FIXED-POINT AND COINCIDENCE THEOREMS FOR SET-VALUED MAPS WITH NONCONVEX OR NONCOMPACT DOMAINS IN TOPOLOGICAL VECTOR SPACES

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A technique, based on the investigations of the images of maps, for obtaining fixed-point and coincidence results in a new class of maps and domains is described. In particular, we show that the problem concerning the existence of fixed points of expansive set-valued maps and inner set-valued maps on not necessarily convex or compact sets in Hausdorff topological vector spaces has a solution. As a consequence, we prove a new intersection theorem concerning not necessarily convex or compact sets and its applications. We also give new coincidence and section theorems for maps defined on not necessarily convex sets in Hausdorff topological vector spaces. Examples and counterexamples show a fundamental difference between our results and the well-known ones.

1. Introduction

Suppose that E is a Hausdorff topological vector space, $C \subset E$, $C \neq \emptyset$. In fixed-point and coincidence theory and its applications, a great part of the vast literature in the last century concerns conditions on C, E, F, and G guaranteeing the existence of fixed points or coincidences of set-valued maps $F : C \to 2^E$ and $G : C \to 2^E$. In various methods of investigations, the assumptions that the maps are inner and C are convex compact subsets of E play the crucial role (see, e.g., [5, 6, 7, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 30, 33, 41]). The general topic of fixed points and coincidences for set-valued maps on convex compact sets, originating mainly with the work of Kakutani [30], Bohnenblust and Karlin [5], Glicksberg [23], Fan [15, 16, 17, 18, 19, 20, 21, 22], and Browder [6, 7], has been well developed in various directions.

In the past decade, there was a renewed interest in the fixed-point and coincidence theory of set-valued maps in topological vector spaces (see, e.g., [2, 3, 8, 9, 10, 11, 12, 28, 36, 37, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48]), partially due to new and powerful methods of investigations introduced into it (notably

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based on those introduced by Fan and Browder). Most of the work has centered around the fixed-point and coincidence theory of maps on convex compact sets, but there are also a considerable number of papers devoted to maps on nonconvex and noncompact sets (see, e.g., [8, 45]).

There exist a number of introductions to and surveys of fixed-point and coincidence theory. We mention [47] among the more recent ones but also some elder ones [14, 49, 50]. See also many references therein.

A natural question arises: whether expansive set-valued maps and inner set-valued maps on not necessarily convex or compact sets have fixed points and, as a consequence, theorems of intersection type hold and whether the maps $F : C \to 2^E$ and $G : C \to 2^E$ on not necessarily convex sets in Hausdorff topological vector spaces have coincidences. The affirmative answers are given in this paper. Using a technique based on the investigation of the images of maps, we obtain a number of new fixed-point, coincidence, intersection, and section theorems of Fan-Browder type. Examples and counterexamples show a fundamental difference between our results and the known results of the above-mentioned authors.

2. Fixed points and coincidences of expansive set-valued maps on not necessarily convex or compact sets in topological vector spaces

Let C be a subset of a Hausdorff topological vector space E over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A set-valued map $F : C \to E$ (which will always be denoted by capital letters) is a map which assigns a unique $F(c) \in 2^E$ (here 2^E denotes the family of all subsets of E) to each $c \in C$. We say that $c \in C$ is a fixed point of $F : C \to 2^E$ if $c \in F(c)$. We say that a map $F : C \to 2^E$ is expansive if $C \subset F(C)$ where $F(C) = \bigcup_{c \in C} F(c)$.

Maps in the usual sense will be considered as special (single-valued) set-valued maps and these ordinary maps will always be denoted by small letters $f : C \to E$.

We prove the following theorem.

Theorem 2.1. *Let C be a nonempty subset of a Hausdorff topological vector space E over $\mathbb{R}$, let $F : C \to 2^E$, and let K be a convex subset of E. Assume that the following conditions hold:*

(i) *$C \subset K \subset F(C)$;*
(ii) *$F(C)$ is a compact subset of E;*
(iii) *for each $c \in C$, $F(c)$ is open in $F(C)$;*
(iv) *for each $y \in K$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is nonempty and convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. By (iii), the compact set $F(C)$ is covered by the sets $F(c)$, $c \in C$, which are open in $F(C)$. Clearly, there exists a finite set $\{c_1, \ldots, c_n\} \subset C$ such that $F(c_i)$ are nonempty, $1 \leq i \leq n$, and $F(C) = \bigcup_{i=1}^{n} F(c_i)$. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a partition of unity with respect to this cover, that is, a finite family of real-valued nonnegative continuous maps φ_i on $F(C)$ such that φ_i vanish outside $F(c_i)$ and are less than or equal to one everywhere, $1 \leq i \leq n$, and $\sum_{i=1}^{n} \varphi_i(y) = 1$ for all $y \in F(C)$.

Let σ be a simplex spanned by points $c_1, \dots, c_n$ and let $\varphi : F(C) \to \sigma$ be a continuous map defined by the formula $\varphi(y) = \sum_{i=1}^{n} \varphi_i(y)c_i$, $y \in F(C)$. Clearly, $\sigma \subset K \subset F(C)$ and hence $\varphi(\sigma) \subset \varphi(K) \subset \varphi(F(C)) \subset \sigma$.

If $y \in K$ is arbitrary and fixed and $\varphi_i(y) \neq 0$ for some $i \in \{1, \dots, n\}$, then $y \in F(c_i)$, so $c_i \in F^{-1}(y)$. As a consequence, for each $y \in K$, $\varphi(y)$ is a convex linear combination of points of $F^{-1}(y)$ and by (iv), we get for each $y \in K$,

$$\varphi(y) \in F^{-1}(y), \quad \varphi(y) \in C. \tag{2.1}$$

From Brouwer's theorem, we get $u = \varphi(u)$ for some $u \in \sigma$ and hence, since $\sigma \subset K$, by (2.1), $u = \varphi(u) \in F^{-1}(u) \subset C$, and therefore, $u \in F(u)$ and $u \in C$, as required. □

By using various sets K, a number of variations of Theorem 2.1 can be obtained, of which the following two are typical.

