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BASE CHANGE FUNCTORS IN THE $\mathbb{A}^1$-STABLE HOMOTOPY CATEGORY

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(*communicated by Gunnar Carlsson*)

Abstract

We relate base change functors of sheaves in $\mathbb{A}^1$-homotopy theory to group change functors from equivariant homotopy theory, and use these functors to construct elements of the Picard group of the $\mathbb{A}^1$-stable homotopy category. We also prove an analogue of the Wirthmüller isomorphism from equivariant homotopy theory in the $\mathbb{A}^1$-context.

Introduction

In this note, we will discuss certain functors between the $\mathbb{A}^1$-stable homotopy categories over fields. These are in a sense analogous to the change of groups functors in equivariant stable homotopy theory. Section 1 contains some preliminaries in doing $\mathbb{A}^1$-stable homotopy theory. In Section 2, we define the change of base field functors between the $\mathbb{A}^1$-homotopy categories over fields L and k, where L is a finite extension of k. In Section 3, We give a application of these functors in constructing objects of the $\mathbb{A}^1$-stable homotopy category over k that are invertible under the smash product, by constructing a functor from the category of equivariant spaces with respect to the Galois group of k to the category of algebraic spaces over k. In Section 4, we prove an analogue of the Wirthmüller isomorphism, a classical result from algebraic topology. We will also show that in the present setting, this result is in fact not only an analogue, but a consequence of the equivariant result. This gives a very explicit formulation of the Wirthmüller isomorphism map. It is also analogous to classical results from the theory of sheaves. Finally, in Section 5, we give some technical details of the definitions of join and smash powers of a k-space, which are needed in showing that the examples constructed in Section 3 are indeed invertible.

1. Preliminaries

We recall some fundamental constructions needed to do stable homotopy theory in the algebraic geometrical setting, due to Morel and Voevodsky [**11**]. Let S be a

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Noetherian scheme of finite dimension, and Sm/S the category of smooth schemes of finite type over S. The Nisnevich topology on Sm/S is defined to be the subtopology of the étale topology generated by diagrams of the following form.

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "i"'] & X. \end{tikzcd}$$

Here, we require i to be an open embedding, p to be an étale map, such that $p|_{p^{-1}(X\backslash\{0\})}$ is an isomorphism.

The Nisnevich topology makes Sm/S into a small Grothendieck site [**12**]. The category of S-spaces

$$Spc(S) = \Delta^{op} Sh(Sm/S)_{Nis}$$

is the category of simplicial sheaves of sets over Sm/S with respect to the Nisnevich topology. In particular, a smooth scheme X over S is the sheaf represented by X, concentrated in simplicial degree 0. The category $Spc(S)$ has all small colimits and limits, and is the analogue of topological spaces. Also, it is generated by Sm/S in the sense that every S-space is a colimit of smooth schemes over S in $Spc(S)$.

To do homotopy theory in $Spc(S)$, one defines a closed model structure on the category $Spc(S)$ in two steps as follows. First, one says that a map f is a simplicial cofibration if it is a monomorphism, a simplicial weak equivalence if it is a weak equivalence of simplicial sets at every point of the site Sm/S, and a simplicial fibration if it has the right lifting property with respect to all acyclic simplicial cofibrations. This defines the simplicial model structure on $Spc(S)$. Let $\mathcal{H}_s(S)$ be the homotopy category on $Spc(S)$ associated to this model structure. The simplicial model structure has too few weak equivalences for our purposes. In particular, we would like the affine line $\mathbb{A}^1_S$ over S to be weakly equivalent to the S-space S, which is the analogue in the category $Spc(S)$ of a single point. To obtain this, one localizes in the sense of Bousfield [**2**] at all projection maps $\mathbb{A}^1 \times X \to X$. Specifically, an S-space Y is said to be $\mathbb{A}^1$-local, if for all X, the map of morphism sets in $\mathcal{H}_s$

$$\mathcal{H}_s(X, Y) \to \mathcal{H}_s(X \times \mathbb{A}^1, Y)$$

is a bijection. A map $f : X \to Y$ is an $\mathbb{A}^1$-weak equivalence if for all $\mathbb{A}^1$-local objects Z, the map of morphism sets

$$\mathcal{H}_s(Y, Z) \to \mathcal{H}_s(X, Z)$$

is a bijection. A map f is an $\mathbb{A}^1$-cofibration if it is a simplicial cofibration, and it is an $\mathbb{A}^1$-fibration if it satisfies the right lifting property with respect to all acyclic $\mathbb{A}^1$-weak equivalences. These three classes of maps define the $\mathbb{A}^1$-local model structure on $Spc(S)$, the model structure with which one works in homotopy theory.

One can also consider the category $Spc(S)_\bullet$ of based S-spaces, an object of which is a S-space X together with a given map from S to X, called the basepoint of X. The $\mathbb{A}^1$-local model structure on $Spc(S)_\bullet$ is defined in the unbased category $Spc(S)$ [**11, 6**]. We also have analogues of certain basic constructions from topology.

In particular, we have the disjoint basepoint functor $(-)_+ : Spc(S) \to Spc(S)_\bullet$, which takes an unbased S-space X to the based S-space $X \amalg S$, where the basepoint is the disjoint copy of S. Also, for $X, Y \in Spc(S)_\bullet$, the smash product of X and Y over S is given by

$$X \wedge_S Y = (X \times_S Y)/_S(X \vee_S Y)$$

where $X \vee_S Y$ is the pushout in $Spc(S)$ of the basepoint maps $S \to X$ and $S \to Y$, and $/_S$ is the quotient space functor in $Spc(S)$, obtained by collapsing $X \wedge_S Y$ to S.

To do stable homotopy theory, one stabilizes with respect to the one-dimensional projective space $\mathbb{P}^1_S$ over S. The category of S-spectra is defined in a similar manner as in topology. Namely, an S-prespectrum D is a sequence of based S-spaces $\{D_n\}$, along with a give structure map

$$\Sigma^{\mathbb{P}^1} D_n \to D_{n+1}$$

for each n. Here, $\Sigma^{\mathbb{P}^1}$ denotes the suspension functor $\mathbb{P}^1_S \wedge -$. Equivalently, a structure map is a map

$$D_n \to \Omega^{\mathbb{P}^1} D_{n+1} \tag{1.1}$$

where $\Omega^{\mathbb{P}^1} = \underline{Hom}_\bullet(\mathbb{P}^1_S, -)$ is the internal Hom object from $\mathbb{A}^1_S$, i. e. the right adjoint to the functor $\Sigma^{\mathbb{P}^1}$ in the category of based S-spaces. An S-prespectrum is an S-spectrum if its structure maps (1.1) are isomorphisms in $Spc(S)_\bullet$. There is a spectrification functor L from the category of S-prespectra to the category of S-spectra, which is analogous to the spectrification functor from inclusion prespectra to spectra in topology. Namely, given an S-prespectrum $D = \{D_n\}$,

$$(LD)_n = \mathrm{colim}_k \Omega^{(\mathbb{P}^1)^{\wedge k}} D_{n+k}.$$

We denote the category of S-spectra by $Spectra(S)$. Stabilizing the $\mathbb{A}^1$-local model structure on $Spc(S)_\bullet$ in the manner of Bousfield and Friedlander [**3**] gives the stable $\mathbb{A}^1$-local model structure on the category of S-spectra (see [**6, 14**]). In particular, for any S-prespecturm D, the unit map $D \to LD$ is always an $\mathbb{A}^1$-weak equivalence [**6**]. We will call the homotopy category associated with this model structure the *stable homotopy category over* S, denoted by $\mathcal{SH}(S)$. It is the algebraic analogue of the stable homotopy category in topology. As in topology, we can think of S-spectra as indexed on an universe $\mathcal{U} \cong \mathbb{A}^\infty_S$ over S, then for two S-spectra E and E', the internal smash product $E \wedge E'$ is given by first taking the external smash product $E \overline{\wedge} E'$, which is an S-spectrum indexed on $\mathcal{U}^{\oplus 2}$, then taking the change of universe functor back to spectra indexed on $\mathcal{U}$ via an linear injection $\mathcal{U}^{\oplus 2} \to \mathcal{U}$ [**6**]. The operad of such linear injections is $\mathbb{A}^1_S$-contractible. Hence, the smash product of spectra are well-defined in $\mathcal{SH}(S)$. In this note, we will work only with the case where $S = Spec(k)$ for an arbitrary field k.

2. Change of Bases Functors

Let k be an arbitrary field, and let L be a finite separable extension of k. We have a canonical map

$$i : Spec(L) \to Spec(k) \tag{2.1}$$

from the inclusion of k in L.

Definition 2.2. *Define the functor*

$$i^* : Spc(k) \to Spc(L)$$

by

$$i^*(X) = X \times_{Spec(k)} Spec(L).$$

The structure map $i^*(X) \to Spec(L)$ is the pullback of the structure map $X \to Spec(k)$ along i.

On the level of affine schemes, i^* corresponds to the extension of scalars functor

$$i^* = L \otimes_k - : Algebras(k) \to Algebras(L).$$

Considering k-spaces as simplicial Nisnevich sheaves over Sm/k, Morel and Voevodsky [**11**] defined the inverse image functor with respect to a map of base schemes. If $f : S_1 \to S_2$ is a morphism of schemes, it gives a continuous map of the Nisnevich sites $(Sm/S_1)_{Nis} \to (Sm/S_2)_{Nis}$. Thus, there is an inverse image of sheaves functor $f^* : Spc(S_2) \to Spc(S_1)$. In our case, $f = i : Spec(L) \to Spec(k)$ is a smooth map. So for a smooth scheme $X \in Sm/k$, the inverse image of the sheaf represented by X is the sheaf on Sm/L represeted by $X \times_{Spec(k)} Spec(L)$. Also, recall that every k-space is a colimit of sheaves represented by smooth schemes (see [**6**], Appendix). Thus, our functor i^* is the same as the inverse image functor for all objects of Sm/k.

The functor i^* is analogous to a change of groups functor from equivariant homotopy theory in the following sense. Recall from Lewis, May and Steinberger [**9**] that if we have a compact Lie group G, and a (closed) subgroup $H \subseteq G$, then there is a forgetful functor from the category of G-equivariant topological spaces to the category of H-equivariant topological spaces. On the other hand, we can also consider the category of G-equivariant spaces parametrized over a given G-equivariant based space X, i. e. the comma category of G-equivariant spaces Z, together with a given continuous G-map to X. In particular, there is a equivalence of categories between the category of G-equivariant spaces parametrized over the homogenous G-space G/H, and the category of H-equivariant spaces. Namely, given an H-space T, we have a G-space

$$G \times_H T = \{(g,t) \mid g \in G,\ t \in T\}/(gh,x) \sim (g,hx) \tag{2.3}$$

where the action of G is induced by the multiplication of G on itself from the left. This has a natural G-map to G/H, induced from collapsing T to a single fixed point. Conversely, for a G-space Z with a G-map $Z \to G/H$, the fiber Z_{eH} of Z over the coset eH of G/H is an H-equivariant space. It is straightforward to check that these two functors are inverse equivalences. Also, let $f : G/H \to *$ be the collapse map. Then similarly as in the algebraic case, we also have the functor f^* from the category of G-spaces to the category of G-spaces over G/H, give by $f^*(T) = G/H \times T$, with the diagonal G-action, and mapping to G/H via the first projection. Also, f^* corresponds to the forgetful functor from G-spaces to H-spaces, with respect to the equivalence of categories between G-spaces over G/H and H-spaces. Now let $G = Gal(E/k)$ for some finite Galois extension E of L, and let $H = Gal(E/L)$. Then the E-points of a k-space has a natural G-action, and the E-points of an L-space has a natural H-action. The counterpart of the homogenous space G/H is $Spec(L)$, whereas L-spaces, i. e. k-spaces over $Spec(L)$, corresponding to G-spaces over G/H, and $i : Spec(L) \to Spec(k)$ corresponds to the G-map $G/H \to *$. In this sense, i^* is analogous to the forgetful functor with respect to the inclusion map $H \to G$. For instance, $i^*Spec(k) = Spec(L)$, which is analogous to the fact that the G-space consisting of a single fixed point forgets to the H-space consisting of a single fixed point.

In the equivariant topological context, the forgetful functor has both a left adjoint and a right adjoint. The left adjoint is $G \times_H -$ from H-equivariant spaces to G-equivariant spaces, as given in (2.3). The right adjoint is $Maps_H(G, -)$, the space of H-equivariant maps from G, with the G-action induced by the action of G on itself from the right.

In the algebraic context, we also have both a left and a right adjoint to i^*. We denote the left adjoint to i^* by

$$i_\sharp : Spc(L) \to Spc(k).$$

An L-space X has a structure map $X \to Spec(L)$, which is also a map over $Spec(k)$. Composition with i gives a structure map $X \to Spec(k)$, which completely determines a k-space structure on X. This gives $i_\sharp X$. If we have a map $f : i_\sharp X \to Y$ over $Spec(k)$ for an L-space X and a k-space Y, then taking the pullback of f along the $i : Spec(L) \to Spec(k)$ gives a map over $Spec(L)$ from X to $i^*(Y) = Spec(L) \times_{Spec(k)} Y$. Conversely, given a map $g : X \to Spec(L) \times_{Spec(k)} Y$, composition with the map $Spec(L) \times_{Spec(k)} Y$ gives a map from $i_\sharp X$ to Y over $Spec(k)$. It is clear that these corrspondances are inverse to each other, so $i_\sharp$ is the left adjoint to i^*. In particular, on the level of affine schemes, $i_\sharp$ corresponds to a functor

$$Algebras(L) \to Algebras(k).$$

This is the right adjoint to the extension of scalars functor, i. e. the forgetful functor. When there is no possibility of confusion, we will omit $i_\sharp$ from the notation. Since i is a smooth morphism, then by Proposition 3.1.23 of [**11**], when we think of i^* as the inverse image of sheaves functor, we can also define the left adjoint $i_\sharp$ of i^* to be an "extension by zero" functor of sheaves. By the uniqueness of adjoints, this definition of $i_\sharp$ coincide with our definitions.

