A POINTWISE APPROXIMATION THEOREM FOR LINEAR COMBINATIONS OF BERNSTEIN POLYNOMIALS

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Abstract. Recently, Z. Ditzian gave an interesting direct estimate for Bernstein polynomials. In this paper we give direct and inverse results of this type for linear combinations of Bernstein polynomials.

1. Introduction

For the Bernstein polynomial

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{nk}(x),\quad p_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.1}$$

Berens and Lorentz showed in [1] that

$$B_n(f,x)-f(x)=O\left(\left(\frac{1}{\sqrt{n}}\delta_n(x)\right)^{\alpha}\right)\Longleftrightarrow \omega^1(f,t)=O(t^{\alpha}), \tag{1.2}$$

where $0<\alpha<1,\ \delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n}},\ \varphi(x)=\sqrt{x(1-x)}$.
Recently, Ditzian [3] gave the following interesting result.

$$|B_n(f,x)-f(x)|\le C\omega^2_{\varphi^\lambda}(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}),\ 0\le\lambda\le 1.$$

However, Ditzian did not consider the inverse result in [3]. We did give such an inverse result in [6], where we obtained the following equivalence.

$$B_n(f,x)-f(x)=O((n^{-\frac{1}{2}}\varphi(x)^{1-\lambda})^{\alpha})\Longleftrightarrow \omega^2_{\varphi^\lambda}(f,t)=O(t^{\alpha}),\ 0<\alpha<2.$$

In this paper we consider linear combinations of Bernstein polynomials, that is

$$B_{n,r}(f,x)=\sum_{i=0}^{r-1}C_i(n)B_{n_i}(f,x), \tag{1.3}$$

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where n_i and C_i satisfy [5]

$$(1.4)\quad \begin{aligned} &(a)\ n = n_0 < n_1 < \cdots < n_{r-1} \le Kn; \quad (b)\ \sum_{i=0}^{r-1} |C_i(n)| < C \\ &(c)\ \sum_{i=0}^{r-1} C_i(n) = 1; \quad (d)\ \sum_{i=0}^{r-1} C_i(n) n_i^{-\rho} = 0, \quad \rho = 1, 2, \cdots, r-1. \end{aligned}$$

We recall that

$$(1.5)\qquad \omega^r_{\varphi^\lambda}(f,t) = \sup_{0<h\le t}\ \sup_{x\pm\frac{r}{2}h\varphi^\lambda(x)\in[0,1]} \left|\Delta^r_{h\varphi^\lambda(x)} f(x)\right|$$

is equivalent to the K-functional

$$(1.6)\qquad K_{\varphi^\lambda}(f,t^r) = \inf_{g^{(r-1)}\in A.C_{loc}} (\|f-g\|_{c[0,1]} + t^r\|\varphi^{r\lambda} g^{(r)}\|_{C[0,1]}).$$

We write $\omega^r_{\varphi^\lambda}(f,t) \sim K_{\varphi^\lambda}(f,t^r)$, i.e., there exists a constant C such that

$$(1.7)\qquad C^{-1}K_{\varphi^\lambda}(f,t^r) \le \omega^r_{\varphi^\lambda}(f,t) \le CK_{\varphi^\lambda}(f,t^r).$$

Now we state our main result.

Theorem. *For $f \in C[0,1]$, $0 < \alpha < r$, $0 \le \lambda \le 1$, we have*

$$(1.8)\qquad B_{n,r}(f,x) - f(x) = O((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha) \Longleftrightarrow \omega^r_{\varphi^\lambda}(f,t) = O(t^\alpha).$$

Remark. In the case $r = 1$ and $\lambda = 0$ our result is (1.2) of Berens and Lorentz [1].

2. Direct theorem

In this section we prove the direct part of (1.8). We need the K-functional (see [5], p. 24):

$$(2.1)\qquad \overline{K}_{\varphi^\lambda}(f,t^r) = \inf_{g^{(r-1)}\in A.C_{loc}} \{\|f-g\| + t^r\|\varphi^{r\lambda}g^{(r)}\| + t^{r/(1-\frac{\lambda}{2})}\|g^{(r)}\|\}.$$

which is also equivalent to $\omega^r_{\varphi^\lambda}(f,t)$.

