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On the Invariants of Mannheim Offsets of Timelike Ruled Surfaces with Timelike Rulings

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Abstract

In this study, we give dual characterizations for Mannheim offsets of timelike ruled surfaces with timelike rulings in terms of their integral invariants. Also, we give new characterization for the Mannheim offsets of developable timelike ruled surfaces. We show that if the offset surfaces are developable then the striction lines of surfaces are Mannheim curves. Moreover, we obtain relationships between areas of projections of spherical images for Mannheim offsets of timelike ruled surfaces and their integral invariants.

Keywords: *Integral invariants, Mannheim offset, Timelike Ruled surface.*

1 Introduction

In the space, a continuously moving of a straight line generates a surface which is called ruled surface. These surfaces are one of the most important topics of differential geometry. Because of their simple generation, these surfaces arise in a variety of applications including Computer Aided Geometric Design (CAGD), mathematical physics, kinematics for modeling the problems and model-based manufacturing of mechanical products. Furthermore, in general, offset surface, the surface which is offset a specified distance from the original along the parent surface's normal, is used in CAGD widely. Some studies dealing with offsets of surfaces have been given in references [4, 18, 19, 20]. Ravani and Ku have defined and given a generalization of theory of Bertrand curves to ruled surfaces and called Bertrand trajectory ruled surfaces on the line geometry [20]. By using the invariants of ruled surfaces given in [5, 6], Küçük and Gürsoy have given the characterizations of Bertrand trajectory ruled surfaces in dual space and have obtained the relations between the invariants [10].

Furthermore, recently, a new definition of curve pairs has been given by Liu and Wang: Let C and C^* be two space curves. C is said to be a Mannheim partner curve of C^* if there exists a one to one correspondence between their points such that the binormal vector of C is the principal normal vector of C^* [11]. Orbay, Kasap and Aydemir have given a generalization of Mannheim curves to ruled surfaces and called Mannheim offsets [12]. Mannheim offsets of timelike ruled surfaces in the Minkowski 3-space IR_1^3 have been studied in [14].

In this paper, we examine the Mannheim offsets of trajectory timelike ruled surfaces with timelike rulings in view of their integral invariants. We give a result obtained in [14] in short form. Furthermore, using the dual representations of timelike ruled surfaces, we obtain some new results which are not obtained in [14]. Moreover, we obtain that the striction lines of Mannheim offsets of developable timelike trajectory ruled surfaces are Mannheim partner curves in the Minkowski 3-space IR_1^3. Furthermore, we give relations between the integral invariants (such as the angle of pitch and the pitch) of closed timelike trajectory ruled surfaces. Finally, we obtain the relationships between the areas of projections of spherical images of Mannheim offsets of timelike trajectory ruled surfaces and their integral invariants.

2 Differential Geometry of the Ruled Surfaces in the Minkowski 3-Space

Let IR_1^3 be a 3-dimensional Minkowski space over the field of real numbers IR with the Lorentzian inner product $\langle \, , \rangle$ given by

$$\langle \vec{a},\vec{b}\rangle = -a_1b_1 + a_2b_2 + a_3b_3$$

where $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in IR_1^3$. A vector $\vec{a}=(a_1,a_2,a_3)$ of IR_1^3 is said to be timelike if $\langle \vec{a},\vec{a}\rangle<0$, spacelike if $\langle \vec{a},\vec{a}\rangle>0$ or $\vec{a}=0$, and lightlike (null) if $\langle \vec{a},\vec{a}\rangle=0$ and $\vec{a}\neq 0$. Similarly, an arbitrary curve $\alpha(s)$ in IR_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively [13]. The norm of a vector $\vec{a}$ is defined by $\|\vec{a}\|=\sqrt{|\langle \vec{a},\vec{a}\rangle|}$. A vector $\vec{a}\in IR_1^3$ is called a unit vector if $\|\vec{a}\|=\sqrt{|\langle \vec{a},\vec{a}\rangle|}=1$ and the sets of the unit timelike and spacelike vectors are called hyperbolic unit sphere and Lorentzian unit sphere, respectively, and denoted by

$$H_0^2=\{\vec{a}=(a_1,a_2,a_3)\in IR_1^3 : \langle \vec{a},\vec{a}\rangle=-1\}$$

and

$$S_1^2=\{\vec{a}=(a_1,a_2,a_3)\in IR_1^3 : \langle \vec{a},\vec{a}\rangle=1\}$$

respectively [22].

A surface in the Minkowski 3-space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [3].

Let I be an open interval in the real line IR, $\vec{k}=\vec{k}(s)$ be a curve in IR_1^3 defined on I and $\vec{q}=\vec{q}(s)$ be a unit direction vector of an oriented line in IR_1^3. Then we have the following parametrization for a ruled surface N

$$\varphi(s,v)=\vec{k}(s)+v\,\vec{q}(s) \tag{1}$$

where $\vec{q}(s)$ is called ruling and $\vec{k}=\vec{k}(s)$ is called base curve or generating curve of the surface. In particular, if the direction of $\vec{q}$ is constant, the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

The striction point on a ruled surface N is the foot of the common normal between two consecutive rulings. The set of the striction points constitute a curve $\vec{c}=\vec{c}(s)$ lying on the ruled surface and is called striction curve. The parametrization of the striction curve $\vec{c}=\vec{c}(s)$ on a ruled surface is given by

$$\vec{c}(s) = \vec{k}(s) - \frac{\langle d\vec{q},\, d\vec{k} \rangle}{\langle d\vec{q},\, d\vec{q} \rangle} \vec{q} \tag{2}$$

So that, the base curve of the ruled surface is its striction curve if and only if $\langle d\vec{q},\, d\vec{k} \rangle = 0$.