Theorem 2.2. *Let C be a nonempty subset of a Hausdorff topological vector space E over $\mathbb{R}$ and let $F : C \to 2^E$. Assume that the following conditions hold:*

(i) *F is expansive, that is, $C \subset F(C)$;*
(ii) *$F(C)$ is convex;*
(iii) *$F(C)$ is a compact subset of E;*
(iv) *for each $c \in C$, $F(c)$ is open in $F(C)$;*
(v) *for each $y \in F(C)$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is nonempty and convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. We use Theorem 2.1 for $K = F(C)$. □

Theorem 2.3. *Let C be a nonempty subset of a Hausdorff topological vector space E over $\mathbb{R}$ and let $F : C \to 2^E$. Assume that the following conditions hold:*

(i) *F is expansive, that is, $C \subset F(C)$;*
(ii′) *C is convex;*
(iii) *$F(C)$ is a compact subset of E;*
(iv) *for each $c \in C$, $F(c)$ is open in $F(C)$;*
(v′) *for each $y \in C$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is nonempty and convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. Indeed, if φ and σ are as in the proof of Theorem 2.1 and C is convex, then $\sigma \subset C \subset F(C)$, and we may use Theorem 2.1 for $K = C$. □

Example 2.4. (a) Let $E = \mathbb{R}^2$, let T be a closed triangle with vertices (0,0), (1,0) and (0,1), and let $C = C_1 \cup \{(0,0)\} \cup C_2$ where $C_1 = \{c = (c_1, 0) : 0 < c_1 \leq 1\}$ and $C_2 = \{c = (0, c_2) : 0 < c_2 \leq 1\}$. Define sets $P = \{c = (c_1, c_2) : |c_2 - c_1| < 1/2\}$, $H_1 = \{c = (c_1, c_2) : c_2 < c_1\}$ and $H_2 = \{c = (c_1, c_2) : c_2 > c_1\}$. If $F(C) = T$ where

$F(0,0) = T \cap P$, $F(c) = T \cap H_1$ for $c \in C_1$, and $F(c) = T \cap H_2$ for $c \in C_2$, then the assumptions of Theorem 2.2 are satisfied, C is nonconvex and $\mathrm{Fix}(F) = C$.

(b) Let $E = \mathbb{R}$, $C = (1;3)$, $F(C) = [0;4]$ where $F(c) = [0;2)$ for $c \in (1;2)$, $F(c) = (2;4]$ for $c \in (2,3)$ and $F(2) = (1;3)$. Then the assumptions of Theorem 2.2 are satisfied, C is noncompact and $\mathrm{Fix}(F) = C$.

If in Theorems 2.2 or 2.3 we omit at least one of the assumptions, then we can construct a counterexample.

Example 2.5. (a) The condition on $F(C)$ in (i) cannot be omitted as the following two counterexamples show:

(A_1) if $E = \mathbb{R}$, $C = (1;4)$, $F(C) = [2;5]$ where $F(c) = [2;5]$ for $c \in (1;2)$ and $F(c) = (4;5]$ for $c \in [2;4)$, then assumptions (ii), (iii), (iv), and (v) are satisfied, $C \not\subset F(C)$, $F(C) \not\subset C$, and $\mathrm{Fix}(F) = \varnothing$;

(A_2) if $E = \mathbb{R}^2$, $C = \{c = (c_1, c_2) : |c_1| \le 1,\ |c_2| \le 2\}$, $C = C_1 \cup C_2 \cup C_3 \cup (-C_3) \cup C_4$ where $C_1 = \{(0,-2)\}$, $C_2 = \{(0,2)\}$, $C_3 = \{c = (c_1, c_2) : -1 \le c_1 < 0,\ |c_2| \le 2\}$, and $C_4 = \{c = (c_1, c_2) : c_1 = 0,\ |c_2| < 2\}$, $F(C_1) = D_1$, $F(C_2) = D_2$, $F(C_3) = D_3$, $F(-C_3) = -D_3$, $F(C_4) = D_3 \cup (-D_3)$, and if $F(C) = \{y = (y_1, y_2) : |y_1| \le 1, |y_2| \le 1\}$ where $D_1 = \{y = (y_1, y_2) : |y_1| < 1/2,\ |y_2| \le 1\}$, $D_2 = \{y = (y_1, y_2) : 1/2 < |y_1| \le 1,\ |y_2| \le 1\}$, and $D_3 = \{y = (y_1, y_2) : 0 < y_1 \le 1,\ |y_2| \le 1\}$, then assumptions (ii), (iii), (iv), and (v) are satisfied, $F(C) \subset C$, $F(C) \ne C$, and $\mathrm{Fix}(F) = \varnothing$.

(b) The assumption that $F(C)$ is convex or C is convex is necessary. Indeed, let $E = \mathbb{R}^2$, $C = C_0 \cup (-C_0)$ where $C_0 = \{c = (c_1, 0) : c_1 \in (1;2)\}$, $F(c) = D_0$ for $c \in -C_0$, and $F(c) = -D_0$ for $c \in C_0$ where $D_0 = \{y = (y_1, y_2) : \|(y_1, y_2) - (3/2, 0)\| \le 1\}$. Then assumptions (i), (iii), (iv), and (v) are satisfied, while (ii) and (ii′) are not, and $\mathrm{Fix}(F) = \varnothing$.

(c) The assumption that $F(C)$ is compact is necessary. Indeed, let $E = \mathbb{R}$, $C = (0;1)$, $F(c) = (0;c)$ for $1/2 \le c < 1$, $F(c) = (1/2 + c; 1)$ for $0 < c < 1/2$. Then assumptions (i), (ii), (iv), and (v) are satisfied and $\mathrm{Fix}(F) = \varnothing$.

(d) Assumption (iv) is necessary. Indeed, let $E = \mathbb{R}$, $C = (1;4)$, $F(C) = [1;4]$ where $F(c) = [1;2) \cup (3;4]$ for $c \in (2;3)$, $F(c) = (2;5/2)$ for $c \in (1;2)$, $F(c) = [5/2;3)$ for $c \in (3;4)$, $F(2) = \{3\}$ and $F(3) = \{2\}$. Thus assumptions (i), (ii), (iii), and (v) are satisfied and $\mathrm{Fix}(F) = \varnothing$.

(e) Assumption (v) (or (v′)) is necessary. Indeed, let $E = \mathbb{R}$, $C = (1;5)$, $F(C) = [1;5]$ where $F(c) = (2;3) \cup (4;5]$ for $c \in (1;2)$, $F(c) = [1;2) \cup (3;4)$ for $c \in (2;3)$, $F(c) = [1;3) \cup (4;5]$ for $c \in (3;4)$, $F(c) = [1;2) \cup (3;4]$ for $c \in (4;5)$, $F(2) = (2;5]$, $F(3) = [1;3) \cup (3;5]$, and $F(4) = [1;4)$. Then $F^{-1}(4) = \{2,3\}$ is nonconvex. Thus assumptions (i), (ii′), (iii), and (iv) are satisfied, while (v′) is not, and $\mathrm{Fix}(F) = \varnothing$.

