Gen. Math. Notes, Vol. 2, No. 1, January 2011, pp. 40-44

www.i-csrs.org
Available free online at http://www.geman.in

A Class of Inequealities on Matrix Norms and Applications

Feixiang Chen and Zhanfei Zuo

Department of Mathematics and Computer Science,
Chongqing Three Gorges University Wanzhou 404000,P.R. China
E-mail: cfx2002@126.com
Department of Mathematics and Computer Science,
Chongqing Three Gorges University Wanzhou 404000,P.R. China
E-mail:zuozhanfei@139.com

(Received 09.11.2010, Accepted 23.11.2010)

Abstract

Some inequalities regarding matrix norms are established, which are related to numerical computations and optimization, respectively.

Keywords: *matrix norm; Frobenius-norm; spectral norm.*

2000 MSC No: Use appropriate MSC Nos.

1 Introduction

If one has several vectors in R^n or several matrices in M_n, what might it mean to say that some are "small" or that others are "large" ? Under what circumstances might we say that two matrices are "close together" or "far apart" ? One way to answer these questions is to study norms, or measures of size, of matrices. It is essential in the analysis and assessment of algorithms for numerical computations and optimization.

2 notation and terminology

Throughout this paper, uppercase roman letters denote matrices, lowercase roman letters denote vectors, and lowercase Greek letters denote scalars. We let R^n denote the set of n-dimensional vectors having real components. Also,we let M_n denote the set of $n \times n$ matrices with real entries. For a matrix $Q \in M_n$ with all real eigenvalues, we denote its eigenvalues by $\lambda_i[Q]$, $i = 1, \ldots, n$, and its smallest eigenvalue and largest eigenvalue by $\lambda_{min}[Q]$ and $\lambda_{max}[Q]$, respectively. The trace of a matrix $Q \in M_n$ is denoted by tr$Q \equiv \sum_{i=1}^{n} Q_{ii}$. The set of all symmetric $n \times n$ is denoted by S^n. The Euclidean norm and its associated operator norm are both denoted by $\|\cdot\|$.

Definition 2.1 *A function $\|\cdot\| : M_n \to R$ is called a matrix norm if for all $A, B \in M_n$ when it satisfies the following axioms:*
(1) $\|A\| \geq 0$, with equality holding if and only if $A = 0$;
(2) $\|\alpha A\| = \alpha\|A\|$, for all complex scalar α;
(3) $\|A + B\| \leq \|A\| + \|B\|$, for all $A, B \in M_n$;
(4) $\|AB\| \leq \|A\|\|B\|$, for all $A, B \in M_n$.

Definition 2.2 *The Frobenius norm defined for $A \in M_n$ by*

$$\|A\|_F \equiv (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2}.$$

Definition 2.3 *Let $\|\cdot\|$ be a vector norm on R^n. Define operator norm or spectral norm $\|\cdot\|$ on M_n by*

$$\|A\| \equiv max_{\|x\|=1}\|Ax\| = \sqrt{\lambda_{max}(A^T A)}.$$

3 The Main Results

In the next result, we collect some useful facts about symmetric matrices. For its proof, we refer the reader to Golub and Van Loan [1] or Horn and Johnson [2]. In the next result, we collect some useful facts about symmetric matrices. For its proof, we refer the reader to Golub and Van Loan [1] or Horn and Johnson [2].

Lemma 3.1 *For all $E \in S^n$, we have:*

$$\lambda_{max}(E) = max_{\|u\|=1} u^T E u.. \tag{1}$$

$$\lambda_{min}(E) = min_{\|u\|=1} u^T E u.. \tag{2}$$

$$\|E\| = max_{i=1,\ldots,n} |\lambda_i(E)|. \tag{3}$$

$$\|E\|_F^2 = \sum_{i=n}^{n} [\lambda_i E]^2. \tag{4}$$

Lemma 3.2 *For all $A \in M_n$, the following relations hold:*

$$max_{i=1,\dots,n}Re[\lambda_i(A)] \leq \frac{1}{2}\lambda_{max}(A + A^T). \tag{5}$$

$$min_{i=1,\dots,n}Re[\lambda_i(A)] \geq \frac{1}{2}\lambda_{min}(A + A^T). \tag{6}$$

$$\sum_{i=1}^{n} |\lambda_i(A)|^2 \leq \|A\|_F^2 = \|A^T\|_F^2; \tag{7}$$

$$\lambda_{max}(A^TA)\| = \|A^TA\| = \|A\|^2 = \|A^T\|^2. \tag{8}$$

Proof. Inequality (5) is stated as an exercise in Horn and Johnson; see [1]. Inequality (6) follows from (5) applied to the matrix $-W$. To prove (7), we prove that $tr(A^2) \leq tr(A^TA)$. we first consider the case $n = 2$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

hence, $tr(A^2) = a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} \leq a_{11}^2 + a_{22}^2 + a_{12}^2 + a_{22}^2 = tr(A^TA)$. We consider $n = k$ and $n = k + 1$, respectively.

$$A = \begin{bmatrix} B & \alpha^k \\ \beta^k & a_{k+1,k+1} \end{bmatrix}.$$

where $B \in M_k$ and $\alpha^k, \beta^k \in R_k$. Then, $tr(A^2) = tr(B^2) + 2(\beta^k)^T\alpha^k + a_{k+1,k+1}^2 \leq tr(B^2) + a_{k+1,k+1}^2 + \|\beta^k\|^2 + \|\alpha^k\|^2 = tr(A^TA)$. Hence, (7) holds. Considering that $A^TA \in S^n$, which clearly implies (8).