The distribution parameter (or drall) of the ruled surface in (1) is given as

$$\delta_\varphi = \frac{\left| d\vec{k},\, \vec{q},\, d\vec{q} \right|}{\langle d\vec{q},\, d\vec{q} \rangle} \tag{3}$$

and a ruled surface is developable if and only if at all its points the distribution parameter is $\delta_\varphi = 0$ [1, 9].

For the unit normal vector $\vec{m}$ of the ruled surface N we have $\vec{m} = \dfrac{\vec{\varphi}_s \times \vec{\varphi}_v}{\|\vec{\varphi}_s \times \vec{\varphi}_v\|}$. So,

at the points of a nontorsal ruling $s = s_1$ we have $\vec{a} = \lim\limits_{v \to \infty} \vec{m}(s_1, v) = \dfrac{(d\vec{q}/ds) \times \vec{q}}{\|d\vec{q}/ds\|}$.

The plane of ruled surface N which passes through its ruling s_1 and is perpendicular to the vector $\vec{a}$ is called the *asymptotic plane* α. The tangent plane γ passing through the ruling s_1 which is perpendicular to the asymptotic plane α is called the *central plane*. Its point of contact C is *central point* of the ruling. The straight lines which pass through point C and are perpendicular to the planes α and γ are called the *central tangent* and *central normal*, respectively.

Since the vectors $\vec{q},\, d\vec{q}/ds$ are perpendicular to the vector $\vec{a}$, representation of the unit vector $\vec{h}$ of central normal is given by $\vec{h} = \dfrac{d\vec{q}/ds}{\|d\vec{q}/ds\|}$. The orthonormal system $\{C; \vec{q}, \vec{h}, \vec{a}\}$ is called Frenet frame of the ruled surfaces N such that $\vec{h}$ and $\vec{a} = \vec{h} \times \vec{q}$ are the central normal and the central tangent of N, respectively, and C is the striction point.

Let now consider the ruled surface N. According to the Lorentzian characters of ruling and central normal, we can give the following classifications for the ruled surface N:

i) If the central normal vector $\vec{h}$ is spacelike and the ruling $\vec{q}$ is timelike, then the ruled surface N is said to be of type N_-^1.

ii) If both the central normal vector $\vec{h}$ and the ruling $\vec{q}$ are spacelike, then the ruled surface N is said to be of type N_+^1.

iii) If the central normal vector $\vec{h}$ is timelike and the ruling $\vec{q}$ is spacelike, then the ruled surface N is said to be of type N_+^2 [16, 23].

The ruled surfaces of types N_-^1 and N_+^1 are clearly timelike and the ruled surface of type N_+^2 is spacelike [8].

By using these classifications, the parametrization of N can be given as follows,

$$\varphi(s,v) = \vec{k}(s) + v\,\vec{q}(s) \tag{4}$$

where

$$\left\langle \vec{h},\vec{h} \right\rangle = \varepsilon_1\,(=\pm 1), \ \ \left\langle \vec{q},\vec{q} \right\rangle = \varepsilon_2\,(=\pm 1) \tag{5}$$

The set of all bound vectors $\vec{q}(s)$ at the origin 0 constitutes the *directing cone* of the ruled surface N. If $\varepsilon_2 = -1$ (resp. $\varepsilon_2 = 1$), the end points of the vectors $\vec{q}(s)$ drive a spherical spacelike (resp. spacelike or timelike) curve k_1 on hyperbolic unit sphere H_0^2 (resp. on Lorentzian unit sphere S_1^2), called the *hyperbolic (resp. Lorentzian) spherical image* of the ruled surface N [16,23].

Let $\left\{\vec{q},\vec{h},\vec{a}\right\}$ be a moving orthonormal trihedron making a spatial motion along a closed space curve $\vec{k}(s)$ in IR_1^3 where $s \in IR$ and $\vec{h}$ is assumed spacelike. In this motion, the oriented line $\vec{q}$ generates a closed timelike ruled surface called closed timelike trajectory ruled surface (CTTRS). A parametric equation of a closed trajectory timelike ruled surface generated by $\vec{q}$-axis is

$$\varphi_q(s,v) = \vec{k}(s) + v\,\vec{q}(s), \quad \varphi(s+2\pi,v) = \varphi(s,v), \ \ s,v \in IR \tag{6}$$

Consider the moving orthonormal system $\left\{\vec{q},\vec{h},\vec{a}\right\}$ which represents a timelike ruled surface of the type N_+^1 or N_-^1 generated by the vector $\vec{q}$. Then, the axes of trihedron intersect at striction point of $\vec{q}$-generator of φ_q-CTTRS. The structural equations of this motion are

$$d\vec{q} = k_1\vec{h}, \quad d\vec{h} = -\varepsilon_2 k_1\vec{q} + k_2\vec{a}, \quad d\vec{a} = \varepsilon_2 k_2\vec{h} \tag{7}$$

And

$$\frac{d\vec{b}}{ds} = \mu(\sigma)\vec{q} + \vartheta(\sigma)\vec{a} \tag{8}$$

where $\vec{b} = \vec{b}(s)$ is the striction line of φ_q -CTTRS and $\mu(\sigma) = \cosh\sigma$, $\vartheta(\sigma) = \sinh\sigma$, if both $\vec{q}$ and $\vec{b}(s)$ are timelike; and $\mu(\sigma) = \sinh\sigma$, $\vartheta(\sigma) = \cosh\sigma$, if both $\vec{q}$ and $\vec{b}(s)$ are spacelike. The differential forms k_1, k_2 and σ are the natural curvature, the natural torsion and the striction of φ_q -CTTRS, respectively. Here, s is the arclength of the striction line [16].