We say that a single-valued map $f : C \to E$ and a set-valued map $F : C \to 2^E$ have a coincidence if $f(c) \in F(c)$ for some $c \in C$.

The following theorem is a generalization of the above one.

Theorem 2.6. *Let C be a nonempty convex subset of a Hausdorff topological vector space E over $\mathbb{R}$, let $F : C \to 2^E$ be an expansive map, and let $f : C \to E$ be a single-valued continuous map such that $f(C) \subset F(C)$. Assume that the following conditions hold:*

(i) *$F(C)$ is a compact subset of E;*
(ii) *for each $c \in C$, $F(c)$ is open in $F(C)$;*
(iii) *for each $y \in f(C)$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is nonempty and convex.*

Then there exists $u \in C$ such that $f(u) \in F(u)$.

Proof. Let φ and σ be as in the proof of Theorem 2.1. We have $\varphi : F(C) \to \sigma$, $\sigma \subset C$, and $y \in F(\varphi(y))$ for each $y \in f(C)$. On the other hand, $\varphi \circ f : \sigma \to \sigma$ and, by the theorem of Brouwer, $(\varphi \circ f)(u) = u$ for some $u \in \sigma$. Consequently, $f(u) \in F(\varphi(f(u))) = F(u)$. □

3. Fixed points and coincidences of set-valued inner maps on not necessarily convex sets in topological vector spaces

We say that a map $F : C \to 2^E$ is inner if $F(C) \subset C$. This section is devoted to new fixed-point and coincidence theorems for set-valued inner maps on not necessarily convex sets.

We have the following theorem.

Theorem 3.1. *Let C be a nonempty compact subset of a Hausdorff topological vector space E over $\mathbb{R}$ and let $F : C \to 2^E$ be an inner map such that $F(C)$ is a convex subset of E. Assume that the following conditions hold:*

(i) *for each $c \in C$, $F(c)$ is nonempty and convex;*
(ii) *for each $y \in F(C)$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is open in C.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. In virtue of (i) and (ii), there exists a finite set $\{y_1, \dots, y_n\} \subset F(C)$ such that $F^{-1}(y_i)$ are nonempty, $1 \le i \le n$, and $C = \bigcup_{i=1}^n F^{-1}(y_i)$. Let $\{\varphi_1, \dots, \varphi_n\}$ be a partition of unity with respect to this cover, let σ be a simplex spanned by points $y_1, \dots, y_n$, and let a continuous map $\varphi : C \to \sigma$ be defined by the formula $\varphi(c) = \sum_{i=1}^n \varphi_i(c) y_i$, $c \in C$. Note that $\sigma \subset F(C) \subset C$ and, consequently, $\varphi(\sigma) \subset \sigma$.

If $c \in C$ and $\varphi_i(c) \neq 0$ for some $i \in \{1, \dots, n\}$, then $c \in F^{-1}(y_i)$, thus $y_i \in F(c)$. By (i), for each $c \in C$,

$$\varphi(c) \in F(c). \tag{3.1}$$

On the other hand, from Brouwer's theorem, we get that $u = \varphi(u)$ for some $u \in \sigma$ and, by (3.1), we have $u = \varphi(u) \in F(u)$, as required. □

Recall that a map $F : C \to 2^E$ is called upper semicontinuous if, for each $c \in C$ and any open set V containing $F(c)$, there is an open set U containing c such that $F(U \cap C) \subset V$ (for details, see [4]). A map $F : C \to 2^E$ is called upper demicontinuous on C (after Fan [20]) if, for each $c \in C$ and any open half-space H in E containing $F(c)$, there is a neighbourhood $N(c)$ of c in C such that $F(x) \subset H$ for each $x \in N(c)$. It is clear that the condition of upper semicontinuity is stronger than that of upper demicontinuity.

Let C and D be nonempty sets. The maps $F : C \to 2^D$ and $G : D \to 2^C$ are said to have a coincidence if there exists $(u, v) \in C \times D$ such that $v \in F(u)$ and $u \in G(v)$.

We now establish the following theorem.

Theorem 3.2. *Let C be a nonempty compact subset of a Hausdorff locally convex topological vector space E over $\mathbb{R}$, let $F : C \to 2^E$ be an inner map, and let $G : C \to 2^E$ be an upper demicontinuous map such that $G(C) \subset F(C)$. Assume that the following conditions hold:*

(i) *$F(C)$ is a closed convex subset of E;*
(ii) *for each $c \in C$, $F(c)$ is nonempty and convex;*
(iii) *for each $y \in F(C)$, $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is open in C;*
(iv) *for each $c \in C$, $G(c)$ is a nonempty closed convex subset of $F(C)$.*

Then there exists $(u, v) \in C \times F(C)$ such that $u \in F(v)$ and $v \in G(u)$.

Proof. Let φ and σ be as in the proof of Theorem 3.1. Since $\sigma \subset F(C) \subset C$ and $\varphi : C \to \sigma$, then $G \circ \varphi : F(C) \to 2^{F(C)}$ and $G \circ \varphi$ is upper demicontinuous on the compact convex set $F(C)$. By [20, Theorem 6], there exists $v \in F(C)$ such that $v \in G(\varphi(v))$. Moreover, in virtue of (3.1), $\varphi(v) \in F(v)$. This implies the assertion for $u = \varphi(v)$. □

4. Intersection theorem with applications on not necessarily convex or compact sets in topological vector spaces

Various intersection theorems concerning convex and compact sets, with their applications, are given in [6, 17, 18, 21, 22, 33, 35]. From Theorem 2.2, we get the following new intersection theorem.