Theorem 1. *For $f \in C[0,1]$, $0 \le \lambda \le 1$,*

$$(2.2)\qquad |B_{n,r}(f,x) - f(x)| \le A\omega_{\varphi^\lambda}(f, n^{-\frac{1}{2}}\delta_n(x)^{1-\lambda}).$$

Proof. By (2.1), we may choose $g_n \equiv g_{n.x.\lambda}$ for a fixed x and λ such that

$$(2.3)\qquad \|f - g_n\| \le A_1\omega^r_{\varphi^\lambda}(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)),$$

$$(2.4)\qquad (n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^r\|\varphi^{r\lambda}g_n^{(r)}\| \le A_2\omega^r_{\varphi^\lambda}(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)),$$

$$(2.5)\qquad (n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^{2r/(2-\lambda)}\|g_n^{(r)}\| \le A_3\omega^r_{\varphi^\lambda}(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)).$$

We recall that [5, p. 134]

$$(2.6)\qquad B_{n,r}((\cdot - x)^k,\ x) = 0, \quad k = 1, 2, \cdots, r-1$$

and that [5, p. 138]

$$B_n((\cdot - x)^{2j},\ x) = \sum_{m=0}^{j-1} \frac{\varphi(x)^{2(j-m)}}{n^{j-m}} n^{-2m} q_m(x),$$

where $q_m(x)$ are fixed bounded polynomials. Therefore

$$B_n((\cdot - x)^{2j},\ x) \le M n^{-j} \delta_n^{2j}(x). \tag{2.7}$$

From the definition of the $B_{n,r}$ we have

$$|B_{n,r}(f,x) - f(x)| \le |B_{n,r}(f - g_n, x)| + |f(x) - g_n(x)| + |B_{n,r}(g_n, x) - g_n(x)|$$

$$\le (C+1)\|f - g_n\| + |B_{n,r}(g_n, x) - g_n(x)|. \tag{2.8}$$

As in [4, Lemma 5.3], we obtain

$$\left| \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du \right| \le \left| \frac{(t-x)^{r-1}}{\delta_n^{r\lambda}(x)} \int_x^t \delta_n^{r\lambda}(u) g_n^{(r)}(u) du \right|, \tag{2.9}$$

$$\left| \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du \right| \le \left| \frac{(t-x)^{r-1}}{\varphi^{r\lambda}(x)} \int_x^t \varphi^{r\lambda}(u) g_n^{(r)}(u) du \right|. \tag{2.10}$$

Using (2.6), (2.7) and (2.9), we get

$$\begin{aligned} &|B_{n,r}(g_n, x) - g_n(x)| \\ &= \left| B_{n,r}\Big(\frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du, x\Big) \right| \\ &\le \sum_{i=0}^{r-1} |C_i(n)| \|\delta_n^{r\lambda} g_n^{(r)}\|_\infty \frac{1}{(r-1)!} B_{n_i}\left(\frac{|t-x|^r}{\delta_n^{r\lambda}(x)}, x \right) \\ &\le \sum_{i=0}^{r-1} |C_i(n)| \|\delta_n^{r\lambda} g_n^{(r)}\|_\infty \delta_n^{-r\lambda}(x) M_1 n^{-\frac{r}{2}} \delta_n^r(x) \\ &\le C M_1 n^{-\frac{r}{2}} \delta_n^{r(1-\lambda)}(x) \|\delta_n^{r\lambda} g_n^{(r)}\|_\infty. \end{aligned} \tag{2.11}$$

Similarly, by (2.10) we have

$$|B_{n,r}(g_n, x) - g_n(x)| \le C M_1 n^{-\frac{r}{2}} \delta_n^r(x) \varphi^{-r\lambda}(x) \|\varphi^{r\lambda} g_n^{(r)}\|. \tag{2.12}$$

If $x \in E_n = [\frac{1}{n},\ 1 - \frac{1}{n}]$, then $\delta_n(x) \sim \varphi(x)$, and, by (2.4) and (2.12),

$$\begin{aligned} |B_{n,r}(g_n, x) - g_n(x)| &\le M_2 (n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x))^r \|\varphi^{r\lambda} g_n^{(r)}\| \\ &\le M_2 A_2 \omega_{\varphi^\lambda}^r (f, n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x)). \end{aligned} \tag{2.13}$$

