \tag{5}$$

$$\big(0 \le \lambda < 1, \mu, \eta \in R, r \in R^{+} and \ r > (\max\{0,\mu\} - \eta)\big),$$

where f is an analytic function in a simply–connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when$(z-t) > 0$, provided further that

$$f(z) = O(|z|^r) \qquad (z \to 0)\,, \tag{6}$$

and $I_{0,z}^{-\lambda,\mu,v,\eta}$ is the generalized fractional integral operator of order $-\boldsymbol{\lambda}$ $(-\infty < \lambda < 0)$ defined by

$$I_{0,z}^{\lambda,\mu,v,\eta} f(z) = \frac{z^{-(\lambda+\mu)}}{\Gamma(\lambda)} \int_0^z t^{\eta-1} \ (z-t)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -v; \lambda; 1-\frac{t}{2}\right) f(t)dt \qquad (7)$$

$\left(\lambda > 0, \mu, \eta \in R, r \in R^+ \ and \ r > (\max\{0,\mu\} - \eta)\right)$,

where f is constrained and the multiplicity of $(z-t)^{\lambda-1}$ is removed as above and r is given by the order estimate (6).

It follows from (5) and (7) that

$$J_{0,z}^{\lambda,\mu,v,1} f(z) = J_{0,z}^{\lambda,\mu,v} f(z), \qquad (8)$$

and

$$I_{0,z}^{\lambda,\mu,v,1} f(z) = I_{0,z}^{\lambda,\mu,v} f(z), \qquad (9)$$

where $J_{0,z}^{\lambda,\mu,v}$ and $I_{0,z}^{\lambda,\mu,v}$ are the familiar Owa-Saigo-Srivastava generalized fractional derivative and integral operators (see, e.g., [4] and [8] see also [7]).

Also

$$J_{0,z}^{\lambda,\lambda,v,1} f(z) = D_z^{\lambda} f(z), \qquad (0 \le \lambda < 1) \qquad (10)$$

and

$$I_{0,z}^{\lambda,-\lambda,v,1} f(z) = D_z^{-\lambda} f(z), \qquad (\lambda > 0) \qquad (11)$$

where D_z^{λ} and $D_z^{-\lambda}$ are the familiar Owa-Srivastava fractional derivative and integral of order $\boldsymbol{\lambda}$, respectively (cf. Owa [3]; see also Srivastava and Owa [6]).

Furthermore, in terms of Gamma function, we have

$$J_{0,z}^{\lambda,\mu,v,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta-\lambda+v)} z^{k+\eta-\mu-1} \qquad (12)$$

$(0 \le \lambda < 1, \mu, \eta \in R, v \in R^+ \ and \ k > (max\,\{0,\mu\} - \eta))$,

and

$$I_{0,z}^{\lambda,\mu,v,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta+\lambda+v)} z^{k+\eta-\mu-1} \qquad (13)$$

$(\lambda > 0, \mu, \eta \in R, v \in R^{+}$ *and* $k > (max\{0, \mu\} - \eta))$.

Now using (1), (12) and (13) in (4), we find that

$$W_{0,z}^{\lambda,\mu,v,\eta} f(z) = z^{-p} + \sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,v,\eta} a_n z^n , \tag{14}$$

Provided that $-\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, \eta > 0, p \in N, f \in \Sigma_P$ and

$$\Gamma_n^{\lambda,\mu,v,\eta} = \frac{(\mu+\eta)_{n+p}(v+\eta)_{n+p}}{(\mu+v+\eta-\lambda)_{n+p}(\eta)_{n+p}} . \tag{15}$$

It may be worth noting that, by choosing $\mu = \lambda$, $\eta = 1$ and p=1, the operator $W_{0,z}^{\lambda,\mu,v,\eta} f(z)$ reduces to the well-known Ruscheweyh derivative $D^{\lambda} f(z)$ for meromorphic univalent functions [5].

In this paper, we shall study a subclass of (1) define below.