Theorem 4.1. *Let E be a Hausdorff topological vector space over $\mathbb{R}$ and let $n \geq 2$. Let $C_1, \dots, C_n$ be nonempty (not necessarily convex or compact) subsets of E, let $K_1, \dots, K_n$ be compact and convex subsets of E, let $S_1, \dots, S_n$ be nonempty subsets of E^n, and let $C = \prod_{j=1}^n C_j$, $K = \prod_{j=1}^n K_j$, and $S = \bigcup_{j=1}^n S_j$. Assume that the following properties hold:*

(i) *$C \subset K = S$;*
(ii) *for each i, $1 \leq i \leq n$, and for each point $(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ of $\prod_{j \neq i}^n K_j$, the section $S_i(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$, formed by all points $c_i \in C_i$ such that $(y_1, \dots, y_{i-1}, c_i, y_{i+1}, \dots, y_n) \in S_i$, is a nonempty convex subset of C_i;*

(iii) *for each* i, $0 \le i \le n$, *and for each point* $c_i \in C_i$, *the section* $S_i(c_i)$, *formed by all points* $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ *of* $\prod_{j\neq i}^n K_j$ *such that*

$$(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) \in S_i, \tag{4.1}$$

is an open subset of $\prod_{j\neq i}^n K_j$.

Then $C \cap \bigcap_{i=1}^n S_i \neq \emptyset$.

Proof. Define $F : C \to 2^K$ as follows. Fix a point $c \in C$ and let $y \in K$. We say that $y \in F(c)$ if and only if, for each $i \in \{1,\dots,n\}$, $(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) \in S_i$.

Write $c \in C$ in the form $c = (c_1,\dots,c_{i-1},c_i,c_{i+1},\dots,c_n)$, $1 \le i \le n$. Using condition (iii) and taking into consideration that, for each $i \in \{1,\dots,n\}$, the section $S_i(c_i)$ is an open subset of $\prod_{j\neq i}^n K_j$, we obtain that $S_i(c_i) \times K_i$ is an open subset of K. Therefore, the set $F(c) = \bigcap_{i=1}^n (S_i(c_i) \times K_i)$ is open in K.

Suppose that $y \in K$. Write $c = (c_1,\dots,c_{i-1},c_i,c_{i+1},\dots,c_n)$ and note that since c belongs to $F^{-1}(y)$ if and only if, for each $i \in \{1,\dots,n\}$,

$$c_i \in S_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n), \tag{4.2}$$

we have $F^{-1}(y) = \prod_{i=1}^n S_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$. But, for each $i \in \{1,\dots,n\}$, the sections $S_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ are nonempty convex subsets of C_i by condition (ii), and thus, $F^{-1}(y)$ is a nonempty convex subset of C. We conclude that $F(C) = K$.

It follows from Theorem 2.2 that $u \in F(u)$ for some $u \in C$. This shows that, for each $i \in \{1,\dots,n\}$, $u = (u_1,\dots,u_{i-1},u_i,u_{i+1},\dots,u_n) \in S_i$, that is, $u \in C \cap \bigcap_{i=1}^n S_i$. □

Example 4.2. Let $E = \mathbb{R}^2$, $n = 2$, $K = K_1 \times K_2$ where $K_1 = K_2 = [0;1]$, $C_1 = C_{1,0} \cup C_{1,1}$ where $C_{1,0} = [0;1/3)$ and $C_{1,1} = (2/3;1]$, $C_2 = [0;1]$, $C = C_1 \times C_2$, $S = S_1 \cup S_2$ where $S_1 = K \cap \{(x_1,x_2) : x_2 > -3x_1+2\}$, and $S_2 = K \cap \{(x_1,x_2) : x_2 < -x_1 + 3/2\}$. Hence C_1 is a noncompact and nonconvex subset of K_1, the assumptions of Theorem 4.1 are satisfied and $C \cap \bigcap_{i=1}^2 S_i \neq \emptyset$.

As an application of Theorem 4.1 we obtain the following theorem.

Theorem 4.3. *Let E be a Hausdorff topological vector space over $\mathbb{R}$ and let $n \ge 2$. Let $C_1,\dots,C_n$ be nonempty (not necessarily convex or compact) subsets of E, let $K_1,\dots,K_n$ be convex compact subsets of E, and let $C = \prod_{j=1}^n C_j$, $K = \prod_{j=1}^n K_j$. Let $f_1,\dots,f_n$ be real-valued maps defined on K, let $t_1,\dots,t_n$ be real numbers, and let the following conditions hold:*

(i) $C \subset K$;

(ii) *for each* i, $1 \le i \le n$, *and for each point* $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ *of* $\prod_{j\neq i}^n K_j$, *the set* $\{c_i \in C_i : f_i(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) > t_i\}$ *is a nonempty convex subset of* C_i;

(iii) *for each i, $1 \le i \le n$, and for each point $c_i \in C_i$, the set*

$$\left\{(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n) \in \prod_{j\neq i} K_j : f_i(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) > t_i\right\} \tag{4.3}$$

is an open subset of $\prod_{j\neq i}^n K_j$.

Then there is a point u in C such that $f_i(u) > t_i$ for each i, $1 \le i \le n$.

Proof. Define the subsets S_i of K to be $S_i = \{y : y \in K, f_i(y) > t_i\}$, $i \in \{1,\dots,n\}$. Clearly, (ii) is equivalent to the condition:

(ii′) for each $i \in \{1,\dots,n\}$ and for each point $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ of $\prod_{j\neq i}^n K_j$, the section $S_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$, formed by all points $c_i \in C_i$ such that $(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) \in S_i$, is a nonempty convex subset of K_i,

and (iii) is equivalent to the condition:

(iii′) for each $i \in \{1,\dots,n\}$ and for each point $c_i \in C_i$, the section $S_i(c_i)$, formed by all points $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$ of $\prod_{j\neq i}^n K_j$ such that

$$(y_1,\dots,y_{i-1},c_i,y_{i+1},\dots,y_n) \in S_i, \tag{4.4}$$

is an open subset of $\prod_{j\neq i}^n K_j$. □

We can apply Theorem 4.1 to obtain $C \cap \bigcap_{i=1}^n S_i \neq \varnothing$. Hence, by the definition of S_i, the point u from this intersection satisfies $f_i(u) > t_i$ for each $i \in \{1,\dots,n\}$.