If $x \in E_n^c = [0, \frac{1}{n}) \bigcup (1 - \frac{1}{n}, 1]$, then $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, and by (2.4), (2.5) and (2.11) we obtain

$$
\begin{aligned}
|B_{n,r}(g_n, x) - g_n(x)| &\le M_3(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^r(\|\varphi^{r\lambda}g_n^{(r)}\| + \|n^{-\frac{r\lambda}{2}}g_n^{(r)}\|) \\
&\le M_4((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^r\|\varphi^{r\lambda}g_n^{(r)}\| \\
&\quad + (n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^{2r/(2-\lambda)}\|g_n^{(r)}\|) \\
&\le M_5\omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)).
\end{aligned} \tag{2.14}
$$

From (2.3), (2.8), (2.13) and (2.14) we get (2.2). ∎

3. Inverse Theorem

In this section we prove the inverse part of (1.8).

Theorem 2. *For $f \in C[0,1],\ 0 < \alpha < r,\ 0 \le \lambda \le 1$, if*

$$|B_{n,r}(f, x) - f(x)| \le B(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha$$

then

$$\omega_{\varphi^\lambda}^r(f, t) = O(t^\alpha). \tag{3.1}$$

To prove Theorem 2 we need some new notation and some lemmas. We use the following notation.

$$C_0 = \{f \in C[0,1],\ \ f(0) = f(1) = 0\},$$

$$\|f\|_0 = \sup_{x\in(0,1)} |\delta_n^{\alpha(\lambda-1)}(x)f(x)|,$$

$$C_\lambda^0 = \{f \in C_0,\ \ \|f\|_0 < \infty\},$$

$$\|f\|_r = \sup_{x\in(0,1)} |\delta_n^{r+\alpha(\lambda-1)}(x)f^{(r)}(x)|,$$

$$C_\lambda^r = \{f \in C_0,\ \|f\|_r < \infty,\ f^{(r-1)} \in A.C_{loc}\}.$$

For $f \in C_0$ we define the K-functional as follows:

$$K_\lambda^\alpha(f, t^r) = \inf_{g\in C_\lambda^r}\{\|f - g\|_0 + t^r\|g\|_r\}. \tag{3.2}$$

We also need the following lemmas which will be proved in the next section.

Lemma 3.1. *If $n \in N,\ \ 0 < \alpha < r$, then*

$$\|B_nf\|_r \le B_1n^{\frac{r}{2}}\|f\|_0 \quad (f \in C_\lambda^0). \tag{3.3}$$

and

$$\|B_nf\|_r \le B_2\|f\|_r \qquad (f \in C_\lambda^r). \tag{3.4}$$

Lemma 3.2. *For $0 < t < \frac{1}{8r},\ \ \frac{rt}{2} \le x \le 1 - \frac{rt}{2}$ and $0 \le \beta \le r$, we have*

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \delta_n^{-\beta}(x + \sum_{k=1}^r u_k)du_1 \cdots du_r \le C(\beta)t^r\delta_n^{-\beta}(x). \tag{3.5}$$

Now we prove (3.1).

Proof of (3.1). Since $B_n(f,x)$ preserves linear functions, we consider only $f \in C_0$ for $r > 1$. If $r = 1$, $f(x) = ax + b$ then $\omega^1_{\varphi^\lambda}(ax+b,t) = a\varphi^\lambda(x)t \le at^\alpha$ $(0 < \alpha < 1)$.
So, we may assume $f \in C_0$. From (3.2) we have

$$K^\alpha_\lambda(f,t^r) \le \|B_{n,r}(f) - f\|_0 + t^r\|B_{n,r}f\|_r \tag{3.6}$$

and we may choose $g \in C^r_\lambda$ such that

$$\|f-g\|_0 \le 2K^\alpha_\lambda(f,n^{-\frac{1}{2}}) \quad \text{and} \quad n^{-\frac{r}{2}}\|g\|_r \le 2K^\alpha_\lambda(f,n^{-\frac{1}{2}}). \tag{3.7}$$