There is also a right adjoint

$$i_* : Spc(L) \to Spc(k)$$

to i^*. To write down i_*, recall that the category of k-spaces is closed, i. e. for any

Y, the functor

$$Y \times_{Spec(k)} - : Spc(k) \to Spc(k)$$

has a right adjoint

$$\underline{Hom}_{Spec(k)}(Y, -) : Spc(k) \to Spc(k).$$

Let X be an L-space, we think of $Spec(L)$ and X as k-spaces via the forgetful functor $i_\sharp$. We define i_*X to be $Maps_L(Spec(L), X)$, the k-space of maps $Spec(L) \to X$ over L. More precisely, the structure map

$$p_X : X \to Spec(L)$$

of X gives a map of k-spaces

$$(p_X)_* : \underline{Hom}_{Spec(k)}(Spec(L), X) \to \underline{Hom}_{Spec(k)}(Spec(L), Spec(L)).$$

Also, the identity map $Id : Spec(k) \times_{Spec(k)} Spec(L) \to Spec(L)$ over $Spec(k)$ is adjoint to

$$\iota : Spec(k) \to \underline{Hom}_{Spec(k)}(Spec(L), Spec(L)).$$

We define $i_*X = Maps_L(Spec(L), X)$ by the pullback diagram over $Spec(k)$

$$\begin{tikzcd} Maps_L(Spec(L), X) \arrow[r] \arrow[d] & \underline{Hom}_{Spec(k)}(Spec(L), X) \arrow[d, "(p_X)_*"] \\ Spec(k) \arrow[r, "\iota"'] & \underline{Hom}_{Spec(k)}(Spec(L), Spec(L)). \end{tikzcd} \tag{2.4}$$

Lemma 2.5. *The functor $i_* = Maps_L(Spec(L), -)$ is the right adjoint to i^*.*

Proof. For a k-space Y and an L-space X, a map over $Spec(k)$

$$f : Y \to Maps_L(Spec(L), X)$$

is equivalent to a commutative diagram over $Spec(k)$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & \underline{Hom}_{Spec(k)}(Spec(L), X) \arrow[d, "(p_X)_*"] \\ Spec(k) \arrow[r, "\iota"'] & \underline{Hom}_{Spec(k)}(Spec(L), Spec(L)). \end{tikzcd}$$

By naturality of the adjunction $(Spec(L) \times_{Spec(k)} -,\ \underline{Hom}_{Spec(k)}(Spec(L), -))$ in $Spc(k)$, this is equivalent to a commutative diagram

$$\begin{tikzcd} Y \times_{Spec(k)} Spec(L) \arrow[r] \arrow[d] & X \arrow[d, "p_X"] \\ Spec(k) \times_{Spec(k)} Spec(L) \arrow[r, "Id"'] & Spec(L) \end{tikzcd}$$

i. e. a map $Y \times_{Spec(k)} Spec(L) \to X$ over $Spec(L)$. □

Again, by the uniqueness of adjoints, i_* is the same as the direct image functor of sheaves [**11**], if we think of spaces over k and over L as simplicial Nisnevich sheaves.
Remark: For any L-spaces X and Y, there is another mapping space from X to Y over $Spec(L)$, namely

$$\underline{Hom}_{Spec(L)}(X,Y)$$

which is the right adjoint to the functor

$$X \times_{Spec(L)} - : Spc(L) \to Spc(L).$$

In particular, for $X = Spec(L)$, $Spec(L) \times_{Spec(L)} -$ is the identity functor, thus, so is $\underline{Hom}_{Spec(L)}(Spec(L), -)$. For a general L-space Y, the k-space

$$i_* Y = Maps_L(Spec(L), Y)$$

is not the same as $i_\sharp(\underline{Hom}_{Spec(L)}(Spec(L), Y)) = i_\sharp Y$.

On the level of affine schemes,

$$i_* : Algebras(L) \to Algebras(k)$$

is the left adjoint to the extension of scalars functor. If $[L : k] = n$, and we choose a basis $\alpha_1, \dots, \alpha_n$ of L as a vector space over k, then for a finitely generated L-algebra

$$R = L[y_1, \dots, y_r]/I$$

we can write down the k-algebra $i_* R$ in terms of generators and relations to be

$$i_* R = k[x_{1,1}, \dots, x_{1,r}, x_{2,1}, \dots, x_{n,r}]/J$$

where J is the following ideal of $k[x_{1,1}, \dots, x_{1,r}, x_{2,1}, \dots, x_{n,r}]$. For f a polynomial in $L[y_1, \dots, y_r]$, we have unique $g_1(f), \dots, g_n(f)$ in $k[x_{1,1}, \dots, x_{1,r}, x_{2,1}, \dots, x_{n,r}]$ such that

$$f\left(\sum_{j=1}^{n} x_{j,1}\alpha_j, \dots, \sum_{j=1}^{n} x_{j,r}\alpha_j\right) = \sum_{j=1}^{n} g_j(f)(x_{1,1}, \dots, x_{n,r})\alpha_j.$$

Then set $J = \{g_j(f) \mid f \in I\}$. It is routine to check that this is independent of the choice of basis, and is indeed the left adjoint to the functor $L \otimes_k -$.

Examples:
1. Consider $\mathbb{A}^1_L = Spec(L[x])$ over $Spec(L)$. Then on the level of algebras,

$$i_*(L[x]) = k[x_1, \dots, x_{[L:k]}]$$

so $i_*(\mathbb{A}^1_L) = \mathbb{A}^{[L:k]}_k$.

2. Suppose $L = k[\sqrt{a}]$ is an extension of degree 2 over k, for some $a \in k^\times$, $a \notin (k^\times)^2$, and choose the basis $\{1, \sqrt{a}\}$ of L as a vector space over k. Consider the affine variety

$$(\mathbb{G}_m)_L = Spec(L[x,y]/xy = 1).$$

Then on the level of algebras

$$i_*(L[x,y]/xy = 1) = k[x_1, x_2, y_1, y_2]/\sim$$

If we think of $x = x_1 + x_2\sqrt{a}$, $y = y_1 + y_2\sqrt{a}$, then the relation $\sim$ is given by

$$xy = (x_1 + x_2\sqrt{a})(y_1 + y_2\sqrt{a}) = 1,$$

so $x_1y_1 + ax_2y_2 = 1$, and $x_1y_2 + x_2y_1 = 0$. Hence, the k-space $i_*(\mathbb{G}_m)_L$ is

$$Spec(k[x_1, x_2, y_1, y_2]/(x_1y_1 + ax_2y_2 - 1,\ x_1y_2 + x_2y_1).$$

This gives the adjoint functors $i_\sharp, i^*$ and i_* between the categories of unbased spaces $Spc(k)$ and $Spc(L)$. We would also like to have these functors on the categories of based k-spaces and L-spaces. In the equivariant context, the forgetful functor from based G-spaces to based H-spaces is given by applying the unbased forgetful functor to the diagram of G-spaces $* \to X$, where $*$ denotes a single fixed point. This suggests that

$$i^* : Spc(k)_\bullet \to Spc(L)_\bullet$$

should be defined similarly. Thus, for $X \in Spc(k)_\bullet$, we define $i^*(X)$ by applying the unbased i^* to the diagram $Spec(k) \to X$. We have $i^*(Spec(k)) = Spec(L)$, $i^*X = Spec(L)\times_{Spec(k)}X$, and the induced basepoint $Spec(L) \to Spec(L)\times_{Spec(k)}X$ is the pullback along i of the basepoint map $Spec(k) \to X$.

Given a based L-space X, we can define the based k-space $i_\sharp X$ by the following pushout diagram

$$\begin{tikzcd} Spec(L) \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ Spec(k) \arrow[r] & i_\sharp X \end{tikzcd}$$

where the top horizontal map is the basepoint of X. This is a diagram over k, so strictly speaking, the top right corner of the square is the unbased version of $i_\sharp X$. It is routine to check that the functor $i_\sharp : Spc(L)_\bullet \to Spc(k)_\bullet$ is the left adjoint to $i^* : Spc(k)_\bullet \to Spc(L)_\bullet$.

The based right adjoint $i_* : Spc(L)_\bullet \to Spc(k)_\bullet$ is also defined, same as in the unbased case, to be $Maps_L(Spec(L), -)$. For a based L-space Y, the basepoint map $Spec(k) \to Maps_L(Spec(L), Y)$ is the adjoint to the basepoint map $Spec(L) \to Y$ of Y. The functor $i_* : Spc(L)_\bullet \to Spc(k)_\bullet$ is the right adjoint to i^* since it is the right adjoint to i^* in the categories of unbased spaces.

This gives the functors $i_\sharp$, i^* and i_* for based spaces. We would also like the spectra versions of these functors. For this, we need the following results. First, note that $i^*(\mathbb{P}^1_k) = \mathbb{P}^1_L$.

Lemma 2.6. *If X is a based space over $Spec(k)$, then*

$$i^*(\Sigma^{\mathbb{P}^1_k} X) \cong \Sigma^{\mathbb{P}^1_L} i^* X$$

naturally.

Proof. Let $i_X : Spec(k) \to X$ be the basepoint of X, and $i_{\mathbb{P}^1_k} : Spec(k) \to \mathbb{P}^1_k$ be the basepoint of $\mathbb{P}^1_k$. We have that

$$\Sigma^{\mathbb{P}^1_k} X = \frac{\mathbb{P}^1_k \times_{Spec(k)} X}{\mathbb{P}^1_k \vee_{Spec(k)} X}$$

where the quotient is in the category of k-spaces, and the map of $\mathbb{P}^1_k \vee_{Spec(k)} X$ into $\mathbb{P}^1_k \times_{Spec(k)} X$ is induced by i_X and $i_{\mathbb{P}^1_k}$. Since i^* is a left adjoint, it preserves pushouts, so

$$i^*\Sigma^{\mathbb{P}^1_k} X \cong \frac{i^*\mathbb{P}^1_k \times_{Spec(k)} X}{i^*\mathbb{P}^1_k \vee_{Spec(k)} X} \tag{2.7}$$

naturally, where the quotient takes place in $Spc(L)$. We have that $i^*(Spec(k)) = Spec(L)$, and $i^*(\mathbb{P}^1_k) = \mathbb{P}^1_L$, so

$$i^*\mathbb{P}^1_k \vee_{Spec(k)} X \cong \mathbb{P}^1_L \vee_{Spec(L)} i^*X$$

naturally. Also, since i^* is a right adjoint, it preserves pullbacks, so

$$i^*\mathbb{P}^1_k \times_{Spec(k)} X \cong \mathbb{P}^1_L \times_{Spec(L)} i^*X$$

naturally. Finally, as L-spaces, the basepoints of $\mathbb{P}^1_L$ and i^*X are i^* applied to $i_{\mathbb{P}^1_k}$ and i_X respectively. Hence, (2.7) is naturally isomorphic to

$$\Sigma^{\mathbb{P}^1_L} i^*X = \frac{\mathbb{P}^1_L \times_{spec(L)} i^*X}{\mathbb{P}^1_L \vee_{Spec(L)} i^*X}.$$

□

Proposition 2.8. *Let X be a based k-space, and Y a based L-space. Then there are natural isomorphisms of based k-spaces*

$$\zeta : i_\sharp(Y \wedge_{Spec(L)} i^*X) \cong (i_\sharp Y) \wedge_{Spec(k)} X$$
$$\varphi : i_*(\underline{Hom}_{Spec(L)}(i^*X, Y)) \cong \underline{Hom}_{Spec(k)}(X, i_*Y).$$

Proof. The first statement follows by explicitly checking the definitions of the smash products and of the based $i_\sharp$. For the second statement, note that the functor $i_*(\underline{Hom}_{Spec(L)}(i^*X, -))$ from $Spc(L)_\bullet$ to $Spc(k)_\bullet$ is right adjoint to the functor $i^*(-) \wedge_{Spec(L)} i^*X$, and the functor $Hom_{Spec(k)}(X, i_*(-))$ is right adjoint to the functor $i^*(- \wedge_{Spec(k)} X)$. By explicitly checking the definitions of the smash product, we have that for any based k-space Z,

$$i^*Z \wedge_{Spec(L)} i^*X \cong i^*(Z \wedge_{Spec(k)} X).$$

So the two left adjoints coincide, and the statement follows by the uniqueness of adjoints. □

Taking the adjoint of Lemma 2.6, we get that for a based L-space Y,

$$\Omega^{\mathbb{P}^1_k} i_*Y \cong i_*(\Omega^{\mathbb{P}^1_L} Y) \tag{2.9}$$

naturally. Also, taking $X = \mathbb{P}^1_k$ in the first statement of Proposition 2.8 gives that for a based L-space Y,

$$i_\sharp(\Sigma^{\mathbb{P}^1_L} Y) \cong \Sigma^{\mathbb{P}^1_k} i_\sharp Y \tag{2.10}$$

naturally. Taking the adjoint of this gives that for a based k-space X,

$$\Omega^{\mathbb{P}^1_L} i^* X \cong i^*(\Omega^{\mathbb{P}^1_k} X) \tag{2.11}$$

naturally. Hence, we can define i^* and i_* on the level of spectra, and use the spectrification to define $i_\sharp$ on spectra. More precisely, we have the following definition.

Definition 2.12. *The functors*

$$\begin{aligned} i_\sharp &: Spectra(L) \to Spectra(k) \\ i^* &: Spectra(k) \to Spectra(L) \\ i_* &: Spectra(L) \to Spectra(k) \end{aligned}$$

are given as follows. Suppose D is a k-spectrum, and E is an L-spectrum. For $i_\sharp E$, let the k-prespectrum $i_\sharp^{pre} E$ be given by

$$i_\sharp^{pre}(E)_n = i_\sharp(E_n)$$

with structure maps

$$\Sigma^{\mathbb{P}^1_k} i_\sharp(E_n) \cong i_\sharp(\Sigma^{\mathbb{P}^1_L} E_n) \xrightarrow{i_\sharp \overline{\rho}_n} i_\sharp E_{n+1}$$

where $\overline{\rho}_n : \Sigma^{\mathbb{P}^1_L} E_n \to E_{n+1}$ is the adjoint structure map of E. Define $i_\sharp E$ to be the spectrification of $i_\sharp^{pre}(E)$.