5. Coincidence theorems for set-valued maps and section theorems on not necessarily convex sets in topological vector spaces

Using his infinite-dimensional version of the KKM theorem as a tool, Fan [16] established a geometrical "lemma" concerning convex and compact sets. Next, Browder [6] restated it in the more convenient form of a fixed-point theorem. A weaker form (with a relaxed compactness assumption) of this theorem was afterwards obtained by Fan [21]. Finally, Lassonde [33] extended these results. He gave a proof of the following interesting coincidence theorem:

Theorem 5.1. *Let X be a convex space (i.e., a convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets), Y a topological space, and F the map of X into 2^Y for which the following conditions hold:*

(i) *for each $x \in X$, $F(x)$ is compactly open in Y;*

(ii) *for each $y \in Y$, $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is nonempty and convex;*

(iii) *for some c-compact set $K \subset X$, the set $Y \setminus \bigcup_{x\in K} F(x)$ is compact. Then, for each single-valued continuous map f of X into Y, there exists an $x \in X$ such that $f(x) \in F(x)$.*

Our new coincidence theorem does not require convexity.

Theorem 5.2. *Let C be a nonempty compact set in a Hausdorff topological vector space E over $\mathbb{R}$ and let $f : C \to E$ be a continuous single-valued map on C such that $f(C)$ is a convex set. Let $F : C \to 2^{f(C)}$ be a map such that $f(C) = F(C)$. Suppose that*

(i) *for each $y \in f(C)$, the set $F^{-1}(y) = \{c \in C : y \in F(c)\}$ is open in C;*
(ii) *for each $c \in C$, the set $\{y \in f(C) : y \in F(c)\}$ is nonempty and convex.*

Then there exists a point $u \in C$ such that $f(u) \in F(u)$.

Remark 5.3. If $f = I_E$ (the identity map) and C is convex, then Theorem 5.2 becomes the Browder theorem [6, Theorem 1]. However, his method of proving this fact (based on the partition of unity) is absolutely different from ours.

Section theorems concerning convex compact sets in Hausdorff topological vector spaces, with various applications, are given by Fan [18, 20]. In the proof of Theorem 5.2, we need the following two new auxiliary section theorems of Fan type.

Theorem 5.4. *Let C be a nonempty compact set (not necessarily convex) in a Hausdorff topological vector space E over K. Let $f : C \to E$ and $g : C \to E$ be continuous maps on C, and let $f(C)$ be convex. Let K be a subset of $g(C) \times f(C)$ having the following properties:*

(i) *for each fixed $w \in f(C)$, the set $\{t \in C : (g(t), w) \in K\}$ is closed in C;*
(ii) *for each $t \in C$, $(g(t), f(t)) \in K$;*
(iii) *for any fixed $t \in C$, the set $\{w \in f(C) : (g(t), w) \notin K\}$ is convex (or empty).*

Then there exists a point $c \in C$ such that $\{g(c)\} \times f(C) \subset K$.

Proof. We use KKM set-valued maps. Define a map $H : f(C) \to 2^E$ as follows:

$$H(w) = \{t \in C : (g(t), w) \in K\}, \quad w \in f(C). \tag{5.1}$$

Obviously, by (i), $H(w)$ is a compact subset of C and thus $f(H(w))$ is a compact subset of $f(C)$ for each $w \in f(C)$. Let $\{w_1, \ldots, w_m\}$ be any finite and fixed subset of $f(C)$. We prove that $\operatorname{conv}\{w_1, \ldots, w_m\} \subset f(H(w_1)) \cup \cdots \cup f(H(w_m))$. To this goal, we assume that $f(s) \in \operatorname{conv}\{w_1, \ldots, w_m\}$ but $f(s) \notin f(H(w_1)) \cup \cdots \cup f(H(w_m))$ for some $s \in C$. Then $s \notin H(w_i)$ for all $i = 1, \ldots, m$, that is, $(g(s), w_i) \notin K$ for any $i = 1, \ldots, m$. Therefore, by (iii), w_i, $i = 1, \ldots, m$, are contained in a convex set $U = \{w \in f(C) : (g(s), w) \notin K\}$. Consequently, $\operatorname{conv}\{w_1, \ldots, w_m\} \subset U$ and, in particular, $f(s) \in U$, that is, $(g(s), f(s)) \notin K$, which, by (ii), is impossible. We must have $f(s) \in f(H(w_1)) \cup \cdots \cup f(H(w_m))$. By virtue of [16, Lemma 1, page 305], this yields $f(c) \in \bigcap\{f(H(w)) : w \in f(C)\}$ for some $c \in C$ and we conclude that $\{g(c)\} \times f(C) \subset K$ for some $c \in C$. $\square$

Theorem 5.5. *Let C be a nonempty compact set (not necessarily convex) in a Hausdorff topological vector space E over $\mathbb{K}$. Let $f : C \to E$ and $g : C \to E$ be continuous maps on C, and let $f(C)$ be convex. Let B be a subset of $g(C) \times f(C)$ and suppose that*

(i) *for each fixed $y \in f(C)$, the set $\{c \in C : (g(c), y) \in B\}$ is open in C;*
(ii) *for any fixed $c \in C$, the set $\{y \in f(C) : (g(c), y) \in B\}$ is nonempty and convex.*

Then there exists a point $u \in C$ such that $(g(u), f(u)) \in B$.

Proof. Here, B denotes a complement of the set K in $g(C) \times f(C)$ where K is defined in Theorem 5.4 □

Proof of Theorem 5.2. We define a set $B = \{(c, y) \in C \times f(C) : y \in F(c)\}$ and apply Theorem 5.5 for $g = I_E$. □

6. Coincidences for upper semicontinuous set-valued maps on not necessarily convex sets in locally convex spaces

Let $F : C \to 2^E$ and $G : C \to 2^E$ and let $\Phi : G(c) \times F(c) \to E$ for each $c \in C$. We say that maps F and G have a Φ-coincidence if there exist $c \in C$ and $(u, v) \in G(c) \times F(c)$ such that $\Phi(u, v) = 0$; this point c is called a Φ-coincidence point for F and G. In particular, a Φ-coincidence point is a coincidence point if Φ is of the form $\Phi(u, v) = u - v$ for $(u, v) \in G(c) \times F(c)$, and $c \in C$.

We use these notations in the following theorem.