The pole vector and the Steiner vector of the motion are given by

$$\vec{p} = \frac{\vec{\psi}}{\|\vec{\psi}\|}, \quad \vec{d} = \oint \vec{\psi} \tag{9}$$

respectively, where $\vec{\psi} = \varepsilon_2 k_2 \vec{q} - k_1 \vec{a}$ is the instantaneous Pfaffian vector of the motion. The real invariants of φ_q -CTTRS are defined by

$$\ell_q = \oint d\mu = -\varepsilon_2 \oint \left\langle d\vec{k}, \vec{q} \right\rangle \tag{10}$$

which is the pitch, and

$$\lambda_q = \oint d\theta = \varepsilon_2 \oint \left\langle d\vec{h}, \vec{a} \right\rangle = \varepsilon_2 \left\langle \vec{q}, \vec{d} \right\rangle = 2\pi - a_q \tag{11}$$

which is the angle of pitch of φ_q -CTTRS, respectively, where a_q is the measure of the spherical surface area bounded by the spherical image of φ_q -CTTRS. The pitch and the angle of pitch are well-known real integral invariants of closed timelike trajectory ruled surfaces [2, 16].

The area vector of a x -closed space curve in IR_1^3 is given by

$$\vec{v}_x = \oint \vec{x} \times d\vec{x} \tag{12}$$

and the area of projection of a x -closed space curve in direction of the generator of a y -CTRS is

$$2 f_{x,y} = \left\langle \vec{v}_x, \vec{y} \right\rangle \tag{13}$$

(See [25]).

3 Dual Lorentzian Vectors and E. Study Mapping

In this section, we give a brief summary of the theory of dual numbers and dual Lorentzian vectors.

A dual number has the form $\bar{\lambda} = \lambda + \varepsilon\lambda^*$, where λ and λ^* are real numbers and ε stands for dual unit which is subject to the rules $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$. We denote set of all dual numbers by D:

$$D = \left\{ \bar{\lambda} = \lambda + \varepsilon\lambda^* : \ \lambda, \lambda^* \in IR, \ \varepsilon^2 = 0 \right\}$$

Equality, addition and multiplication are defined in D by

$$\lambda + \varepsilon\lambda^* = \beta + \varepsilon\beta^* \quad \text{if and only if } \lambda = \beta \text{ and } \lambda^* = \beta^*,$$

$$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$$

and

$$(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = \lambda\beta + \varepsilon(\lambda\beta^* + \lambda^*\beta),$$

respectively. Then it is easy to show that $(D, +, \cdot)$ is a commutative ring with unity. The numbers $\varepsilon\lambda^*$ $(\lambda \in IR)$ are divisors of zero [7].

Now let f be a differentiable function with dual variable $\bar{x} = x + \varepsilon x^*$. Then the Maclaurin series generated by f is

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x),$$

where $f'(x)$ is the derivative of f with respect to x [7]. Let D^3 be the set of all triples of dual numbers, i.e. $D^3 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \,\middle|\, \bar{a}_i \in D, \ 1 \le i \le 3 \right\}$. The elements of D^3 are called dual vectors. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$, where $\vec{a}$ and $\vec{a}^*$ are the vectors of IR^3.

Now let $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$, $\tilde{b} = \vec{b} + \varepsilon\vec{b}^* \in D^3$ and $\bar{\lambda} = \lambda + \varepsilon\lambda^* \in D$. Then we define

$$\tilde{a} + \tilde{b} = (\vec{a} + \vec{b}) + \varepsilon(\vec{a}^* + \vec{b}^*), \quad \bar{\lambda}\tilde{a} = \lambda\vec{a} + \varepsilon(\lambda\vec{a}^* + \lambda^*\vec{a}).$$

Then, D^3 becomes a unitary module with these operations. It is called D-module or dual space [7].

The Lorentzian inner product of two dual vectors $\tilde{a}=\vec{a}+\varepsilon\vec{a}^{*}$, $\tilde{b}=\vec{b}+\varepsilon\vec{b}^{*}\in D^{3}$ is defined by

$$\left\langle \tilde{a},\tilde{b}\right\rangle=\left\langle \vec{a},\vec{b}\right\rangle+\varepsilon\left(\left\langle \vec{a},\vec{b}^{*}\right\rangle+\left\langle \vec{a}^{*},\vec{b}\right\rangle\right),$$

where $\left\langle \vec{a},\vec{b}\right\rangle$ is the Lorentzian inner product of the vectors $\vec{a}$ and $\vec{b}$ in the Minkowski 3-space IR_1^3. Then a dual vector $\tilde{a}=\vec{a}+\varepsilon\vec{a}^{*}$ is said to be timelike if $\vec{a}$ is timelike, spacelike if $\vec{a}$ is spacelike or $\vec{a}=0$ and lightlike (null) if $\vec{a}$ is lightlike (null) and $\vec{a}\neq 0$ [21, 24].

The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by $D_1^3=\left\{\ \tilde{a}=\vec{a}+\varepsilon\vec{a}^{*}:\ \vec{a},\vec{a}^{*}\in IR_1^3\ \right\}$.

The Lorentzian cross product of dual vectors $\tilde{a},\tilde{b}\in D_1^3$ is defined by

$$\tilde{a}\times\tilde{b}=\vec{a}\times\vec{b}+\varepsilon(\vec{a}^{*}\times\vec{b}+\vec{a}\times\vec{b}^{*}),$$

where $\vec{a}\times\vec{b}$ is the Lorentzian cross product in IR_1^3 .

Let $\tilde{a}=\vec{a}+\varepsilon\vec{a}^{*}\in D_1^3$. Then $\tilde{a}$ is said to be dual unit timelike (resp. spacelike) vector if the vectors $\vec{a}$ and $\vec{a}^{*}$ satisfy the following equations:

$$<\vec{a},\vec{a}>=-1\ \ (\textit{resp.}\ <\vec{a},\vec{a}>=1),\ \ <\vec{a},\vec{a}^{*}>=0.$$

The set of all dual unit timelike vectors is called the dual hyperbolic unit sphere, and is denoted by $\tilde{H}_0^2$. Similarly, the set of all dual unit spacelike vectors is called the dual Lorentzian unit sphere, and is denoted by $\tilde{S}_1^2$ [21, 24].