We define the L-spectrum $i^* D$ by

$$(i^* D)_n = i^*(D_n)$$

with structure maps

$$i^*(D_n) \xrightarrow{i^* r_n} i^*(\Omega^{\mathbb{P}^1_k} D_{n+1}) \cong \Omega^{\mathbb{P}^1_L} i^* D_{n+1}$$

where $r_n : D_n \to \Omega^{\mathbb{P}^1_k} D_{n+1}$ is the structure map of D. Similarly, define the k-spectrum $i_* E$ by

$$(i_* E)_n = i_*(E_n).$$

The structure maps are

$$i_*(E_n) \xrightarrow{i_* \rho_n} i_*(\Omega^{\mathbb{P}^1_L} E_{n+1}) \cong \Omega^{\mathbb{P}^1_k} i_* E_{n+1}$$

where $\rho : E_n \to \Omega^{\mathbb{P}^1_L} E_{n+1}$ is the structure map of E.

The following proposition follows from the adjunction relations between the functors $i_\sharp$, i^* and i_* on based spaces.

Proposition 2.13. *The functor $i_\sharp : Spectra(L) \to Spectra(k)$ is the left adjoint to $i^* : Spectra(k) \to Spectra(L)$, and the functor $i_* : Spectra(L) \to Spectra(k)$ is the right adjoint to i^*.*

Remark: In the equivariant category, the analogues of Proposition 2.8 are the homeomorphisms of G-spaces

$$G_+ \wedge_H (Y \wedge X) \cong (G_+ \wedge_H Y) \wedge X$$
$$F_H(G_+, F(X,Y)) \cong F(X, F_H(G_+, Y))$$

for X a based G-space and Y a based H-space (see [**9**]). As we will see in Section 5, the analogues of these statements in the theory of derived categories of sheaves into an abelian category can also be stated, as a version of Verdier duality [**1, 8**].

By the composition of adjoints, $(i_\sharp i^*, i_* i^*)$ form a adjoint pair of functors from $Spectra(k)$ to $Spectra(k)$. For $X \in Spc(k)_\bullet$, let

$$F(X, -) : Spectra(k) \to Spectra(k)$$

be the function spectrum functor, i. e. the right adjoint of the functor $X \wedge - : Spectra(k) \to Spectra(k)$. The following corollary is analogous to the fact that in equivariant topology, for a G-spectrum E, we have $G_+ \wedge_H E \cong (G/H)_+ \wedge E$, and $F_H(G_+, E) \cong F((G/H)_+, E)$.

Corollary 2.14. *For a k-spectrum E, we have isomorphisms of k-spectra*

$$i_\sharp i^* E \cong Spec(L)_+ \wedge E$$
$$i_* i^* E \cong F(Spec(L)_+, E).$$

Proof. Since the k-spectrum $Spec(L)_+ \wedge E$ is defined spacewise, it suffices to show that $i_\sharp i^* X \cong Spec(L)_+ \wedge X$ for a based k-space X. We have that for $S^0_L = Spec(L) \amalg Spec(L)$ in $Spc(L)_\bullet$, $i_\sharp S^0_L = Spec(L)_+ = Spec(L) \amalg Spec(k)$ in $Spc(K)_\bullet$. Thus, applying Proposition 2.8 to S^0_L gives the first statement. The second statement follows from the uniqueness of adjoints, and the fact that $F(Spec(L)_+, -)$ is the right adjoint to $Spec(L)_+ \wedge -$. □

Remarks:
1. In general, for any smooth Noetherian scheme S of finite dimension over k, consider the category of $Spc(S)$ of S-spaces. For any smooth map of schemes over k

$$f : S' \to S$$

the functors $f_\sharp$, f^* and f_* between the categories $Spc(S)$ and $Spc(S')$ can be defined as for an extension of fields. The based and stable versions of these functors are also defined accordingly. If f is smooth and finite, then f^* is the same as the inverse image functor. Its left adjoint $f_\sharp$ is the "extension by zero" functor constructed in Proposition 3.1.23 of [**11**], and its right adjoint f_* is the direct image functor.
2. If f is smooth and finite, such as $f : Spec(L) \to Spec(k)$, then by Corollary 3.1.24 and Proposition 3.2.9 of [**11**], f^* preserves simplicial and $\mathbb{A}^1$-weak equivalences. Also, By Propositions 3.2.9 and 3.2.12 of [**11**], the left derived functor of $f_\sharp$ and the right derived functor of f_* in the simplicial homotopy categories preserve $\mathbb{A}^1$-weak equivalences. Thus, $f_\sharp$ preserves $\mathbb{A}^1$-weak equivalences between simplicially cofibrant objects, and f_* preserves $\mathbb{A}^1$-weak equivalences between simplicially fibrant objects.

3. An application to $Pic(\mathcal{SH}(k))$

We give an application of the base change functors towards constructing elements of the Picard group $Pic(\mathcal{SH}(k))$ of the stable homotopy category over k. This is the group of objects that are invertible with respect to the smash product. For instance, there are two versions of the circle in $Spc(k)_\bullet$

$$S^1_s = \mathbb{A}^1/\{0,1\}$$
$$\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}.$$

We have that in $Spc(k)_\bullet$,

$$S^1_s \wedge \mathbb{G}_m \simeq \mathbb{P}^1.$$

Hence, in $\mathcal{SH}(k)$, S^1_s and $\mathbb{G}_m$ are both in $Pic(\mathcal{SH}(k))$. In this sense, $\mathbb{P}^1$ is a "mixed 2-sphere".

There are several known classes of elements in $Pic(\mathcal{SH}(k))$ not generated by S^1_s and $\mathbb{G}_m$ [**5**]. For example, let $a \in k^\times$, $a \notin (k^\times)^2$. The affine variety S^a given by the equation $x^2 - ay^2 = 1$ has the property that

$$S^a \wedge \widetilde{Spec(k[\sqrt{a}])} \cong \mathbb{P}^1$$

where $\widetilde{\ }$ denotes the unreduced suspension. Thus, S^a and $\widetilde{Spec(k[\sqrt{a}])}$ are in $Pic(\mathcal{SH}(k)$, and motivic cohomology calculations show that they are not in the subgroup generated by S^1_s and S^1_t.

We would like to find other ways of constructing these objects. One such way is via the category of equivariant topological spaces. For this section, let L be a finite Galois extension of k, and $G = Gal(L/k)$. For the category $Sh(Sm/k)_{Nis}$ of sheaves on the site Sm/k with the Nisnevich topology, we define a functor

$$F_{L/k} : \text{Finite } G\text{−sets} \to Sh(Sm/k)_{Nis}$$

which takes a homogenous G-set G/H to $Spec(L^H)$, and passes to disjoint unions. To give $F_{L/k}$ on morphisms of finite G-sets, it suffices to give $F_{L/k}(\alpha)$ for any G-equivariant map $G/H \to G/K$, where H and K are subgroups of G. By adjunction, we have that the nonequivariant space of G-equivariant maps from G/H to G/K is naturally isomorphic to the nonequivariant space of H-equivariant maps $* \to G/K$, where the single point $*$ is thought of as a fixed H-space. In turn, this is the same as the space of nonequivariant maps $* \to (G/K)^H$, i. e. the nonequivariant space $(G/K)^H$ of the H-fixed points of the homogenous G-set G/K. We have that

$$(G/K)^H = \{gK \ | g^{-1}Hg \subseteq K\}.$$

In particular, it is empty if H is not subconjugate to K. But $G = Gal(L/k)$, so an element $g \in G$ gives $g : L \to L$. If $g^{-1}Hg \subseteq K$, then for any $x \in L^K$ and $h \in H$, we have that $g^{-1}hg(x) = x$, so $hg(x) = g(x)$. Thus, such a g takes L^K to L^H. Also, for every $k \in K$, we have that $gk|_{L^K} = g|_{L^K}$. So for any $gK \in (G/K)^H$, we get a well-defined map $L^K \to L^H$, which gives

$$F_{L/k}(gK) : F_{L/k}(G/H) = Spec(L^H) \to Spec(L^K) = F_{L/k}(G/K).$$

Recall that the category of k-spaces is just the category of simplicial sheaves on the site Sm/k with the Nisnevich topology. So taking the simplicial categories of both the source and the target, $F_{L/k}$ extends to a functor

$$F_{L/k} : \text{Finite } G\text{−simplicial sets} \to Spc(k). \tag{3.1}$$

Since $F_{L/k}$ takes a single fixed point to $Spec(k)$, it also passes to a functor from the category of based G-simplicial sets to $Spc(k)_\bullet$. Namely, a based G-simplicial set is a G-simplicial set X, together with a G-map $i_X : * \to X$, where $* = G/G$ is a single fixed point. We have that $F_{L/k}(*) = Spc(L^G) = Spec(k)$, since the extension L/k is Galois. Hence, applying $F_{L/k}$ to i_X gives a map $Spec(k) \to F_{L/k}(X)$, which makes $F_{L/k}(X)$ a based space over k. Also, if X is a G-space with a triangulation, then applying $F_{L/k}$ to the simplicial model of X gives $F_{L/k}(X)$ as a k-space. Two different triangulations of X are simplicially equivalent, so the k-space $F_{L/k}(X)$ is well-defined up to simplicial weak equivalences. When there is no possibility of confusion, we write just F for $F_{L/k}$.

Suppose L is a finite Galois extension of k, H is a subgroup of $G = Gal(L/k)$, and $E = L^H$. Then the functors $F_{L/E}$ from H-equivariant spaces to $Spc(E)$ and $F_{L/k}$ from G-equivariant spaces to $Spc(k)$ are related in the following manner. Let $i : Spec(E) \to Spec(k)$ be the map corresponding to the inclusion $k \subseteq E$, and let U denote the forgetful functor from G-equivariant topological spaces to H-equivariant topological spaces. In equivariant topology, recall that there is a natural equivalence of categories between the category of G-equivariant spaces over the homogenous G-space G/H and the category of H-equivariant spaces. Let $f : G/H \to *$ be the map collapsing G/H to a single fixed point. Then we have a functor f^* from G-spaces to G-spaces over G/H, given by $f^*(T) = G/H \times T$. Then the diagram of functors

$$\begin{tikzcd} & G\text{−spaces} \arrow[dl, "U"'] \arrow[dr, "f^*"] & \\ H\text{−spaces} \arrow[rr, "\simeq", "G\times_H -"'] & & G\text{−spaces over } G/H \end{tikzcd} \tag{3.2}$$

commutes. An analogous equivalence of categories hold between the categories of G-simplicial sets over G/H and H-simplicial sets.

Now for $G = Gal(L/k)$, $F_{L/k}(G/H) = Spec(L^H) = Spec(E)$. So by passing to comma categories, $F_{L/k}$ induces a functor from G-spaces over G/H to k-spaces with a structure map to $Spec(E)$, i. e. $Spc(E)$. We will denote this functor on comma categories also by $F_{L/k}$. If N is a subgroup of H, then

$$F_{L/E}(H/N) = Spec(L^N) = F_{L/k}(G/N).$$

But the homogenous G-space G/N has a natrural map to G/H, and it is $G \times_H (H/N)$ as a G-space over G/H. Hence, $F_{L/E}$ is the same as the composition

$$H\text{−spaces} \xrightarrow{G\times_H -} G\text{−spaces over } G/H \xrightarrow{F_{L/k}} Spc(E). \tag{3.3}$$

Lemma 3.4. *In the above situation, the diagram of functors*

$$\begin{tikzcd} G\text{−simplicial sets} \arrow[r, "F_{L/k}"] \arrow[d, "U"'] & Spc(k) \arrow[d, "i^*"] \\ H\text{−simplicial sets} \arrow[r, "F_{L/E}"'] & Spc(E) \end{tikzcd}$$

commutes up to natural equivalence.

Proof. By diagram (3.2) and (3.3), it suffices to show that the diagram of functors

$$\begin{tikzcd} G\text{−simplicial sets} \arrow[r, "F_{L/k}"] \arrow[d, "f^*"'] & Spc(k) \arrow[d, "i^*"] \\ G\text{−simplicial sets over } G/H \arrow[r, "F_{L/k}"'] & Spc(E) \end{tikzcd}$$

commutes up to natural equivalence. Since f^* and i^* commute with simplicial structures, it suffices to show this for a homogenous G-set G/K, where K is a subgroup of G. We have that $F_{L/k}(G/K) = Spec(L^K)$, so

$$i^* F_{L/k}(G/K) = Spec(E) \times_{Spec(k)} Spec(L^K)$$

as an E-space. On the other hand, $f^*(G/K) = G/H \times G/K$, which maps to G/H by collapsing G/K. For any $(g_1H, g_2K) \in G/H \times G/K$, an element $g \in G$ fixes (g_1H, g_2K) if and only if $g \in H \cap K$. Thus, the isotropy subgroup of every point of $G/H \times G/K$ is $H \cap K$, i. e. as a G-set,

$$G/H \times G/K \cong \coprod G/(H \cap K)$$

is the disjoint union of n copies of $G/(H \cap K)$, where $n = ([G : H][G : K])/[G : H \cap K]$. Thus,

$$F_{L/k} f^*(G/K) = \coprod Spec(L^{H \cap K}) = \coprod Spec(EL^K).$$

This is naturally isomorphic to $Spec(E) \times_{Spec(k)} Spec(L^K)$ since the extension L over k is Galois.

□

Consider a finite-demensional real representation V of the group G. We will denote the unit sphere of V by $S(V)$, and the one-point compactification of V by S^V. The following theorem give a class of invertible objects in $\mathcal{SH}(k)$.

