*Gen. Math. Notes, Vol. 2, No.2, February 2011, pp. 73-82*
*ISSN 2219-7184; Copyright © ICSRS Publication, 2011*
*www.i-csrs.org*
*Available free online at http://www.geman.in*

# On the Cusp Catastrophe Model and Stability

**Mohammad Nokhas Murad**

Department of Mathematics, College of Science,
University of Sulaimani, Iraq
E-mail: muradkakaee@yahoo.com

(Received: 8-11-10 /Accepted: 21-11-10)

**Abstract**

*In this paper, we present results on the projection of the folding part of the cusp catastrophe model on the control space to find stability and catastrophic phenomenon of the periodic solutions of some nonlinear differential equations by using methods of catastrophe theory. We have shown here, that the occurrence of the folding of the cusp surface is always accompanied with the saddle - node bifurcation, and that the saddle - node bifurcation can be classified as cusp type catastrophe.*

**Keywords**: *Cusp catastrophe model, cusp type catastrophe, nonlinear differential equations, saddle-node bifurcation.*

## 1 Introduction

Some characteristics of the phenomena of discontinuous jumping in reality are hard to be explained by equations . The catastrophe theory can explain these characteristics. A cusp catastrophe model is developed to analyze the stability by drawing graphs for a cusp-catastrophe model of nonlinear differential equations, the bifurcation set or the projection of the folding part of the cusp catastrophe model on the control space is always accompanied with the saddle-node bifurcation.. The study of catastrophic problems (equilibrium points, catastrophic manifold (CM), amplitude, jump phenomena, …,etc.) has been of immense importance since long time in view of its growing applications in physical, biological and social sciences. Several authors, for example Arrow Smith et al. (1983), Cesari (1971), Hale (1969), Hartman (1963), Hayashi (1964), Hirsch & Smale (1974), Marsden et al. (1976), Sale (1969), Smith et al. (1977), Zeeman (1977, and references therein) and Muhammad Nokhas Murad (1985) have made their valuable contributions towards studying some aspects (equilibrium points, periodic solutions, limit cycles, stability (or instability), and phenomena associated With forced oscillations) of the problems. To my knowledge, these authors, have, however, not looked into catastrophic problems like saddle- node bifurcation as an catastrophic set, classification and its type and the stability and semi-stability of periodic solutions of NLDE. In the present work, therefore, an effort has been made to study these phenomenon by catastrophic method which might bridge the gap between the above referred works and others in progress both qualitative and quantitative thoughts have been given to the problem so as to present a more clear picture of the physical phenomena. This work may generate a continuous interest to one feel that he has actually available new investigative technique. As well know, there are elementary and non-elementary types of catastrophes; the formers of seven kinds (fold, cusp, swallowtail, hyperbolic, elliptic , butterfly parabolic ) and the latter has got no classification, we have shown here that saddle – node bifurcation of the averaged system arising from the general from of the nonlinear differential equation (NLDE) which is of the following form:

$$\ddot{u} + \omega_0^2 u = \varepsilon f(t, u, \dot{u})$$

(For example, the saddle-node bifurcation of Van Der Pol oscillator can be said as a particular case of this NLDE which corresponds to a cusp catastrophe dynamic system). And more importantly, we find that the occurrence of folding of CM is always accompanied with the saddle- node bifurcation. We further attempt to fit a catastrophic model to our above nonlinear differential, and then show the catastrophic phenomena of saddle- node bifurcation, (such as splitting (or coalescing) and appearance (or disappearance) of singular points, and their projections on the limit cycles). Evidently, the present work grows from calculation involved in the aforesaid NLDE. We note here that in many physical circumstances, like the one as here, we came across nonlinear differential equation for which exact solutions (closed from solutions) are not possible. In such cases, therefore, we alternatively, take resort to some standard numerical methods with prescribed boundary conditions (R-K method, perturbation analysis, …, etc.). But we have applied here a Liapunov direct method to obtain the stability of the periodic solutions of above mentioned NLDE, and discuss its associated physical features, without using any boundary conditions. We divide the main body of this work into four parts: The first part is introductory. In sections 2, 3 and 4, we have determined, respectively the dynamic is given here for the fist time), CM and saddle node bifurcation. Minimum complexity of a chaotic system.