Theorem 3.1 (E. Study Mapping) ([21]): *The dual timelike (respectively spacelike) unit vectors of the dual hyperbolic (respectively Lorentzian) unit sphere $\tilde{H}_0^2$ (respectively $\tilde{S}_1^2$) are in on-to-one correspondence with the directed timelike (respectively spacelike) lines of the Minkowski 3-space IR_1^3.*

Definition 3.1 ([21, 24]):

i) Dual Hyperbolic Angle: *Let $\tilde{x}$ and $\tilde{y}$ be dual timelike vectors in D_1^3. Then the dual angle between $\tilde{x}$ and $\tilde{y}$ is defined by $<\tilde{x},\tilde{y}>=-\|\tilde{x}\|\|\tilde{y}\|\cosh\bar{\theta}$. The dual number $\bar{\theta}=\theta+\varepsilon\theta^{*}$ is called the dual hyperbolic angle.*

ii) Dual Lorentzian Timelike Angle: *Let $\tilde{x}$ be a dual spacelike vector and $\tilde{y}$ be a dual timelike vector in D_1^3. Then the angle between $\tilde{x}$ and $\tilde{y}$ is defined by $\left|<\tilde{x},\tilde{y}>\right| = \|\tilde{x}\|\|\tilde{y}\|\sinh\overline{\theta}$. The dual number $\overline{\theta} = \theta + \varepsilon\theta^*$ is called the dual Lorentzian timelike angle.*

Let now $\tilde{K}$ be a moving dual unit hyperbolic or Lorentzian sphere generated by a dual orthonormal system

$$\left\{\tilde{q},\ \tilde{h} = \frac{d\tilde{q}}{\|d\tilde{q}\|}(spacelike),\ \tilde{a} = \tilde{q}\times\tilde{h}\right\},\ \tilde{q} = \vec{q} + \varepsilon\vec{q}^*, \tilde{h} = \vec{h} + \varepsilon\vec{h}^*,\ \tilde{a} = \vec{a} + \varepsilon\vec{a}^* \quad (14)$$

and $\tilde{K}'$ be a fixed dual unit sphere with the same center. Then, the derivative equations of the dual spherical closed motion of $\tilde{K}$ with respect to $\tilde{K}'$ are

$$d\tilde{q} = \overline{k}_1\tilde{h}, \quad d\tilde{h} = -\varepsilon_2\overline{k}_1\tilde{q} + \overline{k}_2\tilde{a}, \quad d\tilde{a} = \varepsilon_2\overline{k}_2\tilde{h} \quad (15)$$

where $\overline{k}_1(s) = k_1(s) + \varepsilon k_1^*(s),\ \overline{k}_2(s) = k_2(s) + \varepsilon k_2^*(s),\ (s \in IR)$ are dual curvature and dual torsion, respectively. From the E. Study mapping, during the spherical motion of $\tilde{K}$ with respect to $\tilde{K}'$, the dual unit timelike (resp. spacelike) vector $\tilde{q}$ draws a dual curve on dual unit hyperbolic (resp. Lorentzian) sphere K' and this curve represents a timelike ruled surface with timelike (resp. spacelike) ruling $\vec{q}$ in line space IR_1^3 [15].

Dual vector $\tilde{\psi} = \vec{\psi} + \varepsilon\vec{\psi}^* = \varepsilon_2\overline{k}_2\tilde{q} - \overline{k}_1\tilde{a}$ is called the instantaneous Pfaffian vector of the motion and the vector $\tilde{P}$ given by $\tilde{\psi} = \|\tilde{\psi}\|\tilde{P}$ is called the dual pole vector of the motion. Then the vector

$$\tilde{d} = \oint\tilde{\psi} \quad (16)$$

is the dual Steiner vector of the closed motion.

By considering the E. Study mapping, the dual equations (15) correspond to real equations (7) and (8) of a closed spatial motion in IR_1^3. So, the differentiable dual closed curve $\tilde{q} = \tilde{q}(s)$ corresponds to a closed trajectory timelike ruled surface in the line space IR_1^3 and denoted by φ_q-CTTRS.

A dual integral invariant of a φ_q-CTTRS can be given in terms of real integral invariants as follows and is called the dual angle of pitch of a φ_q-CTTRS

$$\overline{\lambda}_q = -\varepsilon_2\oint\left\langle d\tilde{h},\tilde{a}\right\rangle = -\left\langle\tilde{q},\tilde{d}\right\rangle = 2\pi - \overline{a}_q = \lambda_q + \varepsilon_2\varepsilon\ell_q \quad (17)$$

where $\tilde{d} = \vec{d} + \varepsilon\vec{d}^*$ and $\overline{a}_q = a_q + \varepsilon a_q^*$ are the dual Steiner vector of the motion and the measure of dual spherical surface area of φ_q-CTTRS, respectively. Here, $\varepsilon_2 = \langle \vec{q}, \vec{q} \rangle$ and ε is dual unit [17].