Lemma 3.3 *Suppose that $W \in M_n$ is a nonsingular matrix. then for any $E \in S^n$, the following relations hold:*

$$\lambda_{max}(E) \leq \frac{1}{2}\lambda_{max}(WEW^{-1} + (WEW^{-1})^T). \tag{9}$$

$$\lambda_{min}(E) \geq \frac{1}{2}\lambda_{min}(WEW^{-1} + (WEW^{-1})^T). \tag{10}$$

Proof. Using (5), we obtain

$$\lambda_{max}(E) = \lambda_{max}(WEW^{-1}) \leq \frac{1}{2}\lambda_{max}(WEW^{-1} + (WEW^{-1})^T)$$

for every $E \in M_n$, and hence (9) follows. Inequality (10) is proved in a similar way by using (6). .

4 Applications

The following inequalities are widely used in numerical computations and optimization [3].

Lemma 4.1 *For every $A \in M_n$, and $H \in S_n$, then the equations*

$$AU + UA = H \tag{11}$$

has a unique solution $U \in S_n$. Moveover, this solution satisfies

$$\|AU\|_F \leq \|H\|_F/\sqrt{2}. \tag{12}$$

Proof. The first part of the lemma follows from the fact that the linear map $\Phi_A : S_n \to S_n$ defined by $\Phi_A(U) : AU + UA$ is an isomorphism. Indeed, since Φ_A has the same domain and codomain, it suffices to show that Φ_A is one-to-one, or equivalently that $AU + UA = 0$ implies $U = 0$. In turn, this last implication follows from the fact that any solution U of (11) satisfies (12) (simply set $H = 0$ in (12) to conclude that $U = 0$). To show the last claim, we square both sides of (11) to obtain

$$2\|AU\|_F^2 + 2tr(UAUA) = \|H\|_F^2.$$

Since $tr(UAUA) = \|A^{1/2}UA^{1/2}\|_F^2 \geq 0$, then we complete the proof.

Lemma 4.2 *Suppose that $W \in M_n$ is a nonsingular matrix. then for any $E \in S^n$, the following relations hold:*

$$\|E\| \leq \frac{1}{2}\|WEW^{-1} + (WEW^{-1})^T\|. \tag{13}$$

$$\|E\|_F \leq \frac{1}{2}\|WEW^{-1} + (WEW^{-1})^T\|_F. \tag{14}$$

Proof. Using (5), we obtain

$$\lambda_{max}(E) = \lambda_{max}(WEW^{-1}) \leq \frac{1}{2}\lambda_{max}(WEW^{-1} + (WEW^{-1})^T)$$

and

$$\lambda_{min}(E) = \lambda_{min}(WEW^{-1}) \geq \frac{1}{2}\lambda_{min}(WEW^{-1} + (WEW^{-1})^T).$$

Then we conclude that

$$\frac{1}{2}\lambda_{min}(WEW^{-1}+(WEW^{-1})^T) \leq \lambda_{min}(E) \leq \lambda_{max}(E) \leq \frac{1}{2}\lambda_{max}(WEW^{-1}+(WEW^{-1})^T).$$

From $\|E\| = \|\lambda_i(E)\|, i = 1, 2, \ldots, n$, then we prove the first part. To prove (14) , we use (4) and (7) to get

$$\|E\|_F^2 = \sum_{i=1}^{n}[\lambda_i(E)]^2 = \sum_{i=1}^{n}[\lambda_i(WEW^{-1})] \leq (WEW^{-1})|_F^2.$$

Hence, we obtain

$$\begin{aligned} 4\|E\|_F^2 &\leq 2\|E\|_F^2 + 2\|WEW^{-1}\|_F^2 \\ &= 2\|WEW^{-1}\|_F^2 + 2tr(E^2) \\ &= 2\|WEW^{-1}\|_F^2 + 2tr(WE^2W^{-1}) \\ &= 2\|WEW^{-1}\|_F^2 + 2tr(WEW^{-1})^2 \\ &= \|WEW^{-1} + (WEW^{-1})^T\|_F^2. \end{aligned}$$

Which clearly implies (14).

Theorem 4.3 *For all $A \in M_n$, the following relations hold:*

$$\frac{1}{2}\|A + A^T\|_F \leq \|A\|_F; . \tag{15}$$

$$\frac{1}{2}\|A + A^T\| \leq \|A\|.. \tag{16}$$

Proof. The two inequalities can be proved by using the triangle inequality for norm.

References

[1] R A Horn and C R Johnson, *Topics in matrix analysis*, New York: Cambridge University Press, (1991).

[2] Horn G. H. Golub and C. E. Vanloan, *Matrix computations: Second Edition*, The John Hopkins University Press, Baltimore, MD, (1989).

[3] A.A. Soliman, Primal-dual following interior point algorithms for semidefinite programming. *SIAM J Optimization* 7(3) (1997) 663–678.