Theorem 3.5. *For V a finite-dimensional real representation of $G = Gal(L/k)$, $F_{L/k}(S^V)$ is invertible in $\mathcal{SH}(k)$.*

To prove the theorem, we introduce the notion of join powers and smash powers of a k-space X to the power of T, where T is an étale scheme over $Spec(k)$. Recall that for topological spaces X and Y, the join $X * Y$ is the homotopy pushout of the diagram

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X * Y \end{tikzcd}$$

where $X \times Y$ maps to X and Y by the projections. For k-spaces X and Y, their join $X * Y$ is defined in the same way. In particular, a model for $X * Y$ is the quotient space of $X \times Y \times \mathbb{A}^1$, obtained by collapsing $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. Also, recall that the unreduced suspension $\widetilde{X}$ of a k-space X is defined by the cofiber sequence

$$X_+ \to S^0 \to \widetilde{X}.$$

The join product has the property that

$$\widetilde{X * Y} \simeq \widetilde{X} \wedge \widetilde{Y}.$$

Now as a simplicial set, the 1-simplex in the topological category realizes to the unit interval I, whereas the 1-simplex in the category $Spc(k)$ realizes to the affine line $\mathbb{A}^1$. Thus, for a G-space X with a triangulation, $F_{L/k}(X \times I) = F_{L/k}(X) \times \mathbb{A}^1$. Also, as shown in the proof of Lemma 3.4, for subgroups H and K of G,

$$F_{L/k}(G/H \times G/K) \cong F_{L/k}(G/H) \times F_{L/k}(G/K)$$

naturally. Passing to the simplicial categories, we get that for G-equivariant spaces X and Y with triangulations,

$$F_{L/k}(X \times Y) \simeq F_{L/k}(X) \times F_{L/k}(Y)$$

naturally. Using these facts, and the fact that $F_{L/k}$ preserves pushouts, it is easy to check that F commutes with the join product and the unreduced suspension. Also, the based version of $F_{L/k}$ commutes with the smash product. This is because for based G-spaces X and Y, $X \wedge Y = (X \times Y)/(X \vee Y)$, and similarly for based k-spaces. Since $X \vee Y$ is a pushout of the basepoint maps $* \to X$ and $* \to Y$, and $F_{L/k}(*) = Spec(k)$, we get that $F_{L/k}(X \vee Y) = F_{L/k}(X) \vee_{Spec(k)} F_{L/k}(Y)$. But $F_{L/k}$ also preserves products and quotient by a subspace, so it preserves the smash product.

For the rest of this section, we will abbreviate $F_{L/k}$ to just F. For $X \in Spc(k)$, and $T \to Spec(k)$ étale, X^{*T} is an analogue of the join power X^{*n}, which takes into account the "Galois action" on the parametrizing k-space T. Likewise, if $X \in Spc(k)_\bullet$, we have the smash power $X^{\wedge T}$. We will defer the exact definitions of the join and smash powers to T to Secton 5. The properties of the usual join and smash powers apply to $(-)^{*T}$ and $(-)^{\wedge T}$. For instance, similarly as in the case of X^{*n} and $X^{\wedge n}$, the join and smash powers to T has the property that for $X \in Spc(k)$,

$$\widetilde{X^{*T}} \simeq (\widetilde{X})^{\wedge T}.$$

Likewise, for a G-space X and a G-set T_G, we can define X^{*T_G}. From the definitions of $(-)^{*T_G}$ and $(-)^{*T}$, we will see in Section 5 that if $G = Gal(L/k)$, and X is a triangulated G-space, then there is a weak equivalence of k-spaces

$$F(X^{*T_G}) \simeq (F(X))^{*F_{L/k}(T_G)}.$$

Let $\mathbb{R}[G]$ be the regular real representation of G. We have the following lemma.

Lemma 3.6. *For the triagulated G-space $S(\mathbb{R}[G])$, we have a natural isomorphism*

$$F(S(\mathbb{R}[G])) \cong (S^0)^{*Spec(L)}.$$

Proof. It suffices to show that $S(\mathbb{R}[G])$ is naturally isomorphic to $(S^0)^{*G}$, where S^0 is $* \amalg *$ in the G-equivariant category. In particular, $S^0 = S(\mathbb{R})$ is the unit sphere of the one-dimensional trivial representation $\mathbb{R}$ of G. For the nonequivariant join product, we have that for two representations V and W of G, there is a natural G-equivariant homeomorphism $S(V) * S(W) \cong S(V \oplus W)$. Similarly, for $(-)^{*G}$, one has that

$$S(\mathbb{R})^{*G} \cong S(\mathbb{R}^G)$$

naturally as G-equivariant spaces. But $\mathbb{R}^G$ is just $\mathbb{R}[G]$ as a G-representation. $\square$

We also have the following lemma, whose proof we defer to Section 5.

Lemma 3.7. *For an étale scheme T over k, the functor $(-)^{\wedge T}$ has the property that for $X, Y \in Spc(k)_\bullet$,*

$$X^{\wedge T} \wedge Y^{\wedge T} \simeq (X \wedge Y)^{\wedge T}.$$

Proof of Theorem 3.5. Since $F(S(\mathbb{R}[G])) \simeq (S^0)^{*Spec(L)}$, and F preserves unreduced suspensions, by taking the unreduced suspension of both sides, we get that

$$F(S^{\mathbb{R}[G]}) \simeq \widetilde{(S^0)^{*Spec(L)}} \simeq (S^1_s)^{\wedge Spec(L)}$$

since $S^1_s = \widetilde{S^0}$ by definition. Recall also from [**11**] that there is a $\mathbb{A}^1$-weak equivalence of k-spaces

$$S^1_s \wedge \mathbb{G}_m \simeq \mathbb{P}^1.$$

So by Lemma 3.7,

$$(S^1_s)^{\wedge Spec(L)} \wedge (\mathbb{G}_m)^{\wedge Spec(L)} \simeq (\mathbb{P}^1)^{\wedge Spec(L)}.$$

Also, note that

$$(\mathbb{A}^1_k)^{Spec(L)} = i_* i^* \mathbb{A}^1_k = i_* \mathbb{A}^1_L = \mathbb{A}^n_k.$$

But we also have an $\mathbb{A}^1$-homotopy equivalence $\mathbb{P}^1 \simeq \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ ([**11**]). Thus,

$$\begin{aligned}
(\mathbb{P}^1)^{\wedge Spec(L)} &\simeq (\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\})^{\wedge Spec(L)} \\
&\cong (\mathbb{A}^1)^{Spec(L)}/(\mathbb{A}^1)^{Spec(L)} \setminus \{0\} \\
&\cong (\mathbb{A}^1)^n/(\mathbb{A}^1)^n \setminus \{0\} \\
&\simeq (\mathbb{P}^1)^{\wedge n}.
\end{aligned}$$

This is invertible in $\mathcal{SH}(k)$, so $F(S^{\mathbb{R}[G]}) = (S^1_s)^{\wedge Spec(L)}$ is invertible in $\mathcal{SH}(k)$. But for any irreducible representation V, V is a direct summand of $\mathbb{R}[G]$, so $F(S^V)$ is a smash summand of $F(S^{\mathbb{R}[G]})$, and is therefore also invertible in $\mathcal{SH}(k)$. Finally, for any finite-dimensional representation V of G, we have $V = \oplus_{i=1}^n V_i$ as a finite direct sum of irreducible representations V_i of G, so S^V is a finite smash product of S^{V_i}'s. Therefore, $F(S^V)$ is in $Pic(\mathcal{SH}(k))$ for any finite-dimensional representation V of G. $\square$

Example: Let L be a cyclic extension of degree p over k, and let γ be a generator of $Gal(L/k) = \mathbb{Z}/p$. Define $\mathcal{O}_+$ to be the cofiber in the stable homotopy category over k of the map

$$Spec(L)_+ \xrightarrow{Id-\gamma} Spec(L)_+.$$

It is the suspension spectrum of the homotopy coequalizer $\mathcal{O}$ of the maps

$$Id,\ \gamma : Spec(L) \to Spec(L)$$

together with a disjoint basepoint. Consider the 2-dimensional real representation V of $\mathbb{Z}/2$ given by multiplication by $e^{2\pi i/p}$ in $\mathbb{R}^2$. The simpicial decompositions of $S(V)$ is that it has one 0-cell $\mathbb{Z}/p \times *$, and one 1-cell $\mathbb{Z}/p \times I$. By the definition of the homotopy coequalizer, we see that the simplicial definition of $\mathcal{O}$ is the same, with $\mathbb{Z}/p$ replaced by $Spec(L)$, and I replaced by $\mathbb{A}^1_k$. Since $F(\mathbb{Z}/p) = Spec(L)$, we get

$$F(S(V)) = \mathcal{O}.$$

Hence, $\mathcal{O} = F(S^V)$ is invertible in $\mathcal{SH}(k)$. (More on $\mathcal{O}$ will be in [**7**].)

4. The Wirthmüller Isomorphism

The main result of this section is the following theorem, which is an analogue of the Wirthmüller isomorphism for the $\mathbb{A}^1$-setting ([**15**], see also [**9, 4**]).

Theorem 4.1. *For L a finite separable extension of k, and E an L-spectrum, we have a natural $\mathbb{A}^1$-weak equivalence of k-spectra*

$$i_\sharp E \simeq i_* E.$$

We will give an explicit construction of the equivalence, in terms of the smash-invertible objects considered in the last section.

We begin by definition a natural map of k-spectra

$$\psi : i_\sharp E \to i_* E. \tag{4.2}$$

By adjunction, this is equivalent to a map of L-spectra

$$\overline{\psi} : i^* i_\sharp E \to E. \tag{4.3}$$

Let X be a based L-space. Let

$$F_{Spectra(L)}(X,-) : Spectra(L) \to Spectra(L)$$

denote the right adjoint to the functor $X \wedge - : Spectra(L) \to Spectra(L)$.

To construct $\overline{\psi}$, we will first consider $i^* i_\sharp$ on the level of based spaces. Let X be a based space over $Spec(L)$, with basepoint $i_X : Spec(L) \to X$ and structure map $p_X : X \to Spec(L)$. We have that $i_\sharp(X) = X/_{Spec(k)}Spec(L)$, where $/_{Spec(k)}$ denotes taking quotient of i_X in the category of spaces over $Spec(k)$. So

$$\begin{aligned} i^* i_\sharp(X) &= Spec(L) \times_{Spec(k)} (X/_{Spec(k)}Spec(L)) \\ &\cong (Spec(L) \times_{Spec(k)} X)/_{Spec(L)}(Spec(L) \times_{Spec(k)} Spec(L)) \end{aligned}$$

since i^* commutes with pushouts as a left adjoint. Note that on the right hand side, both $Spec(L) \times_{Spec(k)} X$ and $Spec(L) \times_{Spec(k)} Spec(L)$ are spaces over $Spec(L)$ via the first projection map.

We have the following property of $Spec(L) \times_{Spec(k)} X$.

Lemma 4.4. *If X is a space over $Spec(L)$, then for $Spec(L) \times_{Spec(k)} X$ as a space over $Spec(L)$ via the first projection, consider the embedding over $Spec(L)$*

$$\Delta_X = p_X \times_{Spec(k)} Id : X \to Spec(L) \times_{Spec(k)} X$$

which is the structure map of X on the first coordinate, and the identity on X on the second. Then we have

$$Spec(L) \times_{Spec(k)} X = \Delta_X \amalg ((Spec(L) \times_{Spec(k)} X) \setminus \Delta_X)$$

as spaces over $Spec(L)$, where $(Spec(L) \times_{Spec(k)} X) \setminus \Delta_X$ is a space over $Spec(L)$ by the first projection.

Proof. For the case when X is an affine scheme of finite type over $Spec(L)$, say $X = Spec(R)$ for a finitely generated L-algebra R, we have that $Spec(L) \times_{Spec(k)} X = Spec(L \otimes_k R)$, and the map Δ_X corresponds to the map of L-algebras

$$L \otimes_k R \to R$$

which is the multiplication. Hence, passing to the level of L-algebras, we get that the lemma holds in the case when X is an affine scheme of finite type over $Spec(L)$.

For general X, recall that every space X over $Spec(L)$ is a colimit of affine schemes of finite type, say $X = \mathrm{colim}_i X_i$ over $Spec(L)$, where each $X_i = Spec(R_i)$ for some finitely generated L-algebra R_i. Then

$$Spec(L) \times_{Spec(k)} X \cong \mathrm{colim}_i\; Spec(L) \times_{Spec(k)} X_i$$

naturally, since $Spec(L) \times_{Spec(k)} - = i^*$ commutes with colimits. Suppose $f : X_i \to X_j$ is a map of the colimit. Then the diagram

$$\begin{tikzcd} X_i \arrow[r, "\Delta_{X_i}"] \arrow[d, "f"'] & Spec(L) \times_{Spec(k)} X_i \arrow[d, "Spec(L)\times_{Spec(k)} f"] \\ X_j \arrow[r, "\Delta_{X_j}"'] & Spec(L) \times_{Spec(k)} X_j \end{tikzcd} \tag{4.5}$$

commutes, since f is a map over $Spec(L)$. Hence, Δ_X is

$$\mathrm{colim}_i \Delta_i : X = \mathrm{colim}_i X_i \to \mathrm{colim}_i (Spec(L) \times_{Spec(k)} X_i) = Spec(L) \times_{Spec(k)} X.$$

Also, by passing to L-algebras, it is straightforward to check that diagram (4.5) is in fact a pullback square. So the map

$$Spec(L) \times_{Spec(k)} f : Spec(L) \times_{Spec(k)} X_i \to Spec(L) \times_{Spec(k)} X_j$$

restricts to $\Delta_{X_i}(X_i) \to \Delta_{X_j}(X_j)$ and

$$(Spec(L) \times_{Spec(k)} X_i) \setminus \Delta_{X_i}(X_i) \to (Spec(L) \times_{Spec(k)} X_j) \setminus \Delta_{X_j}(X_j)$$

and the diagram

$$\begin{tikzcd}
Spec(L) \times_{Spec(k)} X_i \arrow[r, "\cong"] \arrow[d, "Spec(L)\times_{Spec(k)} f"] & \Delta_{X_i}(X_i) \amalg ((Spec(L) \times_{Spec(k)} X_i) \setminus \Delta_{X_i}(X_i)) \arrow[d] \\
Spec(L) \times_{Spec(k)} X_j \arrow[r, "\cong"'] & \Delta_{X_j}(X_j) \amalg ((Spec(L) \times_{Spec(k)} X_j) \setminus \Delta_{X_j}(X_j))
\end{tikzcd}$$

commutes. So passing to colimits, we get that

$$Spec(L) \times_{Spec(k)} X \cong (\mathrm{colim}_i \Delta_{X_i}(X_i)) \amalg (\mathrm{colim}_i (Spec(L) \times_{Spec(k)} X_i) \setminus \Delta_{X_i}(X_i))$$

naturally. We have $\mathrm{colim}_i \Delta_{X_i}(X_i) \cong \Delta_X(X)$, so $\mathrm{colim}_i (Spec(L) \times_{Spec(k)} X_i) \setminus \Delta_{X_i}(X_i)$ is $(Spec(L) \times_{Spec(k)} X) \setminus \Delta_X(X)$. $\square$