Definition 2: *The function* $f \in \Sigma_p$ *is in the class* $\Sigma_p(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ *if it satisfies the condition*

$$\left| \frac{\dfrac{z(W_{0,z}^{\lambda,\mu,v,\eta} f(z))'}{W_{0,z}^{\lambda,\mu,v,\eta} f(z)} + \gamma}{\dfrac{z(W_{0,z}^{\lambda,\mu,v,\eta} f(z))'}{W_{0,z}^{\lambda,\mu,v,\eta} f(z)} + (2\alpha - \gamma)} \right| < \beta , \tag{16}$$

for some $\alpha(\alpha > 0)$, $\beta(0 < \beta \le 1)$, $\gamma\ (0 \le \gamma \le 1), p \in N, -\infty < \lambda < 1$, $\mu + v + \eta > \lambda, \mu > -\eta, v > -\eta$ *and* $\eta > 0$.

For μ=**λ**=0, p=1; the class $\Sigma_p(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ reduces to the class studied recently by Darus [1].

Definition 3: *Let* Σ_p^{+} *denote the subclass of* Σ_p *defined as*

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n ;\ (a_n \ge 0; p \in N). \tag{17}$$

Then we define a new subclass $\Sigma_p^{+}(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ by

$$\Sigma_p^{+}(\lambda, \mu, v, \eta, \gamma, \alpha, \beta) = \Sigma_p^{+} \cap \Sigma_p(\lambda, \mu, v, \eta, \gamma, \alpha, \beta).$$

2 Coefficient Estimates:

Theorem 1: *Assume that* $f \in \Sigma_p$ *and*

$$\sum_{n=p}^{\infty} 2(n+\alpha)\, \Gamma_n^{\lambda,\mu,v,\eta} |a_n| \leq (p-\gamma) + \beta(p+\gamma-2\alpha), \tag{18}$$

where $\Gamma_n^{\lambda,\mu,v,\eta}$ is defined by (15) and the conditions mentioned with (16) hold. Then $f \in \Sigma_p(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Proof: Let us assume that inequality (18) is true. Further suppose that

$$\boldsymbol{\Omega(f)} = \left|\boldsymbol{z}\left(W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right)' + \boldsymbol{\gamma} W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right| - \boldsymbol{\beta}\left|\boldsymbol{z}(W_{0,z}^{\lambda,\mu,v,\eta} f(z))' + (\boldsymbol{2\alpha} - \boldsymbol{\gamma}) W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right|.$$

Now using (14), we find that

$$\Omega(f) = \left|-pz^{-p} + \sum_{n=p}^{\infty} n\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n + \gamma z^{-p} + \sum_{n=p}^{\infty} \gamma \Gamma_n^{\lambda,\mu,v,\eta} a_n z^n\right| - \beta \left| -pz^{-p} + \sum_{n=p}^{\infty} n\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n + (2\alpha-\gamma)z^{-p} + \sum_{n=p}^{\infty} (2\alpha-\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n \right|$$

$$= \left|(\gamma-p)z^{-p} + \sum_{n=p}^{\infty}(n+\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n\right| - \beta\left|(2\alpha-\gamma-p)z^{-p} + \sum_{n=p}^{\infty}(n+2\alpha-\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^n\right|$$

$$\leq -(p-\gamma)r^{-p} + \sum_{n=p}^{\infty}(n+\gamma)\Gamma_n^{\lambda,\mu,v,\eta}|a_n|r^n - \beta(p+\gamma-2\alpha)r^{-p} + \sum_{n=p}^{\infty}(n+2\alpha-\gamma)\Gamma_n^{\lambda,\mu,v,\eta}|a_n|r^n$$

$$= \sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}|a_n|r^n - (p-\gamma) + \beta(p+\gamma-2\alpha)r^{-p}. \tag{19}$$

Since the above inequality holds for all r, 0 < r < 1. Letting r →1 in (19) we easily get that $\Omega(f) \leq 0$, hence $f \in \Sigma_p(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Theorem 2: *Let $f \in \Sigma_p^+$ .Then $f \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ if and only if*

$$\sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} a_n \leq (p-\gamma) + \beta(p+\gamma-2\alpha) \ , \tag{20}$$

where $\Gamma_n^{\lambda,\mu,v,\eta}$ is defined by (15) and all the parameters are constrained as in Theorem 1.

Proof: In view of Theorem 1, it is sufficient to prove the "only if" part.