Analogue to the real area vector given in (12), the dual area vector of a $\tilde{q}$-closed dual curve is given by

$$\tilde{w}_{\tilde{q}} = \oint \tilde{q} \times d\tilde{q} \tag{18}$$

and from ref. [10], the dual area of projection of a $\tilde{q}$-closed dual curve in direction of the generator of a $\tilde{q}_1$-CTRS is

$$2\overline{f}_{\tilde{q},\tilde{q}_1} = \langle \tilde{w}_{\tilde{q}}, \tilde{q}_1 \rangle \tag{19}$$

4 Mannheim Offsets of Timelike Trajectory Ruled Surfaces with Timelike Rulings

Let φ_q be a timelike trajectory ruled surface of the type N_+^1 or N_-^1 generated by dual unit vector $\tilde{q}$ and let dual orthonormal frame of φ_q be $\{\tilde{q}(s), \tilde{h}(s), \tilde{a}(s)\}$. The trajectory ruled surface φ_{q_1}, generated by dual unit vector $\tilde{q}_1$ with dual orthonormal frame $\{\tilde{q}_1(s_1), \tilde{h}_1(s_1), \tilde{a}_1(s_1)\}$ is said to be Mannheim offset of the timelike trajectory ruled surface φ_q, if

$$\tilde{a}(s) = \tilde{h}_1(s_1) \tag{20}$$

holds, where s and s_1 are arc-lengths of striction lines of φ_q and φ_{q_1}, respectively. By this definition and considering the classifications of the timelike ruled surfaces we have the following cases:

Case 1: If the trajectory ruled surfaces φ_q is of the type N_-^1, then by considering (20), the Mannheim offset φ_{q_1} of φ_q is a timelike trajectory ruled surface of the type N_-^1 or N_+^1. If φ_q is of the type N_-^1 and φ_{q_1} is of the type N_+^1, then we have

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{h}_1 \\ \tilde{a}_1 \end{pmatrix} = \begin{pmatrix} \sinh\overline{\theta} & \cosh\overline{\theta} & 0 \\ 0 & 0 & 1 \\ \cosh\overline{\theta} & \sinh\overline{\theta} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{pmatrix} \tag{21}$$

Case 2: Similarly, if both φ_q and φ_{q_1} are of the type N_-^1, we have

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{h}_1 \\ \tilde{a}_1 \end{pmatrix} = \begin{pmatrix} \cosh\overline{\theta} & \sinh\overline{\theta} & 0 \\ 0 & 0 & 1 \\ \sinh\overline{\theta} & \cosh\overline{\theta} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{pmatrix} \tag{22}$$

In (21) and (22), $\overline{\theta} = \theta + \varepsilon\theta^*$, $(\theta, \theta^* \in IR)$ is dual angle between the generators $\tilde{q}$ and $\tilde{q}_1$ of Mannheim trajectory ruled surfaces φ_q and φ_{q_1}. The angle θ is called offset angle and real number θ^* is called offset distance. Then, $\overline{\theta} = \theta + \varepsilon\theta^*$ is called dual offset angle of the Mannheim trajectory ruled surfaces φ_q and φ_{q_1}. Thus, we can give the followings.

5 N_+^1 -Type Mannheim Offsets of Timelike Ruled Surfaces with Timelike Rulings

Let the timelike trajectory ruled surface φ_{q_1} of the type N_+^1 be a Mannheim offset of timelike trajectory ruled surface φ_q with timelike rulings. Then we can give the followings.

Theorem 5.1: *Let trajectory ruled surface φ_{q_1} of the type N_+^1 be a Mannheim offset of timelike trajectory ruled surface φ_q with timelike rulings. Then offset angle and offset distance are given by*

$$\theta = -\int k_1 ds, \quad \theta^* = -\int k_1^* ds \tag{23}$$

respectively, where $\overline{k}_1 = k_1 + \varepsilon k_1^*$ is the dual curvature of φ_q .

Proof: Let φ_q and φ_{q_1} form a Mannheim offset where φ_q and φ_{q_1} are of the type N_-^1 and N_+^1, respectively. From (21) we have

$$\tilde{q}_1 = \sinh\overline{\theta}\,\tilde{q} + \cosh\overline{\theta}\,\tilde{h} \tag{24}$$

Differentiating (24) and by using (15) and (20), it follows

$$\frac{d\tilde{q}_1}{ds} = \left(\frac{d\overline{\theta}}{ds} + \overline{k}_1\right)\tilde{a}_1 + \overline{k}_2 \cosh\overline{\theta}\,\tilde{h}_1 \tag{25}$$

Since $\dfrac{d\tilde{q}_1}{ds}$ is orthogonal to $\tilde{a}_1$, from (25) we get

$$\overline{\theta} = -\int \overline{k}_1 ds \tag{26}$$

Separating the last equation into real and dual parts we have

$$\theta = -\int k_1 ds, \quad \theta^* = -\int k_1^* ds \tag{27}$$

Theorem 5.2: *The closed timelike trajectory ruled surface φ_q with timelike rulings and the closed timelike trajectory ruled surface φ_{q_1} of the type N_+^1 form a Mannheim offset with a constant dual offset angle if and only if the following relationship holds*

$$\overline{\curlywedge}_{q_1} = \overline{\curlywedge}_q \sinh\overline{\theta} + \overline{\curlywedge}_h \cosh\overline{\theta} \tag{28}$$

Proof: Let the closed timelike trajectory ruled surfaces φ_q and φ_{q_1} (of the type N_+^1) form a Mannheim offset with a constant dual offset angle $\overline{\theta}$. Then, from (17) and (21), the dual angle of pitch of φ_{q_1} -CTTRS is given by

$$\begin{aligned}
\overline{\curlywedge}_{q_1} &= \oint \left\langle d\tilde{h}_1, \tilde{a}_1 \right\rangle \\
&= \oint \left\langle d\tilde{a},\ (\cosh\overline{\theta})\tilde{q} + (\sinh\overline{\theta})\tilde{h} \right\rangle \\
&= \oint \left\langle d\tilde{a}, \tilde{q} \right\rangle \cosh\overline{\theta} + \oint \left\langle d\tilde{a}, \tilde{h} \right\rangle \sinh\overline{\theta} \\
&= \oint \left\langle d\tilde{a}, \tilde{q} \right\rangle \cosh\overline{\theta} - \oint \left\langle d\tilde{h}, \tilde{a} \right\rangle \sinh\overline{\theta} \\
&= \overline{\curlywedge}_q \sinh\overline{\theta} + \overline{\curlywedge}_h \cosh\overline{\theta}
\end{aligned}$$

Conversely, if (28) holds, it is easily seen that φ_q and φ_{q_1} -CTTRS form a Mannheim offsets with a constant dual offset angle.