Theorem 6.1. *Let C be a nonempty compact (not necessarily convex) set in a locally convex Hausdorff topological vector space E over $\mathbb{K}$. Let Γ be the set of all continuous seminorms p on E. Let $F : C \to 2^E$ and $G : C \to 2^E$ be upper semicontinuous maps such that $F(c)$ and $G(c)$ are compact subsets of E for each $c \in C$ and let, for each $c \in C$, the map $\Phi : G(c) \times F(c) \to E$ be continuous on $G(c) \times F(c)$.*

(a) *Then either F and G have a Φ-coincidence or there exist $p \in \Gamma$ and $\lambda > 0$ such that $p(\Phi(u, v)) > \lambda$ for all $c \in C$ and all $(u, v) \in G(c) \times F(c)$.*

(b) *Then either F and G have a Φ-coincidence or there exists $p \in \Gamma$ and, for any $c \in C$ and any $u \in G(c)$, there exists $v \in F(c)$ such that*

$$0 < p\big(\Phi(u, v)\big) = \operatorname{Min}\big\{p\big(\Phi(u, w)\big) : w \in F(c)\big\}. \tag{6.1}$$

Proof. (a) If F and G do not have a Φ-coincidence in C, then, for all $c \in C$, the set $\Phi(G(c) \times F(c))$ is compact and $0 \notin \Phi(G(c) \times F(c))$.

First, observe that

(i) for each $c \in C$, there exist $p_c \in \Gamma$ and $\lambda_c > 0$ such that

$$p_c\big(\Phi(u, v)\big) > 2\lambda_c \quad \forall (u, v) \in G(c) \times F(c). \tag{6.2}$$

Indeed, for an arbitrary and fixed $w \in \Phi(G(c) \times F(c))$, there exists $p_w \in \Gamma$ such that $p_w(w) \neq 0$ and, by the continuity of p_w, there exist a neighbourhood M_w of w and $\mu_w > 0$ such that $\mu_w = \operatorname{Inf}\{p_w(t) : t \in M_w\}$. Since the family $\{M_w : w \in \Phi(G(c) \times F(c))\}$ is an open cover of a compact set of $\Phi(G(c) \times F(c))$, there exists a finite subset $\{w_1, \dots, w_m\}$ of $\Phi(G(c) \times F(c))$ such that the family $\{M_{w_i} : i = 1, 2, \dots, m\}$ covers $\Phi(G(c) \times F(c))$ and we may assume that

$$p_c = \operatorname{Max}\{p_{w_i} : i = 1, \dots, m\}, \qquad \lambda_c = \left(\frac{1}{4}\right)\operatorname{Min}\{\mu_{w_i} : i = 1, \dots, m\}. \tag{6.3}$$

Now we prove that

(ii) for each $c \in C$, there exist $p_c \in \Gamma$, $\lambda_c > 0$, and a neighbourhood W_c of c, such that

$$p_c(\Phi(u,v)) > \lambda_c \quad \forall x \in W_c \cap C,\ (u,v) \in G(x) \times F(x). \tag{6.4}$$

Indeed, let $c \in C$ be arbitrary and fixed and we define open sets A_c and B_c as follows:

$$A_c \times B_c = \{(u,v) : p_c(\Phi(u,v)) > \lambda_c\}, \tag{6.5}$$

where p_c and λ_c are as in (i). Since $F(c) \subset A_c$, $G(c) \subset B_c$, F, and G are upper semicontinuous, there exist neighbourhoods U_c and V_c of c, such that $F(x) \subset A_c$ for $x \in U_c \cap C$, and $G(y) \subset B_c$ for $y \in V_c \cap C$. Consequently, we may assume that $W_c = U_c \cap V_c$.

Finally, for each $c \in C$, let p_c, λ_c, and W_c be as in (ii). Since the family $\{W_c : c \in C\}$ is an open cover of a compact set of C, there exists a finite subset $\{c_1, \dots, c_n\}$ of C such that the family $\{W_{c_i} : i = 1, \dots, n\}$ covers C and we may assume that

$$p = \operatorname{Max}\{p_{c_i} : i = 1, \dots, n\}, \qquad \lambda = \operatorname{Min}\{\lambda_{c_i} : i = 1, \dots, n\}. \tag{6.6}$$

(b) If F and G do not have a Φ-coincidence in C, let p and λ be as in (a) and let $c \in C$ be arbitrary and fixed. Observe that, for any $u \in G(c)$, the continuous map $p(\Phi(u,\cdot))$ attains its minimum on a compact set $F(c)$. Let $k : G(c) \times F(c) \to R$ be a map defined by the formula $k(u,v) = p(\Phi(u,v)) - \operatorname{Min}\{p(\Phi(u,w)) : w \in F(c)\}$. Obviously, $k(u,v) > 0$ for each $(u,v) \in G(c) \times F(c)$ and, for each $u \in G(c)$, there exists $v \in F(c)$ such that $k(u,v) = 0$. □

Example 6.2. Let $E = C$, $U = \{c \in E : |c| = 1,\ |\operatorname{Arg}(c)| \leq \pi/4\}$, $U_1 = -U + 2^{1/2}$, $V_1 = \{w : w = tc,\ 0 \leq t \leq 1,\ c \in U_1\}$. Let $C = U_1 \cup (-U_1)$ and let $F : C \to 2^E$, $G : C \to 2^E$ be defined by $F(c) = -V_1$ for $c \in U_1$, $F(c) = V_1$ for $c \in -U_1$, and $G = -F$. Then F and G satisfy the assumptions of Theorem 6.1(b) for Φ defined by $\Phi(u,v) = u - v$, $(u,v) \in G(c) \times F(c)$, $c \in C$. The sets C, $F(C)$, $G(C)$, $F(c)$, and $G(c)$ are nonconvex for all $c \in C$. Moreover, $C \subset F(C)$, $C \subset G(C)$, the sets $F(C)$ and $G(C)$ are not contained in C and any $c \in C$ is a coincidence of F and G.

7. Coincidences and fixed points for continuous single-valued maps on not necessarily convex sets in locally convex spaces

Two maps $f : C \to E$ and $g : C \to E$ have a Φ-coincidence, where $\Phi : g(C) \times f(C) \to E$, if $\Phi(g(c), f(c)) = 0$ for some $c \in C$; this point c is called a Φ-coincidence point for f and g. In particular, a Φ-coincidence point is a coincidence point if Φ is of the form $\Phi(u, v) = u - v$ for $(u, v) \in g(C) \times f(C)$. We say that $c \in C$ is a Φ-fixed point for $f : C \to E$ where $\Phi : C \times f(C) \to E$, if $\Phi(c, f(c)) = 0$. In particular, a Φ-fixed point is a fixed point if Φ is of the form $\Phi(c, v) = c - v$ for $(c, v) \in C \times f(C)$.

For maps defined on convex sets, there are many variations, generalizations, and applications (see, e.g., [1, 10, 13, 24, 25, 31, 32, 34, 36, 38, 41, 42, 43, 46]) of the well-known Fan minimax inequality [20], Hartman-Stampacchia variational inequality [26], and Iohvidov theorem [29].