Discrete chaotic systems, such as the logistic map, can exhibit strange attractors whatever their dimensionality. However, the Poincaré-Bendixson theorem shows that a strange attractor can only arise in a continuous dynamical system (specified by differential equations) if it has three or more dimensions. Finite dimensional linear systems are never chaotic; for a dynamical system to display chaotic behavior it has to be either nonlinear, or infinite-dimensional.
The Poincaré–Bendixson theorem states that a two dimensional differential equation has very regular behavior. The Lorenz attractor is generated by a system of three differential equations with a total of seven terms on the right hand side, five of which are linear terms and two of which are quadratic (and therefore nonlinear). Another well-known chaotic attractor is generated by the Rossler equations with seven terms on the right hand side, only one of which is (quadratic) nonlinear. Spot found a three dimensional system with just five terms on the right hand side, and with just one quadratic nonlinearity, which exhibits chaos for certain parameter values. Zhang and Heide showed that, at least for dissipative and conservative quadratic systems, three dimensional quadratic systems with only three or four terms on the right hand side cannot exhibit chaotic behavior. The reason is, simply put, that solutions to such systems are asymptotic to a two dimensional surface and therefore solutions are well behaved. While the Poincaré–Bendixson theorem means that a continuous dynamical system on the Euclidean plane cannot be chaotic, two-dimensional continuous systems with non-Euclidean geometry can exhibit chaotic behavior. Perhaps surprisingly, chaos may occur also in linear systems, provided they are infinite-dimensional A theory of linear chaos is being developed in the functional analysis, a branch of mathematical analysis. Note that elementary catastrophes cannot occur in linear systems.

## 2 Systems Arising from General form of the NLDE

General form of the NLDE: - The general form of the NLDE considered here is

$$\ddot{u}= -w_0^2 u + \varepsilon f\ (t,u,\ \dot{u}),\ (.=d\backslash dt) \qquad (1)$$

Where ε is very small parameter and $f$ is periodic with respect to t with period $\frac{2\pi}{\omega}$. If then we have the linear form equation (1) in which case we are not interested because catastrophic phenomenon appear only in NLDE. For, we proceed to obtain the approximate solution of eq. (1) as follows:

$$\text{Let } \dot{u} = v\ , \qquad (2)$$

and, from eqs. (1) And (2), we have

$$\dot{v} = -w_0^2 u + \varepsilon\ f\ (t,u,\ \dot{u}) \qquad (3)$$

To satisfy eqs. (2) and (3), we further assume that

$$u = a\ (t) \sin (wt) + b\ (t) + b\ (t) \cos (wt)$$

$$v = w\,[a(t)\cos(wt) - b(t)\sin(wt)] \tag{4}$$

where a (t) and b (t) are slowly varying functions of t, and therefore $\ddot{a}$ and $\ddot{b}$ can be neglected. In order that the set of equations (4) should be the solutions of equations (2) and (3) it must satisfy the following conditions.

$$\dot{a}\sin(wt) - \dot{b}\cos(wt) = 0 \tag{5}$$

$$\dot{a}\cos(wt) - \dot{b}\sin(wt) = \frac{\varepsilon}{w}\left[\beta u + f(t,u,\dot{u}\right] \tag{6}$$

$$\varepsilon\beta = w^2 - w_0^2 \tag{7}$$

From the foregoing eqs. (5), (6), and (7) we obtain the non- autonomous system:

$$\begin{aligned} \dot{a} &= \frac{\varepsilon}{W}\left\{\beta u + f(t,u,\dot{u})\right\}\cos(\mathrm{wt}) \\ \dot{b} &= -\frac{\varepsilon}{w}\left\{\beta u + f(t,u,\dot{u})\sin(wt)\right\} \end{aligned} \tag{8}$$

Integration of equation (8) with respect to t, for 0 < t < 2*πlw*, give
The autonomous system:

$$\begin{aligned} \dot{a} &= \frac{\varepsilon}{2\pi}\int_0^{\frac{2\pi}{w}}\left[\beta u + f(t,u,\dot{u}\right]\cos(wt)dt \\ \dot{b} &= -\frac{\varepsilon}{2\pi}\int_0^{\frac{2\pi}{w}}\left[\beta u + f(t,u,\dot{u}\right]\sin(wt)dt \end{aligned} \tag{9}$$