This allows us to define a map of unbased spaces over $Spec(L)$

$$\overline{\psi}_u : Spec(L) \times X \cong \Delta_X(X) \amalg ((Spec(L) \times_{Spec(k)} X) \setminus \Delta_X(X)) \to X.$$

This is the identity on $\Delta_X(X) \cong X$, and on the other component, it is

$$(Spec(L) \times_{Spec(k)} (X) \setminus \Delta_X(X) \xrightarrow{\pi_1} Spec(L) \xrightarrow{i_X} X.$$

For $X \in Spc(L)_\bullet$, we define

$$\overline{\psi} : i^* i_\sharp(X) \cong (Spec(L) \times_{Spec(k)} X)/_{Spec(L)}(Spec(L) \times_{Spec(k)} Spec(L)) \to X$$

to be induced from $\overline{\psi}_u$. To check that this is a well-defined map in $Spc(L)_\bullet$, we need that $\overline{\psi}_u$ maps $Spec(L) \times_{Spec(k)} Spec(L)$ into the basepoint of X, i. e. the diagram

$$\begin{tikzcd}
Spec(L) \times_{Spec(k)} Spec(L) \arrow[r, "Id\times i_X"] \arrow[d, "\pi_1"'] & Spec(L) \times_{Spec(k)} X \arrow[d, "\overline{\psi}_u"] \\
Spec(L) \arrow[r, "i_X"'] & X
\end{tikzcd} \tag{4.6}$$

commutes. This follows because we have

$$\begin{aligned} &Spec(L) \times_{Spec(k)} Spec(L) \\ &= \Delta_{Spec(L)}(Spec(L)) \amalg ((Spec(L) \times_{Spec(k)} Spec(L)) \setminus \Delta_{Spec(L)}(Spec(L))) \end{aligned}$$

and

$$Spec(L) \times_{Spec(k)} X = \Delta_X(X) \amalg ((Spec(L) \times_{Spec(k)} X) \setminus \Delta_X(X)).$$

It is straightforward to check that diagram (4.6) commutes on $\Delta_{Spec(L)}(Spec(L))$ and on $(Spec(L) \times_{Spec(k)} Spec(L)) \setminus \Delta_{Spec(L)}(Spec(L))$.

Passing to the stable categories, we can now define $\overline{\psi}$ for a spectrum E over $Spec(L)$

$$i^* i_\sharp(E) \to E$$

to be given by first applying spacewise $\overline{\psi}$ for based spaces, then taking the spectrification functor. To check that applying $\overline{\psi}$ spacewise gives a map of L-prespectra, we need that the diagram

$$\begin{tikzcd}
\Sigma^{\mathbb{P}^1_L}(i^* i_\sharp(E_n)) \arrow[r, "\Sigma^{\mathbb{P}^1_L}\overline{\psi}"] \arrow[d, "\cong"'] & \Sigma^{\mathbb{P}^1_L} E_n \arrow[d, "="] \\
i^* i_\sharp \Sigma^{\mathbb{P}^1_L} E_n \arrow[d, "i^* i_\sharp \rho_n"'] & \Sigma^{\mathbb{P}^1_L} E_n \arrow[d, "\rho_n"] \\
i^* i_\sharp E_{n+1} \arrow[r, "\overline{\psi}"'] & E_{n+1}
\end{tikzcd}$$

commutes, where $\rho : \Sigma^{\mathbb{P}^1_L} E_n \to E_{n+1}$ is the structure map of E. This follows since ρ is a map over $Spec(L)$. This gives the map ψ of (4.2).

To define the inverse to ψ, we use the constructions of [**9**]. Let E be the normal closure of L, and let $G = Gal(E/k)$, and $H = Gal(E/L)$. Consider the functor F given by (3.1) from the category of finite G-simplicial sets to the category of k-spaces. In particular, $F((G/H)_+) = Spec(E^H)_+ = Spec(L)_+$. Let V be a finite-dimensional representation of G. such that G/H embeds in V. This embedding extends to an open tubular neighborhood $U$of G/H in V. Now the quotient space $S^V/(S^V \setminus U)$ is the Thom space of the normal bundle of the embedding of G/H_+ in S^V, so it is G-equivariantly homotopy equivalent to $(G/H)_+ \wedge S^V$. Thus, we have a Pontrjagin-Thom map

$$t : S^V \to S^V/(S^V \setminus U) \simeq (G/H)_+ \wedge S^V$$

in the category of G-equivariant spaces. Taking simplicial approximation and applying the functor F and then the suspension spectrum functor to t gives a map of k-spectra

$$F(t) : F(S^V) \to Spec(L)_+ \wedge F(S^V).$$

Now let D be any k-spectrum, we define a pretransfer map

$$\bar{t} : D \to Spec(L)_+ \wedge D$$

as follows. By Theorem 3.5, $F(S^V)$ is invertible in the stable homotopy category $\mathcal{SH}(k)$ over k. By formal arguments, in $\mathcal{SH}(k)$, $F(S^V)^{-1}$ must be $DF(S^V) = F(S^0, S^V)$, the Spanier-Whitehead dual of $F(S^V)$. In fact, bythe proof of Theorem 3.5, we have a rigid model of $F(S^V)^{-1}$ in the category $Spectra(k)$, which underlies $\mathcal{SH}(k)$. Namely, let $G = Gal(E/k)$, and let $\mathbb{R}[G] - V$ be the complement of V in the regular representation $\mathbb{R}[G]$ of V, and let $\mathbb{G}_m$ be the multiplicative group over k. Then we have a natural $\mathbb{A}^1$-weak equivalence

$$F(S^V) \wedge F_{E/k}(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)} \cong (\mathbb{P}^1_k)^n$$

where $n = [E : k]$. Thus, we can define the model of $F(S^V)^{-1}$ to be

$$\Sigma^{-n}\Sigma^\infty(F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)}). \tag{4.7}$$

Here, $\Sigma^{-n}\Sigma^\infty$ denotes the n-th shift desuspension of the suspension spectrum of the space over k. For the smash product of spectra $F(S^V) \wedge F(S^V)^{-1}$, we choose any linear injection $\alpha : \mathcal{U}^{\oplus 2} \to \mathcal{U}$, where $\mathcal{U} \cong \mathbb{A}^\infty_k$ is the universe over k [**6**]. Then using the rigid model (4.7) of $F(S^V)^{-1}$, and α for the smash product of spectra, similarly as in topology, we get a natural homotopy equivalence

$$F(S^V) \wedge F(S^V)^{-1} \simeq \Sigma^{-n}\Sigma^\infty(F(S^V) \wedge F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(L)}) \cong S^0_k. \tag{4.8}$$

In particular, in $\mathcal{SH}(k)$, we have the coevaluation map

$$c : S^0_k \to F(S^V)^{-1} \wedge F(S^V)$$

and the evaluation map

$$e : F(S^V)^{-1} \wedge F(S^V) \to S^0_k.$$

Then e and c are inverse isomorphisms in $\mathcal{SH}(k)$. Again, we define rigid models in *Spectra*(k) for c and e, using (4.7) for $F(S^V)^{-1}$. Namely, define the rigid coevaluation map to be

$$\begin{aligned} c : S^0_k &\cong \Sigma^{-n}\Sigma^\infty(\mathbb{P}^1)^{\wedge n} \\ &\xrightarrow{\simeq} \Sigma^{-n}\Sigma^\infty(F(S^V) \wedge (F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)})) \\ &\xrightarrow{\simeq} F(S^V) \wedge \Sigma^{-n}\Sigma^\infty(F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)}) \\ &= F(S^V) \wedge F(S^V)^{-1}. \end{aligned} \tag{4.9}$$

Similarly, the rigid model of the evaluation map is

$$\begin{aligned} e : F(S^V) \wedge F(S^V)^{-1} &= F(S^V) \wedge \Sigma^{-n}\Sigma^\infty(F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)}) \\ &\xrightarrow{\simeq} \Sigma^{-n}\Sigma^\infty(F(S^V) \wedge (F(S^{\mathbb{R}[G]-V}) \wedge (\mathbb{G}_m)^{\wedge Spec(E)})) \\ &\xrightarrow{\simeq} \Sigma^{-n}\Sigma^\infty(\mathbb{P}^1)^{\wedge n} \\ &\cong S^0_k. \end{aligned} \tag{4.10}$$

Then the rigid models of c and e pass to c and e in $\mathcal{SH}(k)$, so they are inverse $\mathbb{A}^1$-weak equivalences in *Spectra*(k). We define

$$\begin{aligned} \bar{t} : D &\xrightarrow{c} F(S^V)^{-1} \wedge D \wedge F(S^V) \\ &\xrightarrow{F(t)} F(S^V)^{-1} \wedge D \wedge (Spec(L)_+ \wedge F(S^V)) \\ &\xrightarrow{\tau} Spec(L)_+ \wedge (F(S^V)^{-1} \wedge D \wedge F(S^V)) \\ &\xrightarrow{e} Spec(L)_+ \wedge D \\ &\cong i_\sharp i^* D. \end{aligned}$$

The last (natural) isomorophism is by Corollary 2.14. Also, the transposition map τ is a natural homotopy equivalence, since it is naturally homotopic to taking $\Sigma^{-n}\Sigma^{\infty}$ of a transposition of spaces. By arguments similar to that of [**9**], when we pass to $\mathcal{SH}(k)$, $\bar{t}$ is independent of the choice of V.

Let $D = i_*E$ for an L-spectrum E. We define a map ω of k-spectra by

$$\omega : i_*E \xrightarrow{\bar{t}} i_\sharp i^*(i_*E) \xrightarrow{i_\sharp \epsilon} i_\sharp E$$

where the second map ϵ is the counit of the adjunction (i^*, i_*).

To prove Theorem 4.1, we first make the following reduction.

Lemma 4.11. *If $\psi : i_\sharp E \to i_*E$ is an $\mathbb{A}^1$-weak equivalence for all L-spectra E that are shift desuspensions of suspension spectra, then it is an $\mathbb{A}^1$-weak equivalence for all L-spectra.*

Proof. For any L-spectrum $E = \{E_n\}$, we have a canonical $\mathbb{A}^1$-weak equivalence

$$E \simeq \mathrm{colim}_n \Sigma^{-n}\Sigma^{\infty}E_n$$

(see [**6**]). Since $i_\sharp : Spectra(L) \to Spectra(k)$ is a left adjoint, it commutes with all small colimits. Also, recall that any smooth scheme Y over k is an small object in $Spc(k)$, i. e.

$$Hom_{Spc(k)}(Y, \mathrm{colim}_j X_j) \cong \mathrm{colim}_j Hom_{Spc(k)}(Y, X_j)$$

for all directed systems $\{X_j\}$ of k-spaces. Now let $Y \in Sm/k$, then $Y \times_{Spec(k)} Spec(L) \in Sm/L$, so it is also small in $Spc(L)$. Hence, for any directed system $\{X_j\}$ in $Spc(L)$,

$$\begin{aligned} Hom_{Spc(k)}(Y, i_*(\mathrm{colim}_j X_j)) &\cong Hom_{Spc(L)}(Y \times_{Spec(k)} Spec(L), \mathrm{colim}_j X_j) \\ &\cong \mathrm{colim}_j Hom_{Spc(L)}(Y \times_{Spec(k)} Spec(L), X_j) \\ &\cong \mathrm{colim}_j Hom_{Spc(k)}(Y, i_*X_j) \\ &\cong Hom_{Spc(k)}(Y, \mathrm{colim}_j(i_*X_j)). \end{aligned}$$

Also, recall that every object of $Spc(k)$ is a colimit of smooth schemes (see Appendix of [**6**]). Thus, for any $Y \in Spc(k)$, we have $Y = \mathrm{colim}_r Y_r$, where each Y_r is a smooth scheme over k.

$$\begin{aligned} Hom_{Spc(k)}(Y, i_*(\mathrm{colim}_j X_j)) &\cong \lim_r Hom_{Spc(k)}(Y_r, i_*(\mathrm{colim}_j X_j)) \\ &\cong \lim_r Hom_{Spc(k)}(Y_r, \mathrm{colim}_j(i_*X_j)) \\ &\cong Hom_{Spc(k)}(Y, \mathrm{colim}_j(i_*X_j)) \end{aligned}$$

So $i_* : Spc(L) \to Spc(k)$ commutes with all small directed colimits. The same holds for the case of based spaces. Since colimits of spectra are formed spacewise, $i_* : Spectra(L) \to Spectra(k)$ also commutes with all small directed colimits. Further, since $i : Spec(L) \to Spec(k)$ is a smooth finite morphism, by Propositions 3.2.9 and 3.2.12 of [**11**], $i_\sharp$ and i_* preserve $\mathbb{A}^1$-weak equivalences. Thus, we have canonical

$\mathbb{A}^1$-weak equivalences of k-spectra

$$i_\sharp E \simeq \mathrm{colim}_n i_\sharp(\Sigma^{-n}\Sigma^\infty E_n)$$
$$i_* E \simeq \mathrm{colim}_n i_*(\Sigma^{-n}\Sigma^\infty E_n).$$

By the naturality of ψ, the map $\psi_E : i_\sharp E \to i_* E$ is the directed colimit of the maps

$$\psi_{\Sigma^{-n}\Sigma^\infty E_n} : i_\sharp(\Sigma^{-n}\Sigma^\infty E_n) \to i_*(\Sigma^{-n}\Sigma^\infty E_n).$$

Also, recall that directed colimits of k-spectra preserve $\mathbb{A}^1$-weak equivalences, since they coincide with homotopy colimits of k-spectra ([**11, 6**]). Thus, if each $\psi_{\Sigma^{-n}\Sigma^\infty E_n}$ is an $\mathbb{A}^1$-weak equivalence, then so is ψ_E. □

The heart of the proof of Theorem 4.1 is the following lemma. Let V be a representation of $G = Gal(E/k)$ as above, and let $H = Gal(E/L)$. We will write

$$U : G\text{−spaces} \to H\text{−spaces}$$

for the forgetful functor.

Lemma 4.12. *The composition map in the category of based L-spaces*

$$\begin{aligned} i^*F_{E/k}(S^V) &\overset{i^*(F_{E/k}(t))}{\to} i^*(Spec(L)_+ \wedge F_{E/k}(S^V)) \cong i^*(i_\sharp i^* F_{E/k}(S^V)) \\ &\overset{\overline{\psi}}{\to} i^* F_G(S^V) \end{aligned} \tag{4.13}$$

is $\mathbb{A}^1$-homotopic to the identity.