Equality (28) is a dual characterization of Mannheim offsets of CTTRS with a constant dual offset angle in terms of their dual integral invariants. Separating (28) into real and dual parts, we obtain

$$\begin{cases}
\lambda_{q_1} = \lambda_q \sinh\theta + \lambda_h \cosh\theta \\
\ell_{q_1} = (-\ell_q + \theta^* \lambda_h)\sinh\theta + (\ell_h + \theta^* \lambda_q)\cosh\theta
\end{cases} \tag{29}$$

Then, we may give following result:

Result 5.1: If $\theta^* = 0$, i.e., the generators $\vec{q}$ and $\vec{q}_1$ of the Mannheim offset surfaces intersect, then we have

$$\begin{cases} \lambda_{q_1} = \lambda_q \sinh\theta + \lambda_h \cosh\theta, \\ \ell_{q_1} = -\ell_q \sinh\theta + \ell_h \cosh\theta. \end{cases} \tag{30}$$

In this case, φ_q and φ_{q_1}-CTRS are intersect along their striction lines. It means that their striction lines are the same.

Let now consider that what the condition for the developable Mannheim offset of a CTTRS is. Let φ_q and φ_{q_1}-CTTRS be the Mannheim offset surfaces and let $\vec{\alpha}(s)$ and $\vec{\beta}(s_1)$ be striction lines of φ_q and φ_{q_1}-CTRS, respectively, where φ_q is of the type N_-^1 and φ_{q_1} is of the type N_+^1. Then, we can write

$$\vec{\beta}(s) = \vec{\alpha}(s) + \theta^* \vec{a}(s) \tag{31}$$

where s is the arc-length of $\vec{\alpha}(s)$. Assume that φ_q-CTTRS is developable. Then from (3) and (8) we have

$$\delta_q = \frac{\left\langle \cosh\sigma\vec{q} + \sinh\sigma\vec{a},\ \vec{q}\times k_1\vec{h} \right\rangle}{\left\langle k_1\vec{h}, k_1\vec{h} \right\rangle} = -\frac{\sinh\sigma}{k_1} = 0 \tag{32}$$

which gives that $\sigma = 0$. Thus, from (8) we get

$$\frac{d\vec{\alpha}}{ds} = \vec{q} \tag{33}$$

Hence, along the striction line $\vec{\alpha}(s)$, orthogonal frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ coincides with the Frenet frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ and differential forms k_1 and k_2 turn into curvature κ_α and torsion τ_α of the striction line $\vec{\alpha}(s)$, respectively. Then, by the aid of (8), (31) and (33) we have

$$\frac{d\vec{\beta}}{ds} = \vec{q} - \theta^* \tau_\alpha \vec{h} \tag{34}$$

On the other hand from (20) and (21) we obtain

$$\frac{d\vec{q}_1}{ds} = \left(\frac{d\theta}{ds} + \kappa_\alpha\right)\cosh\theta\vec{q} + \left(\frac{d\theta}{ds} + \kappa_\alpha\right)\sinh\theta\vec{h} + \tau_\alpha \cosh\theta\vec{a} \tag{35}$$

By using (24) and the fact that $k_1 = \kappa_\alpha$, from (35) we have

$$\frac{d\vec{q}_1}{ds} = \tau_\alpha \cosh\theta \vec{a} \tag{36}$$

From (34) and (36) we obtain

$$\delta_{q_1} = \frac{\langle d\vec{\beta},\, \vec{q}_1 \times d\vec{q}_1 \rangle}{\langle d\vec{q}_1, d\vec{q}_1 \rangle} = -\frac{\cosh\theta + \theta^* \tau_\alpha \sinh\theta}{\tau_\alpha \cosh\theta} \tag{37}$$

Thus, from (33) and (37), it can be stated that if the Mannheim offsets φ_q and φ_{q_1} are developable then the following relationship holds

$$\cosh\theta + \theta^* \tau_\alpha \sinh\theta = 0 \tag{38}$$

In [14], equality (38) is also obtained with a different way. If (38) holds, along the striction line $\beta(s_1)$, orthogonal frame $\{\vec{q}_1, \vec{h}_1, \vec{a}_1\}$ coincides with the Frenet frame $\{\vec{T}_1, \vec{N}_1, \vec{B}_1\}$ of $\beta(s_1)$. Thus, the following theorem may be given:

Theorem 5.3: *Let the trajectory timelike ruled surfaces φ_q and φ_{q_1} be of the type N_-^1 and N_+^1, respectively. If φ_q and φ_{q_1} are developable Mannheim offsets, then their striction lines are Mannheim partner curves in the Minkowski 3-space.*

In this case from (38) we have the following corollary:

Corollary 5.1: *If φ_q and φ_{q_1} form a developable Mannheim offset, then the relationship between the torsion of $\alpha(s)$ and offset distance and offset angle is given by $\tau_\alpha \theta^* = -\coth\theta$.*

If φ_q-CTTRS is developable then from the equations (10), (21) and (34) the pitch ℓ_{q_1} of φ_{q_1}-CTTRS is

$$\begin{aligned} \ell_{q_1} &= -\oint \langle d\vec{\beta}, \vec{q}_1 \rangle \\ &= -\oint \langle \vec{q} - \theta^* \tau_\alpha \vec{h},\ (\sinh\theta)\vec{q} + (\cosh\theta)\vec{h} \rangle ds \\ &= \oint (\sinh\theta + \theta^* \tau_\alpha \cosh\theta) ds \end{aligned}$$

Then we can give the following corollary.