By the assumption of Theorem 2, one has

$$\|B_{n,r}(f,x) - f(x)\|_0 \le Bn^{-\frac{\alpha}{2}}. \tag{3.8}$$

Using Lemma 3.1 and (3.7) we have

$$\begin{aligned} \|B_{n,r}(f)\|_r &\le \|B_{n,r}(f-g)\|_r + \|B_{n,r}g\|_r \\ &\le M(n^{\frac{r}{2}}\|f-g\|_0 + \|g\|_r) \\ &\le 2Mn^{\frac{r}{2}}K^\alpha_\lambda(f,n^{-\frac{r}{2}}). \end{aligned} \tag{3.9}$$

From (3.6), (3.8) and (3.9) we obtain

$$K^\alpha_\lambda(f,t^r) \le M_1(n^{-\frac{\alpha}{2}} + t^r n^{\frac{r}{2}} K^\alpha_\lambda(f,n^{-\frac{r}{2}}))$$

and this implies, via the Berens-Lorentz lemma [1], that if $\alpha < r$ then

$$K^\alpha_\lambda(f,t^r) \le M_2t^\alpha. \tag{3.10}$$

On the other hand, notice that $\delta_n^{\alpha(1-\lambda)}(x)$ is concave. So, we have, for $g \in C^r_\lambda$,

$$\begin{aligned} \left|\Delta^r_{t\varphi^\lambda(x)}g(x)\right| &\le \|g\|_0 \sum_{j=0}^{r}\binom{r}{j}\delta_n^{\alpha(1-\lambda)}\left(x + (j-\frac{r}{2})t\varphi^\lambda(x)\right) \\ &\le \|g\|_0 2^r \delta_n^{\alpha(1-\lambda)}(x). \end{aligned} \tag{3.11}$$

Using Lemma 3.2 for $g \in C^r_\lambda$, $0 < t\varphi^\lambda(x) < \frac{1}{8r}$ and $\frac{rt\varphi^\lambda(x)}{2} \le x \le 1 - \frac{rt\varphi^\lambda(x)}{2}$, we have

$$\begin{aligned} &\left|\Delta^r_{t\varphi^\lambda(x)}g(x)\right| \\ &\le \left|\int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)}\cdots\int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} g^{(r)}(x+u_1+\cdots+u_r)du_1\cdots du_r\right| \\ &\le \|g\|_r \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} \\ &\quad \cdots \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} \delta_n^{-r+\alpha(1-\lambda)}(x+u_1+\cdots+u_r)du_1\cdots du_r \\ &\le M_3t^r\varphi^{r\lambda}(x)\delta_n^{-r+\alpha(1-\lambda)}(x)\|g\|_r \le M_3t^r\delta_n^{(\alpha-r)(1-\lambda)}(x)\|g\|_r. \end{aligned} \tag{3.12}$$

From (3.9)-(3.12), $0 < t\varphi^\lambda(x) < \frac{1}{8r}$, $\frac{rt\varphi^\lambda(x)}{2} \le x \le 1 - \frac{rt\varphi^\lambda(x)}{2}$ and by choosing an appropriate g, we have

$$\begin{aligned}\left|\Delta^r_{t\varphi^\lambda(x)} f(x)\right| &\le \left|\Delta^r_{t\varphi^\lambda(x)}(f-g)(x)\right| + \left|\Delta^r_{t\varphi^\lambda(x)} g(x)\right| \\ &\le M_4 \delta_n^{\alpha(1-\lambda)}(x)\left\{\|f-g\|_0 + t^r \delta_n^{r(\lambda-1)}(x)\|g\|_r\right\} \\ &\le 2M_4 \delta_n^{\alpha(1-\lambda)} K_\lambda^\alpha\left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)}\right) \\ &\le 2M_4 \delta_n^{\alpha(1-\lambda)}(x) \frac{t^\alpha}{\delta_n^{\alpha(1-\lambda)}(x)} \\ &= M_5 t^\alpha.\end{aligned}$$

The proof of (3.1) is complete.