Proof. Since in the category of based G-spaces, $G_+ \wedge_H S^V = G/H_+ \wedge S^V$, we have a map $u : G/H_+ \wedge S^V \to S^V$ induced by the map $G_+ \to H_+$, which maps $G \setminus H$ to the disjoint basepoint in H_+. By Lemma II.5.9 of [**9**], the following composition in the category of based G-spaces

$$S^V \overset{t}{\to} G/H_+ \wedge S^V \overset{u}{\to} S^V \tag{4.14}$$

is H-homotopic to the identity. We will show that (4.13) is $F_{E/k} \cdot f^*$ of (4.14). Since $F_{E/k}(G/H_+) = Spec(L)_+$, and $F_{E/k}$ preserves smash products, we get that

$$i^*F_{E/k}(G/H_+ \wedge S^V) = i^*(Spec(L)_+ \wedge F_{E/k}(S^V)).$$

To see that the maps are correct, recall that there is a natural equivalence of categories between H-equivariant spaces and G-equivariant spaces over G/H. If we write $f : G/H \to *$ for the collapse map, then the forgetful functor U corresponds to $f^* = G/H \times -$. By Lemma 3.4, We have that

$$F_{E/L} \cdot U \cong i^* \cdot F_{E/k} \cong F_{E/k} \cdot f^*.$$

Here, $F_{E/k}$ is thought of as a functor from the comma category of G-spaces over G/H to the comma category of k-spaces over $F_{E/k}(G/H) = Spec(L)$, i. e. L-spaces. Hence, checking the definitions of $\overline{\psi}$ and u, we get that (4.13) is indeed $F_{E/k}f^*$ applied to the sequence (4.14).

But $F_{E/k}$ is a simplicial functor, so it takes a homotopy in based G-spaces to an $\mathbb{A}^1$-homotopy in $Spc(k)_\bullet$. Also, $i^*(X \times_{Spec(k)} \mathbb{A}^1_k) = i^*X \times_{Spec(L)} \mathbb{A}^1_L$ for any k-space X, so it takes an $\mathbb{A}^1$-homotopy of k-spaces to an $\mathbb{A}^1$-homotopy of L-spaces. □

Lemma 4.15. *1. For Y a based L-space, the diagram*

$$\begin{tikzcd}
\Sigma^\infty(i^* i_\sharp Y) \arrow[r, "\cong"] \arrow[d, "\Sigma^\infty \overline{\psi}_Y"'] & i_* i_\sharp \Sigma^\infty Y \arrow[d, "\overline{\psi}_{\Sigma^\infty Y}"] \\
\Sigma^\infty Y \arrow[r, "="'] & \Sigma^\infty Y
\end{tikzcd}$$

commutes. Here, the left vertical map $\overline{\psi} Y$ is thought of as in the category of based L-spaces, and the right vertical map $\overline{\psi}_{\Sigma^\infty Y}$ is in the category of L-spectra.

2. For D a k-spectrum and Y a based L-space, the diagram

$$\begin{tikzcd}
i^*(i_\sharp(i^* D \wedge Y)) \arrow[r, "\zeta", "\cong"'] \arrow[d, "\overline{\psi}_{i^* D \wedge Y}"'] & i^*(D \wedge (i_\sharp Y)) \arrow[r, "\cong"] & i^* D \wedge (i^* i_\sharp E) \arrow[d, "Id \wedge \overline{\psi}_Y"'] \\
i^* D \wedge Y \arrow[rr, "="'] & & i^* D \wedge Y
\end{tikzcd}$$

commutes. Here, ζ is the isomorphism from Proposition 2.8.

Proof. The first part follows from the fact that the map $\overline{\psi}$ on spectra is defined spacewise. The second part follows from the analogous statement for based spaces, and stabilizing with respect to D. Here, we use the fact that the functors i^* and $i_\sharp$ commute with the spectrification functor L, so for the L-spectrum $i^* D$ and the L-space Y,

$$i^* i_\sharp (i^{*D} \wedge E) \cong L(\{i^* i_\sharp (i^* D_n \wedge Y)\})$$

naturally, where $\{i^* i_\sharp (i^* D_n \wedge Y)\}$ is the L-prespectrum obtained by applying $i^* i_\sharp$ to the L-prespectrum $\{D_n \wedge Y\}$ spacewise. $\square$

Proof of Theorem 4.1. We will show that ω and ψ are inverse $\mathbb{A}^1$-weak equivalences on all shift desuspensions of suspension spectra. For this, we use arguments similar to that of [**9**]. We first show that $\psi \cdot \omega$ is homotopic to the identity on $i_* E$, for any L-spectrum E. Let $\epsilon : i^* i_* E \to E$ denote the counit of the adjunction (i^*, i_*), then by definition, $\epsilon \cdot (i^* \psi) = \overline{\psi}$. Thus, it suffices to show that $\overline{\psi} \cdot i^* \omega$ is naturally homotopic to $\epsilon : i^* i_* E \to E$. Consider the following diagram of L-spectra.

$$\begin{tikzcd}
i^* i_* E \arrow[r, "i^* \overline{t}"] & i^* i_\sharp i^* i_* E \arrow[r, "i^* i_\sharp \epsilon"] \arrow[d, "\overline{\psi}"'] & i^* i_\sharp E \arrow[d, "\overline{\psi}"] \\
& i^* i_* E \arrow[r, "\epsilon"'] & E.
\end{tikzcd}$$

The top row of this diagram is just $i^* \omega$. By the naturality of ϵ and $\overline{\psi}$, the square commutes. Thus, it suffices to show that the composition

$$i^* i_* E \xrightarrow{i^* \overline{t}} i^* i_\sharp i^* i_* E \xrightarrow{\overline{\psi}} i^* i_* E$$

passes to the identity in the stable homotopy category $\mathcal{SH}(L)$.

We will show that the composition

$$i^*D \xrightarrow{i^*\bar{t}} i^*i_\sharp i^*D \xrightarrow{\overline{\psi}} i^*D \tag{4.16}$$

is the identity in $\mathcal{SH}(L)$ for any k-spectrum D, and apply it to $D = i_*E$. By part 1 of Lemma 4.15, Lemma 4.12 also holds for the suspension spectrum of $F(S^V)$, by stabilizing the homotopy to $\Sigma^\infty F(S^V)$. We have the following diagram of L-spectra.

$$\begin{tikzcd}
i^*D \arrow[r, "\simeq", "c"'] \arrow[dd, "i^*\bar{t}"'] & i^*F(S^V)^{-1}\wedge i^*D\wedge i^*F(S^V) \arrow[d, "i^*F(t)"'] \arrow[dr] & \\
& \begin{matrix} i^*F(S^V)^{-1}\wedge i^*D \\ \wedge i^*(Spec(L)_+\wedge F(S^V)) \end{matrix} \arrow[r, "Id\wedge\overline{\psi}"'] \arrow[d, "e"', "\simeq"] & \begin{matrix} i^*F(S^V)^{-1}\wedge i^*D \\ \wedge i^*F(S^V) \end{matrix} \arrow[d, "\simeq"', "e"] \\
i^*i_\sharp i^*D \arrow[r, "\cong"'] & i^*Spec(L)_+\wedge i^*D \arrow[r, "\overline{\psi}"'] & i^*D.
\end{tikzcd}$$

The left square of this diagram commutes up to homotopy by the definition of $\bar{t}$, and keeping track of the transpositions. The lower right square of this diagram is

$$\begin{tikzcd}
i^*F(S^V)^{-1}\wedge i^*D\wedge i^*(i_\sharp i^*F(S^V)) \arrow[r, "Id\wedge\overline{\psi}_{i^*F(S^V)}"] \arrow[d, "\cong"'] & i^*F(S^V)^{-1}\wedge i^*D\wedge i^*F(S^V) \arrow[d, "="] \\
\begin{matrix} i^*i_\sharp(i^*F(S^V)^{-1}\wedge i^*D \\ \wedge i^*F(S^V)) \end{matrix} \arrow[r, "\overline{\psi}_{i^*F(S^V)^{-1}\wedge i^*D\wedge i^*F(S^V)}"] \arrow[d, "i^*i_\sharp e"'] & \begin{matrix} i^*F(S^V)^{-1}\wedge i^*D \\ \wedge i^*F(S^V) \end{matrix} \arrow[d, "e"] \\
i^*i_\sharp(i^*D) \arrow[r, "\overline{\psi}_{i^*D}"'] & i^*D.
\end{tikzcd}$$

The upper part of this commutes by part 2 of Lemma 4.15, and lower part commutes by the naturality of $\overline{\psi}$. Hence, the composition (4.16) is the composition

$$i^*D \xrightarrow{\simeq} i^*F(S^V)^{-1}\wedge i^*D\wedge i^*F(S^V) \to i^*F(S^V)^{-1}\wedge i^*D\wedge i^*F(S^V) \xrightarrow{\simeq} i^*D.$$

The first and third maps are the inverse homotopy equivalences c and e, respectively. The middle map is homotopic to the identity by Lemma 4.12. Thus, (4.16) is homotopic to $i^*e\cdot i^*c$, which passes to the identity in $\mathcal{SH}(L)$, since by the remark at the end of Section 2, i^* preserves $\mathbb{A}^1$-weak equivalences.

We still have to show that $\omega\cdot\psi$ is the identity on $i_\sharp E$ when we pass to the stable homotopy category over k, if $E \in Spectra(L)$ is the shift desuspension of a suspension spectrum. Let $\eta : E \to i^*i_\sharp E$ be the unit of the adjunction $(i_\sharp, i^*)$. Then it suffices to show that $i^*\omega\cdot i^*\psi\cdot\eta$ is homotopic to η over L. Recall the isomorphism ζ given in Proposition 2.8. For an L-spectrum E, we consider the following diagram of k-spectra.

$$\begin{tikzcd}
i_\sharp E \arrow[r, "\psi"] \arrow[d, "\bar{t}"'] & i_* E \arrow[d, "\bar{t}"'] \arrow[dr, "\omega"'] & \\
i_\sharp i^* i_\sharp E \arrow[r, "i_\sharp i^* \psi"'] & i_\sharp i^* i_* E \arrow[r, "i_\sharp \epsilon"'] & i_\sharp E.
\end{tikzcd} \tag{4.17}$$

This diagram commutes by the naturality of $\bar{t}$. By the fact that $\overline{\psi} = \epsilon \cdot (i^*\psi)$, we get that

$$\omega \cdot \psi = i_\sharp(\overline{\psi}) \cdot \bar{t}.$$

We consider two more diagrams. First, let Y be a based L-space. We have the following diagram of based L-spaces.

$$\begin{tikzcd}
i^*F(S^V) \wedge Y \arrow[r, "Id\wedge\eta"] \arrow[d, "Id\wedge\eta"'] & i^*F(S^V) \wedge i^*i_\sharp Y \arrow[r, "i^*F(t)\wedge Id"] \arrow[d, "="'] & i^*i_\sharp i^*F(S^V) \wedge i^*i^\sharp Y \arrow[d, "\cong"'] \arrow[dl, "\overline{\psi}\wedge Id"] \\
i^*F(S^V) \wedge i^*i_\sharp Y \arrow[r, "="] & i^*F(S^V) \wedge i^*i_\sharp Y \arrow[d, "\cong"'] & i^*(i_\sharp i^*F(S^V) \wedge i_\sharp Y) \arrow[d, "\cong"', "i^*\zeta^{-1}"] \\
i^*(F(S^V) \wedge i_\sharp Y) \arrow[u, "\cong"] & i^*(F(S^V) \wedge i_\sharp Y) \arrow[d, "i^*\zeta^{-1}"', "\cong"] & i^*i_\sharp(i^*F(S^V) \wedge i^*i_\sharp Y) \arrow[ul, "\overline{\psi}"'] \arrow[d, "i^*i_\sharp(Id\wedge\overline{\psi})"] \\
i^*i_\sharp(Y \wedge i^*F(S^V)) \arrow[u, "\cong"', "i^*\zeta"] & i^*i_\sharp(i^*F(S^V) \wedge Y) \arrow[l, "\cong"'] & i^*i_\sharp(i^*F(S^V) \wedge Y). \arrow[l, "="']
\end{tikzcd} \tag{4.18}$$

The two squares on the left of the diagram commute. Also, the upper right triangle commutes up to homotopy by Lemma 4.12. The middle right triangle commutes by part 2 of Lemma 4.15. Also, the lower right square commutes when we restrict to the image of the composition around the top and right side of the diagram, starting with the upper left corner $i^*F(S^V) \wedge Y$. Thus, the large square commutes up to homotopy in the category of based L-spaces.

Let E be an L-spectrum. We also consider the following diagram of k-spectra.

$$\begin{tikzcd}
F(S^V) \wedge i_\sharp E \arrow[r, "Id\wedge\bar{t}"] \arrow[d, "="'] & F(S^V) \wedge i_\sharp i^*(i_\sharp E) \arrow[r, "Id\wedge i_\sharp\overline{\psi}"] & F(S^V) \wedge i_\sharp E \\
F(S^V) \wedge i_\sharp E \arrow[d, "Id\wedge F(t)"'] & i_\sharp(i^*F(S^V) \wedge i^*i_\sharp E) \arrow[u, "\zeta"] \arrow[r, "i_\sharp(Id\wedge\overline{\psi})"'] & i_\sharp(i^*F(S^V) \wedge E) \arrow[u, "\zeta"] \\
i_\sharp i^*F(S^V) \wedge i_\sharp E \arrow[r, "\zeta^{-1}"'] & i_\sharp(i^*F(S^V) \wedge i^*i_\sharp E) \arrow[u, "="] & .
\end{tikzcd} \tag{4.19}$$

The lower left corner of the diagram $i_\sharp i^*F(S^V) \wedge i_\sharp E$ is $Spec(L)_+ \wedge F(S^V) \wedge i_\sharp E$, and the middle top term $F(S^V) \wedge i_\sharp i^*(i_\sharp E)$ is $F(S^V) \wedge Spec(L)_+ \wedge i_\sharp E$. By writing out the definition of $\bar{t}$ on $i_\sharp E$ and keeping track of the transpositions, we get that the left square of the diagram commutes. The right upper square of the diagram commutes by the naturality of ζ and $\overline{\psi}$.