Let us assume that $f \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$. Then

$$\left| \frac{\dfrac{z(W_{0,z}^{\lambda,\mu,v,\eta} f(z))'}{W_{0,z}^{\lambda,\mu,v,\eta} f(z)} + \gamma}{\dfrac{z(W_{0,z}^{\lambda,\mu,v,\eta} f(z))'}{W_{0,z}^{\lambda,\mu,v,\eta} f(z)} + (2\alpha-\gamma)} \right| = \left| \frac{(\gamma-p) + \sum_{n=p}^{\infty}(n+\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^{n+p}}{(2\alpha-\gamma-p) + \sum_{n=p}^{\infty}(n+2\alpha-\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^{n+p}} \right| < \beta \ .$$

Since $Re(z) \leq |z|$ for all z, it follows that

$$Re\left\{ \frac{(\gamma-p) + \sum_{n=p}^{\infty}(n+\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^{n+p}}{(p+\gamma-2\alpha) - \sum_{n=p}^{\infty}(n+2\alpha-\gamma)\Gamma_n^{\lambda,\mu,v,\eta} a_n z^{n+p}} \right\} < \beta \ .$$

Now letting $r \to 1^-$, through real values, we easily obtain the desired result (20).

3 Distortion Theorems:

A distortion property for functions in the class $\Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ is contained in

Theorem 3: *Let $f \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ .Then*

$$\frac{1}{|z|^p} - \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{(p+\gamma)}|z|^p \leq \left| W_{0,z}^{\lambda,\mu,v,\eta} f(z) \right| \leq \frac{1}{|z|^p} + \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{(p+\gamma)}|z|^p \ ,$$

where all the parameters are constrained in as in Theorem 1.

Proof: Since $f \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$. In view of Theorem 2, we have

$$\sum_{n=p}^{\infty} a_n \Gamma_n^{\lambda,\mu,v,\eta} \le \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(p+\alpha)} . \qquad (21)$$

Now

$$\left|W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right| \le \frac{1}{|z|^p} + \sum_{n=p}^{\infty} a_n \Gamma_n^{\lambda,\mu,v,\eta} |z|^n \le \frac{1}{|z|^p} + |z|^p \sum_{n=p}^{\infty} a_n \Gamma_n^{\lambda,\mu,v,\eta} .$$

Now making use of (21), we obtain

$$\left|W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right| \le \frac{1}{|z|^p} + \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(p+\gamma)} |z|^p .$$

Also

$$\left|W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right| \ge \frac{1}{|z|^p} - \sum_{n=p}^{\infty} a_n \Gamma_n^{\lambda,\mu,v,\eta} |z|^n \ge \frac{1}{|z|^p} - |z|^p \sum_{n=p}^{\infty} a_n \Gamma_n^{\lambda,\mu,v,\eta}.$$

Again making use of (21), we get

$$\left|W_{0,z}^{\lambda,\mu,v,\eta} f(z)\right| \ge \frac{1}{|z|^p} - \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(p+\gamma)} |z|^p .$$

This completes the proof of Theorem 3.

4 Radii of Starlikeness and Convexity for the Class $\Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$:

Theorem 4: *Let $f \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ .Then f is meromorphically p-valent starlike of order Ψ $(0 \le \Psi < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_n \left\{ \frac{(p-\Psi)(2(n+\alpha))\Gamma_n^{\lambda,\mu,v,\eta}}{(n+2p-\Psi)(p-\gamma)+\beta(p+\gamma-2\alpha)} \right\}^{\frac{1}{n+p}}, \qquad (22)$$

where all the parameters are constrained as in Theorem 1.

Proof: For $(0 \le \Psi < p)$, we require to show that

$$\left|\frac{zf'(z)}{f(z)}+p\right|<p-\Psi \ . \tag{23}$$

That is

$$\left|\frac{-pz^{-p}+\sum_{n=p}^{\infty} na_n z^n+pz^{-p}+\sum_{n=p}^{\infty} pa_n z^n}{z^{-p}+\sum_{n=p}^{\infty} a_n z^n}\right|=\left|\frac{\sum_{n=p}^{\infty}(n+p)a_n z^{n+p}}{1+\sum_{n=p}^{\infty} a_n z^{n+p}}\right|$$
$$\le\frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum_{n=p}^{\infty} a_n|z|^{n+p}}<p-\Psi \ ,$$

or equivalently

$$\sum_{n=p}^{\infty}\left(\frac{n+2p-\Psi}{p-\Psi}\right)z_n|z|^{n+p}\le 1 \ .$$