4. Proofs of the lemmas

Proof of Lemma 3.1. We first prove (3.3). Suppose that $E_n = [\frac{1}{n},\ 1-\frac{1}{n}]$. For $x \in E_n^c = (0, \frac{1}{n}) \bigcup (1-\frac{1}{n}, 1)$, we have [5] by Hölder's inequality

$$\begin{aligned}\left|B_n^{(r)}(f,x)\right| &= \left|\frac{n!}{(n-r)!}\sum_{k=0}^{n-r} \vec{\Delta}^r_{\frac{1}{n}} f\left(\frac{k}{n}\right) p_{n-r,k}(x)\right| \\ &\le Ln^r \|f\|_0 \sum_{k=0}^{n-r}\sum_{j=0}^{r}\binom{r}{j}\delta_n^{\alpha(1-\lambda)}\left(\frac{k+r-j}{n}\right) p_{n-r,k}(x) \\ &\le L_1 n^r \|f\|_0 \left(\sum_{k=0}^{n-r}\sum_{j=0}^{r}\delta_n^{2r}\left(\frac{k+r-j}{n}\right) p_{n-r,k}(x)\right)^{\alpha(1-\lambda)/2r}.\end{aligned}$$

For $n > 4r$ we have

$$\begin{aligned}&\sum_{k=0}^{n-r}\varphi^{2r}\left(\frac{k+1}{n}\right) p_{n-r,k}(x) \\ &\qquad = \left(\sum_{k=0}^{2r} + \sum_{k=2r+1}^{n-3r} + \sum_{k=n-3r+1}^{n-r}\right)\varphi^{2r}\left(\frac{k+1}{n}\right) p_{n-r,k}(x) \\ &\qquad := I_1 + I_2 + I_3.\end{aligned}$$

Obviously $I_1 + I_3 \le 2((3r+1)!)^r/n^r$ and, by simple computation, we have

$$\begin{aligned}I_2 &= \sum_{k=2r+1}^{n-3r}\left(\frac{k+1}{n}\right)^r\left(\frac{n-k-1}{n}\right)^r \frac{(n-r)!}{(n-r-k)!k!} x^k (1-x)^{n-r-k} \\ &= x^r(1-x)^r \sum_{k=2r+1}^{n-3r} \frac{(n-r)\cdots(n-3r+1)}{n^{2r}} \cdot \frac{(k+1)^r}{k\cdots(k-r+1)} \\ &\quad \cdot \frac{(n-k-1)^r}{(n-r-k)\cdots(n-2r-k+1)} p_{n-3r,k-r}(x) \\ &\le 2\times 2^r \times 3^r \varphi^{2r}(x).\end{aligned}$$

By this and $\delta_n^{2r}(\frac{k+r-j}{n}) \sim \varphi^{2r}(\frac{k+1}{n}) + (\frac{1}{n})^r$, one has

$$\left|B_n^{(r)}(f,x)\right| \le L_2 n^r \|f\|_0 \delta_n^{\alpha(1-\lambda)}(x).$$

Recalling that $x \in E_n^c$ implies $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, we see that

$$\delta_n^{r+\alpha(\lambda-1)}(x) \left|B_n^{(r)}(f,x)\right| \le L_3 n^{\frac{r}{2}} \|f\|_0. \tag{4.1}$$

For $x \in E_n$ we use the expression (cf. [5])

$$B_n^{(r)}(f,x) = (x(1-x))^{-r} \sum_{i=0}^{r} Q_i(x,n) n^i \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^i f\left(\frac{k}{n}\right) p_{n,k}(x)$$

with $Q_i(n,x)$ a polynomials in $nx(1-x)$ of degree $[(r-i)/2]$ with nonconstant bounded coefficients. Thus,

$$\left|(x(1-x))^{-r} Q_i(x,n) n^i\right| \le L_4 \left(\frac{n}{x(1-x)}\right)^{(r+i)/2} \quad \text{for } x \in E_n.$$

If $x \in E_n$ then $\delta_n(x) \sim \varphi(x)$ and, recalling that $\delta_n^2(\frac{k}{n}) \sim \varphi^2(\frac{k}{n}) + \frac{1}{n}$, we have by using Hölder inequality twice

$$\begin{aligned}
\left|B_n^{(r)}(f,x)\right| &\le L_5 \sum_{i=0}^{r} \left(\frac{n}{\varphi^2(x)}\right)^{(r+i)/2} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^i \delta_n^{\alpha(1-\lambda)}\left(\frac{k}{n}\right) p_{n,k}(x) \|f\|_0 \\
&\le L_5 \|f\|_0 \sum_{i=0}^{r} \left(\frac{n}{\varphi^2(x)}\right)^{(r+i)/2} \left(\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2i} p_{n,k}(x)\right)^{\frac{1}{2}} \\
&\quad \left(\sum_{k=0}^{n} \left(\varphi^2\left(\frac{k}{n}\right) + \frac{1}{n}\right)^r p_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{2r}}.
\end{aligned}$$