After integration of eq. (9) we have (10), or, in general, we can assume that (with out any loss of generality) expression (10) takes the form.

$$\begin{aligned} \dot{a} &= \beta b + \mu a - \left\{\chi_2 a r^2 + \chi_4 a r^4 + \ldots + \chi_{2n} a r^{2n}\right\} \\ \dot{b} &= -\beta a + \mu b - \left\{\chi_2 b r^2 + \chi_4 b r^4 + \ldots + \chi_{2n} b r^{2n}\right\} - B \end{aligned} \tag{11}$$

Where $\mu, \beta, B$ and $\chi_2, \chi_4, \ldots, \chi_{2n}$ are parameters and $r = \sqrt{a^2 + b^2}$ is the Amplitude. Thus the foregoing eq. (11) is our desired averaged system arising from the general from of the NLDE (1).

## 3 Catastrophic Manifold (CM)

The equilibrium points of the averaged system (11) occur when $\dot{a} = \dot{b} = 0$, hence, equaling to zero the right hand side of the equation

(11) We obtain, after some simplifications.

$$\left[\mu r - (\chi_2 r^3 + \chi_4 r^5 + ... + \chi_{2n} r^{2n+1})\right]^2 + \beta^2 r^2 - B^2 = 0, \qquad (12)$$

Where we have made use of the polar coordinate transformations:

$a = r\cos\varphi, b = r\sin\phi.$ . Putting $\gamma = r^2$ and making suitable change of coordinates, we may reduce eq. (12) to the standard form of some type of elementary catastrophe, that is, we may find some standard form of the eq. (12) for CM as follows.

$$\gamma^m + u_1\gamma^{m-2} = u_2\gamma^{m-3} + ... + u_{m-1} = 0$$

This is our desired form of the eq. (12) where $m = 2n+1.$ . Let us define a function $F'$ such that.

After integrating with respect to γ, we can find the nonlinear dynamic model as follows $\dot{\gamma} = -(\gamma^m + u_1\gamma^{m-2} + u_2\gamma^{m-3} + ... + u_{m-1})$ (13)

And we can find the canonical form for the potential function as follows

$$F(\gamma, u_1, u_2, ...) = \frac{1}{m+1}\gamma^{m+1} + -\frac{u_1}{m-1}\gamma^{m-1} + u_{m-1}\gamma \qquad (14)$$

Arising from the averaged system (12), for which if n=1 then m=3
And F represents the potential function for cusp type catastrophe, which may written as follows

$$F(\gamma, u_1, u_2) = \frac{1}{4}\gamma^4 + \frac{1}{2}u_1\gamma^2 + u_2\gamma \qquad (15)$$

The stationary points of F are given by

$$\frac{\partial F}{\partial \gamma} = \gamma^3 + u_1\gamma + u_2 = 0 \qquad (16)$$

We are considering F and γ also to be function of the control variables, in this case, $u_1, u_2.$ . We consider the nonlinear dynamic model

$$\dot{\gamma} = -(\gamma^3 + u_1\gamma + u_2), \qquad (16a)$$

and investigate the Lipsanos function of this dynamic. Construct a function $F(\gamma, u_1, u_2) = \frac{1}{4}\gamma^4 + \frac{1}{2}u_1\gamma^2 + u_2\gamma$ which is the cusp catastrophe. [12].

It is easily seen that 15 is a Liapunov function with $\frac{dF}{dt} = -(\gamma^3 + u_1\gamma + u_2)^2 < 0 \Leftrightarrow \gamma^2 + u_1\gamma + u_2 \neq 0$

So, the solution of the nonlinear dynamic (16a) is asymptotically stable in this section, we also study the set of equations (2) in their various aspects: existence, stability (or instability), splitting (or coalescing) of limit cycles and other qualitative properties of limit cycles, we shall use a diagrammatic device to show these properties, which is shown in Figure 3. This shows the control space, atypical trajectory on the control space, and asset of limit cycles for typical points on that trajectory. This shows clearly how