It is enough letting

$$|z|^{n+p}\le\frac{(p-\Psi)(2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta})}{(n+2p-\Psi)(p-\gamma)+\beta(p+\gamma-2\alpha)} .$$

Therefore,

$$|z|\le\left\{\frac{(p-\Psi)\left(2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}\right)}{(n+2p-\Psi)(p-\gamma)+\beta(p+\gamma-2\alpha)}\right\}^{\frac{1}{n+p}} . \tag{24}$$

Setting $|z|=r_1(\lambda,\mu,v,\eta,\gamma,\alpha,\beta,\Psi)\ in$ (24), we get the radius of starlikeness, which completes the proof of Theorem 4.

Noting the fact that f is convex if and only if zf' is starlike [2], we have

Theorem 5: *Let $f\in\Sigma_p^{+}(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ .Then f is meromorphically p-valently convex of order* Ψ $(0\le\Psi<p)$*in* $|z|<r_{2}$ *, where*

$$r_2=inf_n\left\{\frac{p(p-\Psi)(2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta})}{n(n+2p-\Psi)(p-\gamma)+\beta(p+\gamma-2\alpha)}\right\}^{\frac{1}{n+p}} . \tag{25}$$

Proof: Let $f\in\Sigma_p^{+}(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$. Then by Theorem 2

$$\sum_{n=p}^{\infty}\frac{2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}a_n}{(p-\gamma)+\beta(p+\gamma-2\alpha)}\le 1.$$

For$(0 \leq \Psi < p)$, we show that

$$\left|\frac{zf''(z)}{f'(z)} + (1+p)\right| \leq p - \Psi .$$

That is

$$\left|\frac{p(p+1)z^{-(p+1)} + \sum_{n=p}^{\infty} n(n-1)a_n z^{n-1} - p(p+1)z^{-(p+1)} + \sum_{n=p}^{\infty} n(p+1)a_n z^{n-1}}{-pz^{-(p+1)} + \sum_{n=p}^{\infty} na_n z^{n-1}}\right|$$

$$= \left|\frac{\sum_{n=p}^{\infty} n(n+p)a_n z^{n-1}}{-pz^{-(p+1)} + \sum_{n=p}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p - \sum_{n=p}^{\infty} na_n |z|^{n+p}} < p - \Psi ,$$

or equivalently

$$\sum_{n=p}^{\infty} \frac{n(n+2p-\Psi)}{p(p-\Psi)} a_n |z|^{n+p} \leq 1 .$$

It is enough to consider

$$|z|^{n+p} \leq \left\{\frac{p(p-\Psi)\left(2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}\right)}{n(n+2p-\Psi)\left((p-\gamma)+\beta(p+\gamma-2\alpha)\right)}\right\}.$$

Therefore,

$$|z| \leq \left\{\frac{p(p-\Psi)\left(2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}\right)}{n(n+2p-\Psi)\left((p-\gamma)+\beta(p+\gamma-2\alpha)\right)}\right\}^{\frac{1}{n+p}} . \qquad (26)$$

Setting $|z| = r_2(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)\ in$ (26), we get the radius of convexity, which completes the proof of Theorem 5.

5 Closure Theorems:

Let the functions $f_k(z), (k = 1,2, \ldots, s)$, be defined by

$$f_k(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,k} z^n \ , \left(z \in U^*, a_{n,k} \geq 0\right). \qquad (27)$$

We shall prove the following closure theorems.

Theorem 6: *Let the function* $f_k(z), (k = 1,2, \ldots, s)$, *defined by (27) be in the class* $\Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$. *Then the function* $F \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$, *where*

$$F(z) = \sum_{k=1}^{s} b_k f_k(z); (b_k \geq 0 \; and \; \sum_{k=1}^{s} b_k = 1). \tag{28}$$

Proof: From (28), we can write

$$F(z) = z^{-p} + \sum_{n=p}^{\infty} (\sum_{k=1}^{s} b_k a_{n,k}) z^n . \tag{29}$$

Since $f_k \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)(k = 1,2, \ldots, s)$, therefore

$$\sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} \left(\sum_{k=1}^{s} b_k a_{n,k} \right) z^n = \sum_{k=1}^{s} b_k \left(\sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} a_{n,k} \right)$$

$$\leq \sum_{k=1}^{s} b_k \big((p-\gamma) + \beta(p+\gamma-2\alpha\beta) \big) = (p-\gamma) + \beta(p+\gamma-2\alpha).$$

Hence by Theorem 2, we have $F \in \Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

This completes the proof of Theorem 6.