Proceeding as in (4.1), we obtain

$$\sum_{k=0}^{n} \left(\varphi^2\left(\frac{k}{n}\right) + \frac{1}{n}\right)^r p_{n,k}(x) \le C \delta_n^{2r}(x).$$

It is known that for $m \in N$ [5]

$$\left|\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2m} p_{n,k}(x)\right| \le L_6 n^{-m} \varphi^{2m}(x) \quad \text{for } x \in E_n.$$

Consequently, with $x \in E_n$, $\delta_n(x) \sim \varphi(x)$ and

$$\begin{aligned}
\left|B_n^{(r)}(f,x)\right| &\le L_6 \|f\|_0 \sum_{i=0}^{r} \left(\frac{n}{\varphi^2(x)}\right)^{(r+i)/2} \left(\frac{\varphi^{2i}(x)}{n^i}\right)^{\frac{1}{2}} \varphi^{\alpha(1-\lambda)}(x) \\
&\le L_6 (r+1) \left(\frac{n}{\varphi^2(x)}\right)^{r/2} \varphi^{\alpha(1-\lambda)}(x) \|f\|_0.
\end{aligned} \tag{4.2}$$

Hence, using $\delta_n(x) \sim \varphi(x)$ we obtain (3.1) for $x \in E_n$. We complete our proof by using (4.1) and (4.2).

Proof of (3.4). We recall that [5]

$$\left|B_n^{(r)}(f,x)\right| \le n^r \sum_{k=0}^{n-r} \left|\vec{\Delta}^r_{\frac{1}{n}} f\left(\frac{k}{n}\right)\right| p_{n-r,k}(x). \tag{4.3}$$

For $0 < k < n-r$, by [5, p. 155],

$$\frac{k}{n}\left(1-\frac{k}{n}\right) \le R\left(\frac{k}{n}+y\right)\left(1-\frac{k}{n}-y\right), \quad 0 < y < \frac{r}{n},$$

therefore $\delta_n\left(\frac{k}{n}\right) \le R_1 \delta_n\left(\frac{k}{n}+y\right)$. Hence we have, as in [5],

$$\begin{aligned} \left|\vec{\Delta}^r_{\frac{1}{n}} f(\frac{k}{n})\right| &\le R_2 n^{-r+1} \int_0^{\frac{r}{n}} \left|f^{(r)}\left(\frac{k}{n}+u\right)\right| du \\ &\le R_2 n^{-r+1} \|f\|_r \int_0^{\frac{r}{n}} \delta_n^{-r+\alpha(1-\lambda)}\left(\frac{k}{n}+u\right) du \\ &\le R_2 n^{-r+1} \|f\|_r \frac{r}{n} \delta_n^{-r+\alpha(1-\lambda)}\left(\frac{k}{n}\right). \end{aligned} \tag{4.4}$$

For $k=0$, note that $u \in (0, \frac{r}{n})$ implies $\delta_n(u) \sim \frac{1}{\sqrt{n}}$. Thus, we have

$$\begin{aligned} \left|\vec{\Delta}^r_{\frac{1}{n}} f(0)\right| &\le R_3 \int_0^{\frac{r}{n}} u^{r-1} |f^{(r)}(u)| du \\ &\le R_3 \|f\|_r \int_0^{\frac{r}{n}} u^{r-1} \delta_n^{-r+\alpha(1-\lambda)}(u) du \\ &\le R_4 n^{-(r+\alpha(1-\lambda))/2} \|f\|_r. \end{aligned} \tag{4.5}$$

Similarly for $k = n-r$ we have

$$\left|\vec{\Delta}^r_{\frac{1}{n}} f(\frac{n-r}{n})\right| \le R_5 n^{-(r+\alpha(1-\lambda))/2} \|f\|_r. \tag{4.6}$$