Corollary 5.2: *If φ_q-CTTRS is developable then the relation between the pitch ℓ_{q_1} of φ_{q_1}-CTTRS and the torsion of sitriction line $\alpha(s)$ of φ_q-CTTRS is given by*

$$\ell_{q_1} = \oint (\sinh\theta + \theta^* \tau_\alpha \cosh\theta) ds \tag{39}$$

Let now consider the area of projections of Mannheim offsets. From (18), the dual area vectors of the spherical images of φ_q and φ_{q_1} Mannheim offsets are

$$\begin{cases} \tilde{w}_q = \tilde{d} + \barwedge_q \tilde{q}, \\ \tilde{w}_{q_1} = -\tilde{d} - \barwedge_{q_1} \tilde{q}_1, \end{cases} \tag{40}$$

respectively. Then, from (19), dual area of projection of spherical image of φ_{q_1}-CTTRS in the direction $\vec{q}$ is

$$\begin{aligned} 2\bar{f}_{\tilde{q}_1,\tilde{q}} &= \left\langle \tilde{w}_{q_1}, \tilde{q} \right\rangle = -\left\langle \tilde{d} + \barwedge_{q_1} \tilde{q}_1,\ \tilde{q} \right\rangle \\ &= -\left\langle \tilde{d}, \tilde{q} \right\rangle - \barwedge_{q_1} \left\langle (\sinh\bar{\theta})\tilde{q} + (\cosh\bar{\theta})\tilde{h},\ \tilde{q} \right\rangle \\ &= -\left\langle \tilde{d}, \tilde{q} \right\rangle - \barwedge_{q_1} \sinh\bar{\theta} \\ 2\bar{f}_{\tilde{q}_1,\tilde{q}} &= \barwedge_q - \barwedge_{q_1} \sinh\bar{\theta} \end{aligned} \tag{41}$$

Separating (41) into real and dual parts we have the following theorem.

Theorem 5.4: *Let the trajectory timelike ruled surfaces φ_q and φ_{q_1} be of the type N_-^1 and N_+^1, respectively and let they form a Mannheim offset. The relationships between the area of projections of spherical images of the timelike Mannheim offsets φ_q and φ_{q_1} and their integral invariants are given as follows*

$$2 f_{q_1,q} = \lambda_q - \lambda_{q_1} \sinh\theta, \quad 2 f^*_{q_1,q} = -(\ell_q + \ell_{q_1} \sinh\theta + \lambda_{q_1} \theta^* \cosh\theta) \tag{42}$$

Similarly, the dual area of projection of spherical image of φ_{q_1}-CTTRS in the direction $\tilde{h}$ is

$$\begin{aligned} 2\bar{f}_{\tilde{q}_1,\tilde{h}} &= \left\langle \tilde{w}_{q_1}, \tilde{h} \right\rangle = -\left\langle \tilde{d} + \barwedge_{q_1} \tilde{q}_1, \tilde{h} \right\rangle \\ &= -\left\langle \tilde{d}, \tilde{h} \right\rangle - \barwedge_{q_1} \left\langle (\sinh\bar{\theta})\tilde{q} + (\cosh\bar{\theta})\tilde{h}, \tilde{h} \right\rangle \\ &= -\left\langle \tilde{d}, \tilde{h} \right\rangle - \barwedge_{q_1} \cosh\bar{\theta} \\ 2\bar{f}_{\tilde{q}_1,\tilde{h}} &= \left\langle \tilde{w}_{q_1}, \tilde{h} \right\rangle = \barwedge_h - \barwedge_{q_1} \cosh\bar{\theta} \end{aligned} \tag{43}$$

Separating (43) into real and dual parts we have the following:

Theorem 5.5: *Let the trajectory timelike ruled surfaces* φ_q *and* φ_{q_1} *be of the type* N_-^1 *and* N_+^1 *, respectively and let they form a Mannheim offset. Then the relationships between the area of projections of spherical images of the surfaces* φ_h *and* φ_{q_1} *and their integral invariants are given as follows:*

$$2f_{q_1,h} = \lambda_h - \lambda_{q_1}\cosh\theta, \quad 2f^*_{q_1,h} = \ell_h - \ell_{q_1}\cosh\theta - \lambda_{q_1}\theta^*\sinh\theta, \tag{44}$$

Similarly, dual area of projection of spherical image of φ_{q_1}-CTRS in the direction $\tilde{a}$ is

$$\begin{aligned} 2\overline{f}_{\tilde{q}_1,\tilde{a}} &= \langle \tilde{w}_{q_1}, \tilde{a}\rangle = -\langle \tilde{d} + \overline{\wedge}_{q_1}\tilde{q}_1, \tilde{a}\rangle \\ &= -\langle \tilde{d}, \tilde{a}\rangle - \overline{\wedge}_{q_1}\underbrace{\langle \tilde{q}_1, \tilde{h}_1\rangle}_{0} \\ &= -\langle \tilde{d}, \tilde{a}\rangle \\ &= \overline{\wedge}_a \end{aligned}$$

Since $\tilde{a} = \tilde{h}_1$, we have

$$2\overline{f}_{\tilde{q}_1,\tilde{a}} = \overline{\wedge}_a = \overline{\wedge}_{h_1} \tag{45}$$

Separating (45) into real and dual parts we have the followings:

Theorem 5.6: *Let the trajectory timelike ruled surfaces* φ_q *and* φ_{q_1} *be of the type* N_-^1 *and* N_+^1 *, respectively and let they form a Mannheim offset. Then the relationships between the area of projections of spherical images of the surfaces* φ_a *and* φ_{q_1} *and their integral invariants are given as follows:*

$$f_{q_1,a} = \lambda_{h_1} = \lambda_a, \quad f^*_{q_1,a} = \ell_{h_1} = \ell_a$$

6 N_-^1-Type Mannheim Offsets of Timelike Ruled Surfaces with Timelike Rulings

Let the timelike trajectory ruled surface φ_{q_1} of the type N_-^1 be a Mannheim offset of the timelike trajectory ruled surface φ_q with timelike rulings. Then we can give the followings. The proofs of this section can be given by the similar ways in Section 5.