In this section, we will give further applications of Theorem 5.4. In particular, we derive some minimax theorem (Theorem 7.1), Hartman-Stampacchia type variational inequalities (Theorem 7.2), and a theorem of Iohvidov type (Theorem 7.3(b)) for maps on not necessarily convex sets. One of them will be used later to prove new results concerning Φ-coincidences and Φ-fixed points (in particular, coincidences and fixed points) of continuous single-valued maps on not necessarily convex sets (Theorem 7.4).

A real map ψ, defined on a topological vector space E, is said to be lower semicontinuous (upper semicontinuous) on E if, for each real number μ, the set $\{x \in E : \psi(x) > \mu\}$ ($\{x \in E : \psi(x) < \mu\}$) is open.

A real map, ψ defined on a convex set A of a vector space E, is said to be quasi-concave (quasi-convex) on A if, for each real number μ, the set $\{a \in A : \psi(a) > \mu\}$ ($\{a \in A : \psi(a) < \mu\}$) is convex.

As a consequence of Theorem 5.4, we obtain the following theorem.

Theorem 7.1. *Let C be a nonempty compact set (not necessarily convex) in a Hausdorff topological vector space E over $\mathbb{K}$. Let $f : C \to E$ and $g : C \to E$ be continuous maps on C and let $f(C)$ be convex.*

(a) *Let $\Psi : g(C) \times f(C) \to \mathbb{R}$ be a map such that* (i) *for each $v \in f(C)$, $\Psi(\cdot, v)$ is a lower semicontinuous map on $g(C)$;* (ii) *for each $u \in g(C)$, $\Psi(u, \cdot)$ is a quasi-concave map on $f(C)$. Then there exists $c \in C$ such that*

$$\operatorname{Sup}\{\Psi(g(c), f(t)) : t \in C\} \leq \operatorname{Sup}\{\Psi(g(s), f(s)) : s \in C\}. \tag{7.1}$$

Let, additionally, (iii) *$\Psi(g(s), f(s)) \leq 0$ for all $s \in C$. Then there exists $c \in C$ such that*

$$\Psi(g(c), f(t)) \leq 0 \quad \forall t \in C. \tag{7.2}$$

(b) *Let $\Omega : g(C) \times f(C) \to \mathbb{R}$ be a map such that* (iv) *for each $v \in f(C)$, the map $\operatorname{Min}\{\Omega(\cdot, w) : w \in f(C)\} - \Omega(\cdot, v)$ is lower semicontinuous on $g(C)$;* (v) *for* *each* $u \in g(C)$, $\Omega(u,\cdot)$ *is a quasi-convex map on* $f(C)$. *Then there exists* $c \in C$ *such that*

$$\Omega(g(c), f(c)) \leq \Omega(g(c), w) \quad \forall w \in f(C). \tag{7.3}$$

Proof. (a) If $\mu = \operatorname{Sup}\{\Psi(g(s), f(s)) : s \in C\}$, then the set K defined by $K = \{(g(t), f(s)) \in g(C) \times f(C) : \Psi(g(t), f(s)) \leq \mu\}$ satisfies the assumptions of Theorem 5.4 and thus, there exists $c \in C$ such that $\{g(c)\} \times f(C) \subset K$, that is, $\Psi(g(c), v) \leq \mu$ for all $v \in f(C)$. This yields the assertion.

(b) Let $\Psi(u,v) = \operatorname{Min}\{\Omega(u,w) : w \in f(C)\} - \Omega(u,v)$. The map Ψ satisfies conditions (i) and (ii) on $g(C) \times f(C)$. Moreover, for each $u \in g(C)$, there exists $v \in f(C)$ such that $\Psi(u,v) = 0$ and, by the assertion of (a), we get

$$\operatorname{Sup}\{\Psi(g(s), f(s)) : s \in C\} \geq 0. \tag{7.4}$$

Thus, there exists $c \in C$ such that $\Psi(g(c), f(c)) \geq 0$. □

If E is a locally convex Hausdorff topological vector space over $\mathbb{K}$ and E' denotes the topological vector space of continuous linear functionals on E, let $\langle \lambda; x\rangle$ denote the pairing between λ in E' and x in E.

Now, we show the following theorem.

Theorem 7.2. *Let* C *be a nonempty compact set (not necessarily convex) in a locally convex Hausdorff topological vector space* E *over* $\mathbb{K}$. *Let* $f : C \to E$ *and* $g : C \to E$ *be continuous maps on* C *and let* $f(C)$ *be convex. Let* $\Phi : g(C) \times f(C) \to E$ *be a continuous map such that*

$$\Phi(u, \mu_1 v_1 + \mu_2 v_2) = \mu_1 \Phi(u, v_1) + \mu_2 \Phi(u, v_2) \tag{7.5}$$

holds for all $u \in g(C)$, $v_1, v_2 \in f(C)$ *and* $\mu_1 \geq 0$, $\mu_2 \geq 0$ *with* $\mu_1 + \mu_2 = 1$.

(a) *If* $\Gamma = \{p_\alpha : \alpha \in Z\}$ *is the set of all continuous seminorms* p_α *on* E, $\alpha \in Z$, $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$ *is a finite subset of* Γ *and* $p_\alpha = p_{\alpha_1} + p_{\alpha_2} + \cdots + p_{\alpha_n}$, *then there exists at least one* $c \in C$ *such that, for each* $w \in f(C)$,

$$p_\alpha[\Phi(g(c), f(c))] \leq p_\alpha[\Phi(g(c), w)]. \tag{7.6}$$

(b) *Let* $T : g(C) \to E'$ *be continuous. Then there exists* $c \in C$ *such that*

$$\begin{aligned} &\operatorname{Inf}\{\operatorname{Re}\langle (T \circ g)(c); \Phi(g(c), f(t))\rangle : t \in C\} \\ &\qquad \geq \operatorname{Inf}\{\operatorname{Re}\langle (T \circ g)(s); \Phi(g(s), f(s))\rangle : s \in C\}. \end{aligned} \tag{7.7}$$

Let, additionally, $\Phi(g(s), f(s)) = 0$ *for all* $s \in C$. *Then there exists* $c \in C$ *such that*

$$\operatorname{Re}\langle (T \circ g)(c); \Phi(g(c), f(t))\rangle \geq 0 \quad \forall t \in C. \tag{7.8}$$