From (4.3)-(4.6) we get, with $\delta_n(\frac{k}{n}) \sim \delta_n(\frac{k+1}{n-r+2})$,

$$\begin{aligned} &\left|\delta_n^{r+\alpha(\lambda-1)}(x) B_n^{(r)}(f,x)\right| \\ &\le R_6 n^r \|f\|_r \delta_n^{r+\alpha(\lambda-1)}(x) \Bigg[n^{\frac{-r+\alpha(\lambda-1)}{2}} (p_{n-r,0}(x) + p_{n-r,n-r}(x)) \\ &\quad + \sum_{k=1}^{n-r-1} n^{-r} \delta_n^{-r+\alpha(\lambda-1)}\left(\frac{k}{n}\right) p_{n-r,k}(x) \Bigg]. \end{aligned} \tag{4.7}$$

By a simple computation, it is easy to get (cf. [2])

$$\sum_{k=1}^{n-r-1} \left(\frac{n}{k}\right)^r p_{n-r,k}(x) \le C \frac{1}{x^r},$$

and

$$\sum_{k=1}^{n-r-1} \left(\frac{n}{n-k}\right)^r p_{n-r,k}(x) \le C \frac{1}{(1-x)^r}.$$

Hence

$$\sum_{k=1}^{n-r-1}\varphi^{-2r}\left(\frac{k}{n}\right)p_{n-r,k}(x) \le 2^r\sum_{k=1}^{n-r-1}\left(\left(\frac{n}{k}\right)^r+\left(\frac{n}{n-k}\right)^r\right)p_{n-r,k}(x)\le C_1\varphi^{-2r}(x). \tag{4.8}$$

Note that $\delta_n(\frac{k}{n})\sim\max\{\varphi(\frac{k}{n}),\frac{1}{\sqrt{n}}\}$, and that by (4.7) and (4.8) we have

$$\begin{aligned}
&\left|\delta_n^{r+\alpha(\lambda-1)}(x)B_n^{(r)}(f,x)\right|\\
&\le R_6\|f\|_r\delta_n^{r+\alpha(\lambda-1)}(x)\Bigg[n^r(p_{n-r,0}(x)+p_{n-r,n-r}(x))\\
&\quad+\sum_{k=1}^{n-r-1}\delta_n^{-2r}\left(\frac{k}{n}\right)p_{n-r,k}(x)\Bigg]^{\frac{r-\alpha(1-\lambda)}{2r}}\\
&\le R_7\|f\|_r\delta_n^{r+\alpha(\lambda-1)}(x)\left(\min\left\{n^r,\sum_{k=1}^{n-r-1}\varphi^{-2r}\left(\frac{k}{n}\right)p_{n-r,k}(x)\right\}\right)^{\frac{r-\alpha(1-\lambda)}{2r}}\\
&\le R_8\|f\|_r\delta_n^{r+\alpha(\lambda-1)}(x)\delta_n^{-r+\alpha(1-\lambda)}(x)=R_8\|f\|_r.
\end{aligned}$$

Now we have proved the inequality (3.4). This finishes the proof of Lemma 3.1.

Proof of Lemma 3.2. It is known that, for $0<t<\frac{1}{8r}$, $\frac{rt}{2}\le x\le 1-\frac{rt}{2}$ (cf. [7]),

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}}\varphi^{-r}(x+u_1+\cdots+u_r)du_1\cdots du_r\le Ct^r\varphi^{-r}(x).$$

Using this and Hölder's inequality we obtain

$$\begin{aligned}
&\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}}\delta_n^{-r+\alpha(1-\lambda)}(x+u_1+\cdots+u_r)du_1\cdots du_r\\
&\le C_1\Bigg(\int_{-\frac{t}{2}}^{\frac{t}{2}}\\
&\quad\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}}\min\left\{\varphi^{-r}(x+u_1+\cdots+u_r),n^{\frac{r}{2}}\right\}du_1\cdots du_r\Bigg)^{\frac{r-\alpha(1-\lambda)}{r}}t^{\alpha(1-\lambda)}\\
&\le C_2\left(t^r\min\left\{\varphi^{-r}(x),n^{\frac{r}{2}}\right\}\right)^{\frac{r-\alpha(1-\lambda)}{r}}t^{\alpha(1-\lambda)}\\
&\le C_3t^r\delta_n^{-r+\alpha(1-\lambda)}(x),
\end{aligned}$$

which is the stated result.

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