Theorem 6.1: *Let the timelike trajectory ruled surfaces φ_q and φ_{q_1} with timelike rulings form a Mannheim offset. Then the offset angle and offset distance are given by*

$$\theta = -\int k_1 ds, \quad \theta^* = -\int k_1^* ds \ ,$$

respectively.

Theorem 6.2: *The closed timelike trajectory ruled surfaces φ_q and φ_{q_1} with timelike rulings form a Mannheim offsets with a constant dual offset angle if and only if following relationship holds,*

$$\overline{\curlywedge}_{q_1} = \overline{\curlywedge}_q \sinh \overline{\theta} + \overline{\curlywedge}_h \cosh \overline{\theta} \ .$$

Separating this equation into real and dual parts, we obtain,

$$\lambda_{q_1} = \lambda_q \sinh \theta + \lambda_h \cosh \theta, \quad \ell_{q_1} = (-\ell_q + \theta^* \lambda_h) \sinh \theta + (\ell_h + \theta^* \lambda_q) \cosh \theta \ .$$

Then we have the following results.

Result 6.1: If $\theta = 0$, then the relationships between real integral invariants of φ_h and φ_{q_1}-CTTRS are given by $\lambda_{q_1} = \lambda_h, \ \ \ell_{q_1} = \ell_h + \theta^* \lambda_q$. Furthermore, the measure of spherical surface areas bounded by spherical images of φ_h and φ_{q_1}-CTTRS Mannheim offsets are the same, i.e., $a_{q_1} = a_h$ and $a_{q_1}^* = a_h^* + \theta^*(2\pi + a_q)$.

Result 6.2: If $\theta^* = 0$, i.e., the generators $\vec{q}$ and $\vec{q}_1$ of the Mannheim offset surfaces intersect, then we have

$$\lambda_{q_1} = \lambda_q \cosh \theta + \lambda_h \sinh \theta, \quad \ell_{q_1} = -\ell_q \cosh \theta + \ell_h \sinh \theta \ .$$

In this case, φ_q and φ_{q_1}-CTRS are intersect along their striction lines. It means that their striction lines are the same.

Theorem 6.3: *Let the trajectory timelike ruled surface φ_{q_1} of the type N_-^1 be the Mannheim offset of developable timelike ruled surface φ_q with timelike ruling. Then φ_{q_1} is developable if and only if $\cosh \theta + \theta^* \tau_\alpha \sinh \theta = 0$ holds.*

Theorem 6.4: *Let the developable trajectory timelike ruled surfaces φ_q and φ_{q_1} be of the type N_-^1. If φ_q and φ_{q_1} form a Mannheim offset then their striction lines are Mannheim partner curves in Minkowski 3-space IR_1^3.*

Corollary 6.1: *If φ_q-CTTRS is developable then the relation between the pitch ℓ_{q_1} of φ_{q_1}-CTTRS and the torsion of sitriction line $\alpha(s)$ of φ_q-CTTRS is given by*

$$\ell_{q_1} = -\oint (\cosh\theta + \theta^* \tau_\alpha \sinh\theta) ds .$$

Theorem 6.5: *Let the trajectory timelike ruled surfaces φ_q and φ_{q_1} of the type N_-^1 form a Mannheim offset. The relationships between area of projections of spherical images of the surfaces φ_q and φ_{q_1} and their integral invariants are given as follows,*

$$2\overline{f}_{\tilde{q}_1,\tilde{q}} = \left\langle \tilde{w}_{q_1}, \tilde{q} \right\rangle = -\overline{\lambda}_q + \overline{\lambda}_{q_1} \cosh\overline{\theta} ,$$

or

$$2 f_{q_1,q} = \lambda_q - \lambda_{q_1} \cosh\theta, \quad 2 f^*_{q_1,q} = -(\ell_q - \ell_{q_1} \cosh\theta + \lambda_{q_1} \theta^* \sinh\theta) .$$

Theorem 6.6: *Let the trajectory timelike ruled surfaces φ_q and φ_{q_1} of the type N_-^1 form a Mannheim offset. Then the relationships between area of projections of spherical images of ruled surfaces φ_h and φ_{q_1} and their integral invariants are given as follows*

$$2\overline{f}_{\tilde{q}_1,\tilde{h}} = \left\langle \tilde{w}_{q_1}, \tilde{h} \right\rangle = -\overline{\lambda}_h + \overline{\lambda}_{q_1} \sinh\overline{\theta} ,$$

or

$$2 f_{q_1,h} = \lambda_h \lambda_{q_1} \sinh\theta, \quad 2 f^*_{q_1,h} = \ell_h - \ell_{q_1} \sinh\theta + \lambda_{q_1} \theta^* \cosh\theta .$$

Theorem 6.7: *Let the trajectory timelike ruled surfaces φ_q and φ_{q_1} of the type N_-^1 form a Mannheim offset. Then the relationships between area of projections of spherical images of ruled surfaces φ_a and φ_{q_1} and their integral invariants are given as follows*

$$2\overline{f}_{\tilde{q}_1,\tilde{a}} = \left\langle \tilde{w}_{q_1}, \tilde{a} \right\rangle = \overline{\lambda}_a = \overline{\lambda}_{h_1} \quad \text{or} \quad f_{q_1,a} = \lambda_{h_1} = \lambda_a, \quad f^*_{q_1,a} = \ell_{h_1} = \ell_a .$$

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