There is one possible stable limit cycle outside the critical region and two inside it, and that one of them crashes into the unstable limit cycle to form a semi- stable limit cycle and splitting and bifurcation curve. The surface represented by eq. (16) can be plotted on figure 2. Figure (2) and (3) are sufficiently rich to illustrate the set of phenomenon (splitting (or coalescing), appearance (or disappearance) of limit cycles) to be a catastrophic phenomenon. These phenomenon are related, respectively, to the phenomenon (splitting (or coalescing), appearance (or disappearance) of equilibrium points). The cubic equation (160 can have one or three real roots and then the system (2), other is unstable, since F has two minimum points and one maximum points, and the condition of three limit cycles is

$$4u_1^3 + 27u_2^2 < 0 \qquad (17)$$

The boundary of the region (of one limit cycle or three) is defined by

$$4u_1^3 + 27u_2^2 = 0 \qquad (18)$$

Note that $\partial^2 F l \partial\gamma^2$ is zero on this curve and hence the function F has semi-stable limit cycles in the u, v plane. We have shown in section (4) that the projection of the degene- rate singularities is saddle-node bifurcation for the dynamical system arising from Van Der Pol oscillator which can be said as a particular case of this NLDE which corresponds to a cusp type catastrophe. For example,

If n=1, then we obtain from expression (12)

$$\chi_2^2 r^6 - 2\mu\chi_2 r^4 + (\mu^2 + B^2)r^2 - B^2 = 0 \qquad (19)$$

To obtain standard form of catastrophic manifold of some type

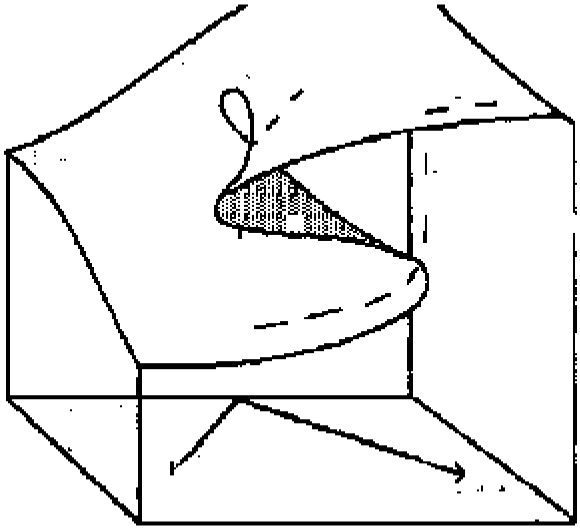
Of elementary catastrophes. We may put $\mu = 4, x_2 = 2, r^2\gamma + 8l3$ Expression (19) becomes

$$\gamma^3 - (16 - 3\beta^2)l3)\gamma + 128l27 + 8l3\beta^2 - B^2 = 0 \qquad (20)$$

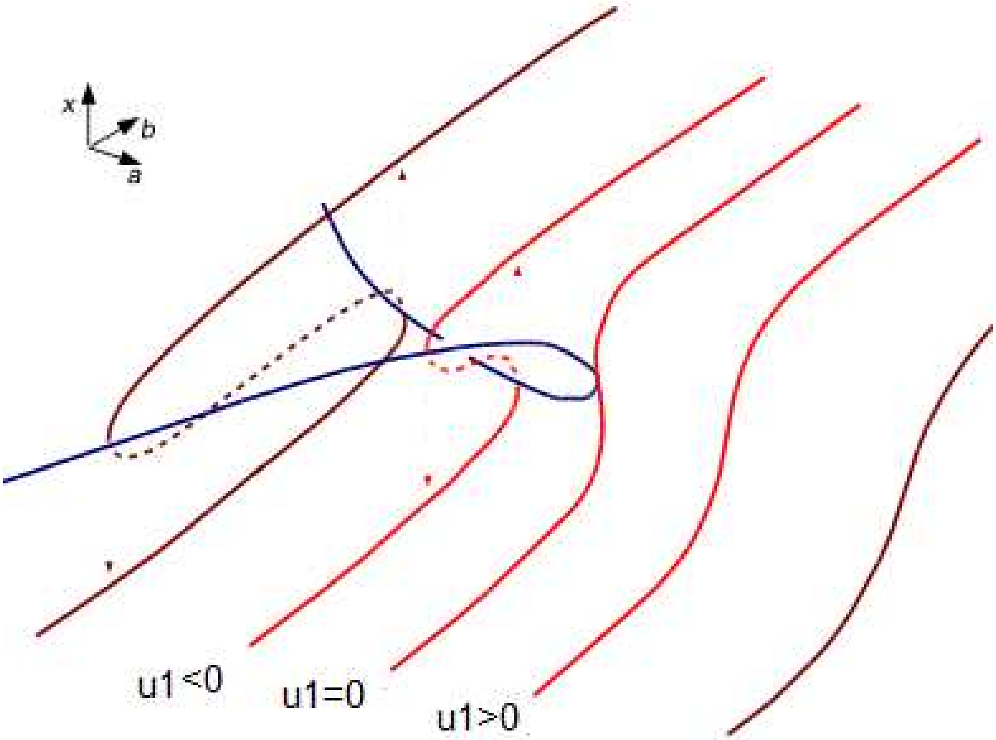
This is our desired form of the expression (12) for RM. Further, the folding curve equation in standard form of cusp catastrophe may be written as