By part 1 of Lemma 4.15, if $E = \Sigma^{-n}\Sigma^\infty Y$ for a based L-space Y, then we can replace the composition around the right of (4.18) by i^* applied to the composition around the bottom of (4.19), which is in turn the top row of (4.19). Hence, we have that

$$Id \wedge \eta : i^*F(S^V) \wedge E \to i^*F(S^V) \wedge i^*i_\sharp E$$

is $\mathbb{A}^1$-homotopic to

$$Id \wedge (i^*i_\sharp\overline{\psi} \cdot i^*\bar{t} \cdot \eta)$$

in the category of L-spectra. Also, since i^* preserves smash products, and $i^*S^0_k = S^0_L$, we get that $i^*F(S^V)$ is invertible in the stable homotopy category over L, with inverse $i^*(F(S^V)^{-1})$. So smashing the homotopy with $i^*(F(S^V)^{-1})$ gives that η is the same as

$$i^*(i_\sharp\overline{\psi}) \cdot i^*(\bar{t}) \cdot \eta$$

in the stable homotopy category $\mathcal{SH}(L)$. But now by (4.17), we can replace $i^*(i_\sharp\overline{\psi}) \cdot i^*(\bar{t})$ by $i^*\omega \cdot i^*(\psi)$. This shows that $\omega \cdot \psi$ pass to the identity in $\mathcal{SH}(k)$, so ω and ψ are inverse $\mathbb{A}^1$-weak equivalences of k-spectra. □

5. The Join and Smash Powers to an Étale Scheme

In this section, we give the exact definition for $X^{*Spec(L)}$ for $X \in Spc(k)$, and $X^{\wedge Spec(L)}$ for $X \in Spc(k)_\bullet$. In fact, for any scheme T over k, such that $T \to Spec(k)$ is étale, we will define X^{*T} for unbased X and $X^{\wedge T}$ for based X. For X^{*T}, we begin by recalling the join power X^{*n} for an integer $n \geqslant 1$. This can be described as follows. Given X^n, for each pair of subsets of $\underline{n} = \{1, \ldots, n\}$ of orders i and j, $i > j$, we take all possible projection maps $X^i \to X^j$. We have a partially ordered set of such projections, and the homotopy pushout of this diagram is X^{*n}. For motivation, recall that for a small category $\mathcal{D}$ and a functor F from $\mathcal{D}$ to the category of topological spaces, the homotopy colimit of F with respect to $\mathcal{D}$ can be described as the classifying space of the following topological category $\mathcal{C}(F)$. The objects space of $\mathcal{C}(F)$ is

$$\coprod_{D\in\mathcal{D}} F(D)$$

and for $x, y \in Obj(\mathcal{C}(F))$, $x \in F(D)$ and $y \in F(D')$, the maps $x \to y$ in $\mathcal{C}(F)$ corresponds to maps $f : D \to D'$ in $\mathcal{D}$, such that $F(f) : x \mapsto y$. This is topologized as

$$\coprod_{D'\in Obj(\mathcal{D})} \coprod_{f:D\to D'} F(D).$$

For a space X and $n \geqslant 1$, we can define the topological category $\mathcal{C}_n(X)$ as follows. Let $\mathcal{A}$ be the set of all nonempty subsets of $\underline{n}$, and for $S \subseteq \underline{n}$, denote by X^S the space of maps from S into X. Then

$$Obj(\mathcal{C}_n(X)) = \coprod_{S \in \mathcal{A}} X^S \tag{5.1}$$

where the disjoint union runs over all pairs S, S' in $\mathcal{A}$ with $S' \subseteq S$. For $a \in X^S$ and $b \in X^{S'}$, we have a map $a \to b$ in $\mathcal{C}_n(X)$ if and only if $S' \subseteq S$, and a projects to b via the projection $X^S \to X^{S'}$. Hence, the morphisms in $\mathcal{C}_n(X)$ are parametrized by pairs $S' \subseteq S$, and points in X^S. So

$$Mor(\mathcal{C}_n(X)) = \coprod_{S' \subseteq S \in \mathcal{A}} X^S.$$

Then X^{*n} is the classifying space of $\mathcal{C}_n(X)$.

For an unbased k-space X and an étale scheme T over $Spec(k)$, we define $X^{*Spec(L)}$ via the interpretation of the homotopy colimit as the classifying space of a category. For motivation, we first consider the case of equivariant topological spaces. Let G be a finite group, and let T be a G-set. Then for a G-equivariant topological space X, we can define X^{*T} in the above manner. Define

$$\mathcal{A} = \{S \subseteq T \mid S \text{ nonempty}\}.$$

Then $\mathcal{A}$ has a natural G-action by the multiplication of G from the right. For each $S \in \mathcal{A}$, let H_S be the stabilizer subgroup of S. Then S is an H_S-equivariant set. Define the H_S-equivariant space $X^S = \underline{Hom}_G(S, X)$ to be the space of nonequivariant maps from S to X, with an H_S-action by conjugation. For each $\alpha \in G$, and $S \in A$, we have $H_{\alpha(S)} = \alpha H_S \alpha^{-1}$, so the stabilizer subgroups are isomorphic for elements in the same G-orbit of $\mathcal{A}$. Also, note that if $a \in X^S$, then $\alpha \cdot a \cdot \alpha^{-1} \in X^{\alpha(S)}$. We have a natural isomorphism of G-equivariant spaces

$$\alpha_* : G \times_{H_S} X^S \xrightarrow{\cong} G \times_{H_{\alpha(S)}} X^{\alpha(S)} \tag{5.2}$$

For $g \in G$ and $a \in X^S$, α_* takes $(g, a) \in G \times_{H_S} (X^S)$ to $(g\alpha^{-1}, \alpha \cdot a \cdot \alpha^{-1})$ in $G \times_{H_{\alpha(S)}} X^{\alpha(S)}$. It is straightforward to check that this is indeed a G-equivariant isomorphism. If $\alpha(S) = \beta(S)$, then α and β differ by $h \in H_S$. But $h_* = Id$ for $h \in H_S$. Hence, the map α_* of (5.2) depends only on the sets $S, \alpha(S)$.

We define the following category $\mathcal{C}_T(X)$, which is enriched over the category of G-equivariant topological spaces. If $\{S_1, \ldots, S_n\}$ is a G-orbit in $\mathcal{A}$, we will replace $\coprod_{i=1}^n X^{S_i}$ in (5.1) by $G \times_{H(S_i)} X^{S_i}$ for any choice of i between 1 and n, to get the G-action. So the object space of $\mathcal{C}_T(X)$ is

$$Obj(\mathcal{C}_T(X)) = \coprod_{S_j} G \times_{H_{S_j}} X^{S_j}$$

where the S_j's range over a set of representatives of the orbits in $\mathcal{A}$. This is a G-equivariant space. For $g(S_j)$ in the orbit of S_j in $\mathcal{A}$, and $a \in X^{g(S_j)}$, a corresponds to $(g^{-1}, g \cdot a \cdot g^{-1})$ in $G \times_{H_{S_j}} X^{S_j}$. By (5.2), $Obj(\mathcal{C}_T(X))$ is independent of the choices of orbit representatives.

For objects $(g, a) \in G \times_{H_{S_j}} X^{S_j}$ and $(g', b) \in G \times_{H_{S_r}} X^{S_r}$, which correspond to

$g^{-1} \cdot a \cdot g \in X^{g^{-1}(S_j)}$ and $(g')^{-1} \cdot b \cdot g' \in X^{(g')^{-1}(S_r)}$, there is a morphism in $\mathcal{C}_T(X)$

$$(g, a) \to (g', b)$$

if and only if $(g')^{-1}(S_r) \subseteq g^{-1}(S_j)$, and $(g')^{-1} \cdot b \cdot g' \in X^{(g')^{-1}(S_r)}$ is the projection of $g^{-1} \cdot a \cdot g \in X^{g^{-1}(S_j)}$. In other words, each morphism of $\mathcal{C}_T(X)$ corresponds uniquely to a pair of sets $S', S \in A$, $S' \subseteq S$, and a point in X^S. Let

$$\mathcal{A}_2 = \{(S', S) \mid S, S' \in \mathcal{A},\ S' \subseteq S\}.$$

Then $\mathcal{A}_2$ has a G-action by multiplication from the right. For each $(S', S) \in \mathcal{A}_2$, let $H_{(S',S)}$ be the stabilizer subgroup of (S', S). Let $\{(S'_i, S_i)\}$ be a set of representatives or the orbits in $\mathcal{A}_2$. Then the G-space of morphisms in $\mathcal{C}_G(T)$ is

$$Mor(\mathcal{C}_T(X)) = \coprod_{(S'_i,S_i)} G \times_{H_{(S'_i,S_i)}} X^{S_i}.$$

For $g \in G$ and $a \in X^{S_i}$, $(g, a) \in G \times_{H_i} X^S$ corresponds to the morphism that comes from the projection of $g^{-1} \cdot a \cdot g \in X^{g^{-1}(S_i)}$ to $X^{g^{-1}(S'_i)}$. There is a natural G-space structure on $Mor(\mathcal{C}_T(X))$. For $S \in \mathcal{A}$, $(S, S) \in \mathcal{A}_2$, $H_S = H_{(S,S)}$, so the

$$Identity : Obj(\mathcal{C}_T(X)) \to Mor(\mathcal{C}_T(X))$$

is given by a disjoint union of identity maps

$$G \times_{H_S} X^S \to G \times_{H_{(S,S)}} X^S$$

composed with isomorphisms of the form (5.2) to make (S, S) one of the chosen representatives of an orbit in $\mathcal{A}_2$. For each pair $(S', S) \in \mathcal{A}_2$, $H_{(S',S)} \subseteq H_S$, so define

$$Source : Mor(\mathcal{C}_T(X)) \to Obj(\mathcal{C}_T(X))$$

to be a disjoint union of quotient maps

$$G \times_{H_{(S',S)}} X^S \to G \times_{H_S} X^S$$

composed with appropriate isomorphisms of the form (5.2). Also, the inclusion $S' \to S$ induces an $H_{(S',S)}$-equivariant map $X^S \to X^{S'}$. so define

$$Target : Mor(\mathcal{C}_T(X)) \to Obj(\mathcal{C}_T(X))$$

to be a disjoint union of compositions

$$G \times_{H_{(S',S)}} X^S \to G \times_{H_{(S',S)}} X^{S'} \to G \times_{H_{S'}} X^{S'}$$

where the first map is induced by $X^S \to X^{S'}$, and the second map is the quotient map. Finally, let $\mathcal{A}_3 = \{S'' \subseteq S' \subseteq S\}$ with a G-action via multiplication from the right, and let $H_{(S'',S',S)}$ be the stabilizer subgroup of (S'', S', S) in $\mathcal{A}_3$. Then $Mor(\mathcal{C}_T(X)) \times_{Obj(\mathcal{C}_T(X))} Mor(\mathcal{C}_T(X))$ is a disjoint union of G-spaces of the form

$$(G \times_{H_{(S',S)}} X^S) \times_{G \times_{H_{S'}} X^{S'}} (G \times_{H_{(S'',S')}} X^{S'}) \cong G \times_{H_{(S'',S',S)}} X^S$$

where (S'', S', S) ranges over a set of representatives of orbits in $\mathcal{A}_3$. So

$$Composition : Mor(\mathcal{C}_T(X)) \times_{Obj(\mathcal{C}_T(X))} Mor(\mathcal{C}_T(X)) \to Mor(\mathcal{C}_T(X))$$

is a disjoint union of quotient maps of the form

$$G \times_{H_{(S'',S',S)}} X^S \to G \times_{H_{(S',S)}} X^S.$$

By definition, the identity, source, target, and composition of morphisms are G-equivariant. Thus, $\mathcal{C}_T(X)$ is a category enriched over G-topological spaces. The classifying space of $\mathcal{C}_T(X)$ is X^{*T}, which has a natural structure as a G-space.

The construction in the algebraic case is similar. Let X be a k-space, and T is a scheme over k, with étale map $T \to Spec(k)$. Then

$$T = \coprod_r Spec(F_r)$$

is a disjoint union of the spectra of finite extension fields F_r over k. Let L be a Galois extension that contains F_r for every r, such as the the algebraic closure of k, and let $G = Gal(L/k)$. In particular, G may be a profinite group. For each F_r, let $H(F_r) = Gal(L/F_r)$, so $F_r = L^{H(F_r)}$. Thus, let the G-set T_G be given by $T_G = \coprod_r G/H(F_r)$, then $T = F_{L/k}(T_G)$. Again, let $\mathcal{A}$ be the collection of nonempty subsets of T_G, with a G-action by the multiplication of G from the right. For $S \in A$, let H_S be the stabilizer subgroup of S, and let $E_S = L^{H_S}$, so $Spec(E_S) = F_{L/k}(G/H_S)$. Also, let $f_S : Spec(E_S) \to Spec(k)$ be the map corresponding to the inclusion of fields. Then $F_{L/E_S}(S) \in Spc(E_S)$, and the analogue of the H_S-equivariant topological space X^S is the E_S-space

$$X^{F_{L/E_S}(S)} = \underline{Hom}_{Spec(E_S)}(F_{L/E_S}(S), (f_S)^*(X)). \tag{5.3}$$

In particular, if $S = T_G$, then $E_S = k$ and $F_{L/E_S}(S) = F_{L/k}(T_G) = T$ as a k-space, so we have

$$X^{T_G} = \underline{Hom}_{Spec(k)}(T, X)$$

by Lemma 2.14. For $\alpha \in G$ and $S \in A$, $H_{\alpha(S)} = \alpha H_S \alpha^{-1}$, so $E_{\alpha(S)} = \alpha(E_S)$. So similarly as for (5.2), we have an isomorphism of k-spaces

$$\alpha_* : (f_S)_\sharp X^{F_{L/E_S}(S)} \xrightarrow{\cong} (f_{\alpha(S)})_\sharp X^{F_{L/E_{\alpha(S)}}(\alpha(S))}. \tag{5.4}$$

For $\alpha \in H_S$, the map (5.4) is the identity map, since $f_S = f_S \cdot \alpha : Spec(E_S) \to Spec(k)$. Thus, in general the map (5.4) depends only on the sets S and $\alpha(S)$. We define the category $\mathcal{C}_{T/k}(X)$, enriched over $Spc(k)$, as follows. Define the k-space of objects to be

$$Obj(\mathcal{C}_{T/k}(X)) = \coprod_{S_j} (f_{S_j})_\sharp X^{S_j}$$

where S_j ranges over the representatives of the orbits in $\mathcal{A}$. For $(S', S) \in \mathcal{A}_2$, set $E_{(S',S)} = L^{H_{(S',S)}}$, so $k \subseteq E_S \subseteq E_{(S,S')}$. Let $f_{(S',S)} : Spec(E_{(S',S)}) \to Spec(k)$, and $a_{(S',S)} : Spec(E_{(S',S)}) \to Spec(E_S)$ be the maps corresponding to the inclusions of fields. Then the analogue of $G \times_{H_{(S',S)}} X^S$ is $(f_{(S',S)})_\sharp a^*_{(S',S)} X^{F_{L/E_S}(S)}$. Thus, define the k-space of morphisms in $\mathcal{C}_{T/k}(X)$ to be

$$Mor(\mathcal{C}_{T/k}(X)) = \coprod_{(S'_i, S_i)} (f_{(S'_i,S_i)})_\sharp (a_{(S'_i,S_i)})^* X^{F_{L/E_S}(S)}$$

where (S_i, S_i') ranges over a set of representatives or the orbits in A_2.