Theorem 7: *The class* $\Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ *is closed under convex linear combination.*

Proof: Let the functions $f_k(k = 1,2)$ given by (28) be in the class $\Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$. Then it is enough to show that the function

$$g(z) = \sigma f_1(z) + (1-\sigma) f_2(z), (0 \leq \sigma \leq 1), \tag{30}$$

is also in the class $\Sigma_p^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Since, for $(0 \leq \sigma \leq 1)$,

$$g(z) = z^{-p} + \sum_{n=p}^{\infty} [\sigma a_{n,1} + (1-\sigma) a_{n,2}] \, z^n ,$$

we observe that

$$\sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} \{\sigma a_{n,1} + (1-\sigma) a_{n,2}\}$$

$$= \sigma \sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} a_{n,1} + (1-\sigma) \sum_{n=p}^{\infty} 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta} a_{n,2}$$

$$\leq (p-\gamma) + \beta(p+\gamma-2\alpha).$$

Hence, by Theorem 2, we have $g \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Theorem 8: Let $f_{p-1}(z) = z^{-p}$,

$$f_p(z) = z^{-p} + \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}} z^n \,, \qquad (31)$$

where all parameters are constrained as in Theorem 1.
Then $f \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ if and only if f can be expressed in the form

$$f(z) = \sigma_{p-1} f_{p-1}(z) + \sum_{n=p}^{\infty} \sigma_n f_n(z), \qquad (32)$$

where $\sigma_{p-1} \geq 0\,, \sigma_n \geq 0 \; and \; \sigma_{p-1} + \sum_{n=p}^{\infty} \sigma_n = 1$.

Proof: Let

$$f(z) = \sigma_{p-1} f_{p-1}(z) + \sum_{n=p}^{\infty} \sigma_n f_n(z)$$

$$= z^{-p} + \sum_{n=p}^{\infty} \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}} \sigma_n z^n \,.$$

Then

$$\sum_{n=p}^{\infty} \frac{\big((p-\gamma)+\beta(p+\gamma-2\alpha)\big) 2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}}{2(n+\alpha)\Gamma_n^{\lambda,\mu,v,\eta}\big((p-\gamma)+\beta(p+\gamma-2\alpha)\big)} \sigma_n$$

$$= \sum_{n=p}^{\infty} \sigma_n = 1 - \sigma_{p-1} \leq 1 \,.$$

Hence by Theorem 2, we have $f \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Conversely, Let $f \in \Sigma_p^+(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Since

$$a_n \leq \frac{(p-\gamma)+\beta(p+\gamma-2\alpha)}{2(n+\alpha)\Gamma_n^{\lambda,\mu,\nu,\eta}}\,, \qquad for \quad n \geq p\,.$$

We may take

$$\sigma_n = \frac{2(n+\alpha)\Gamma_n^{\lambda,\mu,\nu,\eta}}{(p-\gamma)+\beta(p+\gamma-2\alpha)}a_n\,, \qquad for \quad n \geq p$$

and $\sigma_{p-1} = 1 - \sum_{n=p}^{\infty} \sigma_n$. Then

$$f(z) = \sigma_{p-1}f_{p-1}(z) + \sum_{n=p}^{\infty} \sigma_n f_n(z).$$

This completes the proof of Theorem 8.

References

[1] M. Darus, Meromorphic functions with positive coefficients, *Int. J. Math. & Math. Sci.*, 6(2004), 319-324.

[2] P.L. Duren, *Univalent Functions, Grundlehren der Mathematischen Wissenschaften (Vol. 259)*, Springer-Verlag, New York, (1983).

[3] S. Owa, On the distortion theorems, *I. Kyung Pook Math. J.*, 18(1978), 53-59.

[4] S. Owa, M. Saigo and H.M. Srivastava, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, *J. Math. Anal. Appl.*, 140(1989), 419-426.

[5] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 49(1975), 109-115.

[6] H.M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus and their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1989).

[7] H.M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, (1992).

[8] H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.*, 31(1988), 412-420.