$$4((3\beta^2 - 16)l3)^3 + 27(128l27 + 8l3\beta^2 - B^2)^2 = 0 \quad (21)$$

Which is called the bifurcation set. We further proceed to prove that the forgoing expression (21) is nothing but saddle-node bifurcation represents a catastrophic set (cusp type). Thus we should obtain the expression of saddle- node bifurcation and compare it with expression (21).



**Crose Sections of(CM)**



# 4 Saddle-Node Bifurcaion

For the same values of $\chi_2$ =1and μ= 4, equation (11) becomes

$$\dot{a} = \beta b + 4a - ar^2 = X_1(a,b) \qquad \text{(say)} \qquad (22)$$

$$\dot{b} = -\beta a + 4b - br^2 - B = X_2(a,b) \qquad \text{(say)}$$

Let X (a,b) be the vector field of the system (22) defined by X(a,b) = ($X_1$ (a,b), $X_2$ (a,b) and ($a_0$ , $b_0$)be an equilibrium point of X which represents the periodic solution of NLDE {eq.(1)}. Therefore, X ($a_0$ , $b_0$) Must be equal to zero. Further, we define, we define the linear part of X by

$$DX_{(a0,b0)} = \begin{pmatrix} \partial X_1 l \partial a & \partial X_1 / \partial b \\ \partial X_2 / \partial a & \partial X_2 / \partial b \end{pmatrix} = \qquad (23)$$

$$\begin{pmatrix} 4 - 3a^2 - b^2 B - 2ab & \\ -B - 2ab & 4 - a^2 - 3b^2 \end{pmatrix} = A.$$

The stability of the system can be calculated and analyzed. The analysis performed by eigenvalues methods.
The eigenvalues $\lambda_i$ are calculated for the **A**-matrix, which are the non-trivial solutions of the equation

$$Ax = \lambda x \qquad (23a)$$

where x is an *2x1* vector. Rearranging (23a) to solve for $\lambda$ yields

$$\det(A - \lambda I) = 0 \qquad (23b)$$

The *two* solutions of (23b) are the *eigenvalues* $(\lambda_1, \lambda_2)$ of the *2x2* matrix **A**. These eigenvalues may be real or complex, and are of the form $\alpha \pm \beta i$. If **A** is real, the complex eigenvalues always occur in conjugate pair.

The eigne values of the matrix A are

$$\lambda_1 = 4 - 2r^2 - \sqrt{r^4 - \beta^2} \qquad (24)$$

$$\lambda_2 = 4 - 2r^2 + \sqrt{r^4 - \beta^2}$$

Now, we able to determine some kinds of bifurcation sets under certain different condition: Case 1: when $\lambda_1$ and $\lambda_2$ are imaginary, we may obtain Hopf bifurcation set which represents a non- elementary catastrophe set. Case 2: when the determinant of the linear part [(1)] of vector field is zero (or one of the eigne values is zero), we obtain other kind of bifurcation set as follows: Assume $\lambda_1$ =0, we obtain after some simplifications

$$\gamma = \mp 1l3\sqrt{16 - 3\beta^2} \qquad (25)$$

Where $\gamma = r^2$. Eliminating γ between (20) and (25), we obtain

$$-4\{(16 - 3\beta^2)l3\}^3 + 27\{128l27 + 8l3\beta^2 - B^2\}^2 = 0$$

Note that the Eigen- values are not imaginary (Hartman 1963)