The identity, source, target, and composition maps of $\mathcal{C}_{T/k}(X)$ are defined similarly as in the equivariant case. For S in A, $E_{(S,S)} = E_S$, $a : Spec(E_S) \to Spec(E_{(S,S)})$ is the identity, and $f_S = f_{(S,S)} : Spec(E_S) \to Spec(k)$. So

$$Identity : Obj(\mathcal{C}_{T/k}(X)) \to Mor(\mathcal{C})_{T/k}(X)$$

is a disjoint union of identity maps on $(f_S)_\sharp X^{F_{L/E_S}(S)}$, composed with isomorphisms of the form (5.4) to make (S, S) one of the chosen representatives of an orbit in $\mathcal{A}_2$. For the source map, note that for each (S', S) in $\mathcal{A}_2$, $f_{(S',S)} = f_S \cdot a_{(S',S)}$. Let c be the counit of the adjunction pair $((a_{(S',S)})_\sharp, (a_{(S',S)})^*)$. So define

$$Source : Mor(\mathcal{C}_{T/k}(X)) \to Obj(\mathcal{C}_{T/k}(X))$$

to be a disjoint union of maps

$$\begin{aligned}(f_{(S',S)})_\sharp(a_{(S',S)})^* X^{F_{L/E_S}(S)} &= (f_S)_\sharp(a_{(S',S)})_\sharp(a_{(S',S)})^* X^{F_{L/E_S}(S)} \\ &\xrightarrow{c} (f_S)_\sharp X^{F_{L/E_S}(S)}\end{aligned}$$

composed with isomorphisms of the form (5.4) to make (S', S) an orbit representative in $\mathcal{A}_2$. To define the target map, consider the natural map $b_{(S',S)} : Spec(E_{(S',S)}) \to Spec(E_{S'})$. So we have a commuatative diagram

$$\begin{tikzcd} Spec(E_{(S',S)}) \arrow[r, "b_{(S,S')}"] \arrow[d, "a_{(S',S)}"'] \arrow[dr, "f_{(S',S)}"] & E_{S'} \arrow[d, "f_{S'}"] \\ Spec(E_S) \arrow[r, "f_S"'] & Spec(k). \end{tikzcd}$$

Then by Lemma 3.4, the inclusion $S' \to S$ induces a map of $E_{(S',S)}$-spaces

$$(b_{(S',S)})^* F_{L/E_{S'}}(S') = F_{L/E_{(S',S)}}(S') \to F_{L/E_{(S',S)}}(S) = (a_{(S',S)})^* F_{L/E_S}(S).$$

This in turn induces a map of $E_{(S',S)}$-spaces

$$(a_{(S',S)})^* X^{F_{L/E_S}(S)} \to (b_{(S',S)})^* X^{F_{L/E_{S'}}(S')}.$$

This is because by taking the adjoint of Proposition 2.8, $(b_{(S',S)})^*$ and $(a_{(S',S)})^*$ commute with the internal Hom functor. This gives that

$$\begin{aligned} X^{(a_{(S',S)})^* F_{L/E_S}(S)} &= \underline{Hom}_{E_{(S',S)}}((a_{(S',S)})^* F_{L/E_S}(S), (f_{(S',S)})^* X) \\ &\cong (a_{(S',S)})^* \underline{Hom}_{E_S}(F_{L/E_S}(S), f_S^* X) \\ &= (a_{(S',S)})^* X^{F_{L/E_S}(S)} \end{aligned}$$

and similarly for $X^{(b_{(S',S)})^* F_{L/E_{S'}}(S')}$. So

$$Target : Mor(\mathcal{C}_{T/k}(X)) \to Obj(\mathcal{C}_{T/k}(X))$$

is a disjoint union of compositions

$$(f_{(S',S)})_\sharp(a_{(S',S)})^* X^{F_{L/E_S}(S)} \to (f_{(S',S)})_\sharp(b_{(S',S)})^* X^{F_{L/E_{S'}}(S')} \to (f_{S'})^* X^{F_{L/E_{S'}}(S')}$$

where the second map is the counit of the adjunction pair $((b_{(S',S)})_\sharp, (b_{(S',S)})^*)$. Finally, for the composition of morphisms in $\mathcal{C}_{T/k}(X)$, consider $(S'', S', S) \in \mathcal{A}_3$. Let $E_{(S'',S',S)} = L^{H_{(S'',S',S)}}$. We have natural maps

$$f_{(S'',S',S)} : Spec(E_{(S'',S',S)}) \to Spec(k)$$

and

$$c_{(S'',S',S)} : Spec(E_{(S'',S',S)}) \to Spec(E_{(S',S)}).$$

Then $Mor(\mathcal{C}_{T/k}(X)) \times_{Obj(\mathcal{C}_{T/k}(X))} Mor(\mathcal{C}_{T/k}(X))$ is a disjoint union of k-spaces of the form

$$(f_{(S',S)})_\sharp (a_{(S',S)})^* X^{F_{L/E_S}(S)} \times (f_{(S'',S')})_\sharp (a_{(S'',S')})^* X^{L/E_{S'}(S')}$$

where the product is over $(f_{S'})_\sharp X^{L/E_{S'}(S')}$, to which the two factors map by the target and source maps, respectively. This is isomorphic as a k-space to

$$(f_{(S'',S',S)})_\sharp (c_{(S'',S',S)})^* (a_{(S',S)})^* X^{F_{L/E_S}(S)}.$$

Hence,

$$Composition: \; Mor(\mathcal{C}_{T/k}(X)) \times_{Obj(\mathcal{C}_{T/k}(X))} Mor(\mathcal{C}_{T/K}(X)) \to Mor(\mathcal{C}_{T/k}(X))$$

is a disjoint union of counit maps

$$(f_{(S'',S',S)})_\sharp c^*_{(S'',S',S)} a^*_{(S',S)} X^{F_{L/E_S}(S)} \to (f_{(S',S)})_\sharp a^*_{(S',S)} X^{F_{L/E_S}(S)}$$

for the adjunction pairs $((c_{(S'',S',S)})_\sharp, (c_{(S'',S',S)})^*)$, where (S'', S', S) ranges over a set of orbit representatives in $\mathcal{A}_3$. Then $\mathcal{C}_{T/k}(X)$ is a category enriched over $Spc(k)$. The join power X^{*T} is defined to be the classifying space of $\mathcal{C}_{T/k}(X)$, which has a natural structure as a k-space.

To show that X^{*T} is independent of the choice of L, it suffices to consider the case where $T = Spec(F)$, for some separable finite extension F of k. Suppose L and L^{prime} are two Galois extensions of k containing F. We can assume without loss of generality that L' contains L. Let $J = Gal(L'/L)$, $G' = Gal(L'/k)$ and $G = Gal(L/k)$. So we have a short exact sequence of groups

$$1 \to J \to G' \xrightarrow{p} G \to 1.$$

Let $H = Gal(L/F)$, and $H' = Gal(L'/F)$. Then $T = Spec(F) = F_{L/k}(G/H) = F_{L'/k}(G'/H')$. But $H' = p^{-1}(H)$, so there is a canonical isomorphism of G'-sets

$$G'/H' \cong G/H$$

where G/H is thought of as a G'-set fixed by J. So let $\mathcal{A}_G$ be the collection of nonempty subsets of G/H, and $\mathcal{A}_{G'}$ be the collection of nonempty subsets of G'/H'. There is a canonical G'-equivariant bijection between $\mathcal{A}_G$ and $\mathcal{A}_{G'}$. In particular, for any $S \in \mathcal{A}_{G'}$, let $H'_S \subseteq G'$ be the isotropy subgroup of S in $\mathcal{A}_{G'}$, and let $H_S \subseteq G$ be the isotropy subgroup of S in $\mathcal{A}_G$. Then $H'_S = p^{-1}(H_S)$. Then

$$L^{H_S} = (L')^{H'_S}$$

so the definition of $E_S = L^{H_S}$ is independent of the choice of L. For a k-space X, the objects and morphisms of the category $\mathcal{C}_{T/k}(X)$ are build up out of k-spaces of the form $X^{F_{L/E_S}(S)}$ for all $S \in \mathcal{A}_G$. By arguments similar as above,

$$F_{L/E_S}(S) = F_{L'/E_S}(S)$$

as k-spaces, where S is thought of as a subset of G/H on the left hand side, and as a subset of G'/H' on the right hand side. Hence, by the definition of $X^{F_{L/E_S}(S)}$ in 5.3, we see that $X^{F_{L/E_S}(S)}$ is independent of the choice of L.

If X is a triangulated G-equivariant space, T_G is a G-set, and $T = F_{L/k}(T_G)$ is étale over $Spec(k)$. Then from the completely analogous definitions of the categories $\mathcal{C}_G(X)$ and $\mathcal{C}_{T/k}(F_{T/k}(X))$, we get a weak equivalence of k-spaces

$$F_{L/k}(X^{*T_G}) = F_{L/k}(X)^{*T}.$$

Likewise, for $X \in Spc(k)_\bullet$ and $T \to Spec(k)$ étale, we can also define $X^{\wedge T}$, the smash powerof X to T. As before, we have an extension field L of k, with $G = Gal(L/k)$, such that $T = F_{L/k}(T_G)$ for some G-set T_G. For any $S \in \mathcal{A}$, we have $f_S : Spec(E_S) \to Spec(k)$. Then there is a map of unbased E_S-spaces

$$X^{F_{L/E_S}(S)} \to (f_S)^* X^T$$

by inclusions of basepoints, induced by the inclusion map $S \to G$, which is H_S-equivariant. This corresponds to a map of unbased k-spaces

$$(f_S)_\sharp X^{F_{L/E_S}(S)} \to X^{Spec(L)}.$$

In particular, for any $\alpha \in G$, then $E_{\alpha(S)} = \alpha(E_S)$, and the diagram

$$\begin{tikzcd}
(f_{\alpha(S)})_\sharp X^{F_{L/E_{\alpha(S)}}(\alpha(S))} \arrow[rr, "\alpha_*", "\cong"'] \arrow[dr] & & (f_S)_\sharp X^{F_{L/E_S}(\alpha(S))} \arrow[dl] \\
& X^T &
\end{tikzcd}$$

commutes, where α_* is the map of (5.4). Let S_j range over a set of orbit representatives of $\mathcal{A} \setminus \{G\}$. Then define the $X^{\wedge T} \in Spc(k)_\bullet$ to be

$$X^{\wedge T} = X^T /_{Spec(k)} \bigcup_{S_j} (f_{S_j})_\sharp X^{F_{L/E_{S_j}}(S_j)}.$$

Here, $\bigcup$ denotes the union inside X^T.

It remains to prove Lemma 3.7.

Proof of Lemma 3.7. For any $S \subset T_G$, the functor

$$(-)^{F_{L/E_S}(S)} = \underline{Hom}_{Spec(E_S)}(F_{L/E_S}(S), (f_S)^* -) : Spc(k) \to Spc(E_S)$$

commutes with fibered products of unbased k-spaces. For $X, Y \in Spc(k)_\bullet$,

$$\begin{aligned}
&(X \wedge Y)^{\wedge T} \\
&= (X \wedge Y)^T / \bigcup_{S_j} (f_{S_j})_\sharp (X \wedge Y)^{F_{E/S_j}(S_j)} \\
&= (X \times Y / X \cup Y)^T / \bigcup_{S_j} (f_{S_j})_\sharp (X \wedge Y)^{F_{E/S_j}(S_j)}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&X^{\wedge T} \wedge Y^{\wedge T} \\
&= (X^T / \bigcup_{S_j} (f_{S_j})_\sharp X^{F_{E/S_j}(S_j)}) \wedge (Y^T / \bigcup_{S_j} (f_{S_j})_\sharp Y^{F_{E/S_j}(S_j)}) \\
&\cong \frac{X^T \times Y^T}{(\bigcup_{S_j} (f_{S_j})_\sharp X^{F_{E/S_j}(S_j)} Y^T) \cup (X^T \times \bigcup_{S_j} (f_{S_j})_\sharp Y^{F_{E/S_j}(S_j)})}.
\end{aligned}
$$

for all ordered pairs of subsets (S, S') in T_G, $(S, S') \neq (T_G, T_G)$, we have a map

$$((f_S)_\sharp X^{F_{L/E_S}(S)}) \times ((f_{S'})_\sharp Y^{F_{L/E_{S'}}(S')}) \to X^T \times Y^T \cong (X \times Y)^T.$$

Both $(X \wedge Y)^{\wedge T}$ and $X^{\wedge T} \wedge Y^{\wedge T}$ are isomorphic to the quotient

$$(X \times Y)^T / \bigcup_{S_i, S'_j} ((f_S)_\sharp X^{F_{L/E_S}(S)}) \times ((f_{S'})_\sharp Y^{F_{L/E_{S'}}(S')}).$$

$\square$

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