This is the same expression as given in (21) which we wish to prove it to be exactly saddle –node bifurcation set: NOTE: we have proved that $g_{(0,2)}(\beta, B) \neq 0,$ , for proof see [8], where $(\beta, B)$ is any element in bifurcation set. So we have the following proposition. The occurrence of the folding of the RM is always accompanied with the saddle – node bifurcation. If n=2 then m=5 and the response manifold for the averaged system (11) is

$$\gamma^5 + u_1\gamma^3 + ... + u_3\gamma + u_4 = 0$$

Which represents a catastrophic manifold for butterfly. Hence the catastrophic phenomena appears in the averaged system (11) where n=2 is butterfly. And so the projection of the folding part (i.e., the saddle- node bifurcation) represents a butterfly catastrophe. For example take the function f in (1) as follows:

$$f(t, u, u) = \mu u^5 + B\sin(wt) \qquad (27)$$

The averaged system is:

$$\dot{a} = -\frac{\varepsilon}{w^2}(\beta b + 516\,\mu b r^4 + B$$

$$(28)$$

$$\dot{b} = \frac{\varepsilon}{w^2}(\beta a + 516\,\mu a r^4)$$

Let μ=615 and the response manifold (RM) is.

$$\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2 = 0$$

Which is a catastrophic manifold of butterfly. And the nonlinear dynamic system is written as follows

$$\dot{\gamma} = -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2) \qquad (28a)$$

Let $u_1 = 2\beta, u_2 = \beta^2, u_3 = -\beta^2$ and investigate the Liapunov function. Of this dynamic. Construct a function $F(\gamma, u_1, u_2.u_3) = \frac{1}{6}\gamma^6 + \frac{1}{4}u_1\gamma^4 + u_2\gamma^2 + u_3\gamma$

Which is the Butterfly catastrophe. [12]. It is easily seen that $F(\gamma)$ is a Liapunov function with $\dot{\gamma} = -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2)^2 < 0 \Leftrightarrow -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2) \neq 0$

# 5 Conclusion

The solution of the nonlinear dynamic (26a) is asymptotically stable. And there are the following propositions:

**Proposition 4.1** *The saddle- node bifurcation can be classified as cusp type catastrophe).*
**Proposition 4.2** *There exist asymptotically stable solution for any nonlinear dynamical systems arising from NLDE.*
**Proposition 4.3** *The occurrence of the folding of Cusp Catastrophe is always accompanied with saddle- node bifurcation.*

## Acknowledgements

My acknowledgements to Dr. Hemen Dutta for helping me for publishing this paper

## References

[1] K.D. Arrow Smith and K.L. Taha, *Vector Fields Mecanica*, 18(1983), 195-203.
[2] L. Ceseri, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations (3 rd ed.)*, Academic Press New York, (1971).
[3] K.L. Hale, *Ordinary Differential Equations*, John Wiley, (1969).
[4] P. Hartman, A lemma in the theory of structural stability of differential equations, *Proc. A.M.S.*, 14(1963), 568-578.
[5] C. Hayashi, *Nonlinear Oscillations in Physical Systems*, McGraw Hill, New York, (1964).
[6] W. Hirsch and S. Smile, *Differential Equations Dynamical System and Linear Algebra*, Academic Press INC), (1974).
[7] E.J. Marsden and M. McCracken, The Hopf bifurcation and its applications, *Appl. Math. Sci*, 13(1976).
[8] M.N. Mohammad, Treatment of phenomena of instability by method of catastrophe theory, *M. sc. Thesis University of Baghdad*, Baghdad, Iraq, (1985).
[9] S. Small, Am. neurological anatomy in relation to clinical medicine, *Math. Mon.*, (1969) (Jan.), 4-9.
[10] J. Soto Mayor, Generic one-parameter families of vector fields on two-dimensional manifolds, *Publl H.E.S.*, 43(1974), 5-46.
[11] W.D. Jordon and P. Smith, *Nonlinear Ordinary Differential Equation*, Oxford University Press, (1977).
[12] C. Zeeman, *Catastrophe Theory*, Action Wesley, (1977).
[13] X. Zheng, J. Sun and T. Zhong, Study on mechanics of crowd jam based on the cusp-catastrophe model, *University of Chemical Technology*.