(c) *Let* $h : g(C) \to E$ *and* $L : [(I_E - h) \circ g](C) \to E'$ *be continuous. Then there exists* $c \in C$ *such that*

$$\begin{aligned}\operatorname{Inf}\{\operatorname{Re}\langle L[g(c) - h(g(c))]; \Phi(g(c), f(t))\rangle : t \in C\}\\ \geq \operatorname{Inf}\{\operatorname{Re}\langle L[g(s) - h(g(s))]; \Phi(g(s), f(s))\rangle : s \in C\}.\end{aligned} \tag{7.9}$$

Let, additionally, $\Phi(g(s), f(s)) = 0$ *for all* $s \in C$. *Then there exists* $c \in C$ *such that*

$$\operatorname{Inf}\{\operatorname{Re}\langle L[g(c) - h(g(c))]; \Phi(g(c), f(t))\rangle : t \in C\} \geq 0 \quad \forall t \in C. \tag{7.10}$$

Proof. (a) We denote $\Omega = p_\alpha \circ \Phi$ and use Theorem 7.1(b).

(b) Let $\Psi : g(C) \times f(C) \to \mathbb{R}$ be a map of the form

$$\Psi(u, v) = -\operatorname{Re}\langle T(u); \Phi(u, v)\rangle. \tag{7.11}$$

Then Ψ satisfies the conditions of (iv) and (v) and, consequently, there exists $c \in C$ such that

$$\begin{aligned}\operatorname{Inf}\{\operatorname{Re}\langle (T \circ g)(c); \Phi(g(c), f(t))\rangle : t \in C\}\\ \geq \operatorname{Inf}\{\operatorname{Re}\langle (T \circ g)(s); \Phi(g(s), f(s))\rangle : s \in C\}.\end{aligned} \tag{7.12}$$

(c) We use (b) for $T = L \circ (I_E - h)$. □

Our new coincidence theorem does not require convexity.

Theorem 7.3. *Let* C *be a nonempty compact set (not necessarily convex) in a locally convex Hausdorff topological vector space* E *over* $\mathbb{K}$. *Let* Γ *be the set of all continuous seminorms* p *on* E. *If* $f : C \to E$ *and* $g : C \to E$ *are continuous on* C, *if* $f(C)$ *is convex, and if* $\Phi : g(C) \times f(C) \to E$ *is a continuous map such that*

$$\Phi(u, \mu_1 v_1 + \mu_2 v_2) = \mu_1 \Phi(u, v_1) + \mu_2 \Phi(u, v_2) \tag{7.13}$$

holds for all $u \in g(C)$, $v_1, v_2 \in f(C)$ *and* $\mu_1 \geq 0$, $\mu_2 \geq 0$ *with* $\mu_1 + \mu_2 = 1$, *then*

(a) *either* f *and* g *have a* Φ-*coincidence in* C *or there exist* $c \in C$ *and* $p \in \Gamma$, *such that*

$$0 < p[\Phi(g(c), f(c))] = \operatorname{Min}\{p[\Phi(g(c), w)] : w \in f(C)\}; \tag{7.14}$$

(b) *if, for each* $u \in g(C)$, *there is some* $v \in f(C)$ *with* $\Phi(u, v) = 0$, *then* f *and* g *have a* Φ-*coincidence in* C.

Proof. (a) Denote $A_\alpha = \{t \in C : p_\alpha[\Phi(g(t), f(t))] = 0\}$, $\alpha \in Z$. Obviously, A_α is a closed subset of C and thus compact, $\alpha \in Z$. Assume that the second assertion does not hold, that is, for each point $t \in C$ and for any $p_\alpha \in \Gamma$ such that $p_\alpha[\Phi(g(t), f(t))] > 0$, there exists a point $w \in f(C)$ with $p_\alpha[\Phi(g(t), w)] < p_\alpha[\Phi(g(t), f(t))]$.

We then prove that the above yields the first assertion, that is, that

$$\bigcap \{A_\alpha : \alpha \in Z\} \neq \varnothing. \tag{7.15}$$

To this aim, observe that if $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$ is a finite subset of Γ and $p_\alpha = p_{\alpha_1} + p_{\alpha_2} + \cdots + p_{\alpha_n}$, then, by (7.6), there exists at least one $c \in C$ such that, for all $t \in f(C)$, we have $p_\alpha[\Phi(g(c), f(c))] \leq p_\alpha[\Phi(g(c), t)]$. But, by the above assumption, if $p_\alpha[\Phi(g(c), f(c))] > 0$, then, for some $w \in f(C)$, we obtain $0 < p_\alpha[\Phi(g(c), f(c))] \leq p_\alpha[\Phi(g(c), w)] < p_\alpha[\Phi(g(c), f(c))]$, which is impossible and thus, $p_\alpha[\Phi(g(c), f(c))] = 0$. Consequently, $c \in A_{\alpha_1} \cap A_{\alpha_2} \cap \cdots \cap A_{\alpha_n}$. This yields $\bigcap \{A_\alpha : \alpha \in Z\} \neq \varnothing$.

Part (b) is a simple consequence of (a). □

By using various maps Φ, f, and g, a number of variations of Theorem 7.3 can be obtained, of which the following is typical.

Theorem 7.4. *Let C be a nonempty compact set (not necessarily convex) in a locally convex Hausdorff topological vector space E over $\mathbb{K}$. Let Γ be the set of all continuous seminorms p on E.*

(i) *If $f : C \to E$ and $g : C \to E$ are continuous on C and if $f(C)$ is convex, then we have* (a) *either f and g have a coincidence or there exist $c \in C$ and $p \in \Gamma$ such that $0 < p[g(c) - f(c)] = \mathrm{Min}\{p[g(c) - w] : w \in f(C)\}$;* (b) *if, for each $c \in C$, there exists $\lambda \in \mathbb{K}$ such that $|\lambda| < 1$ and $\lambda f(c) + (1-\lambda)g(c) \in f(C)$, then f and g have a coincidence.*

(ii) *If $f : C \to E$ is continuous on C, if $f(C)$ is convex, and if $\Phi : C \times f(C) \to E$ is a continuous map such that $\Phi(c, \mu_1 v_1 + \mu_2 v_2) = \mu_1 \Phi(c, v_1) + \mu_2 \Phi(c, v_2)$ holds for all $c \in C$, $v_1, v_2 \in f(C)$ and $\mu_1 \geq 0$, $\mu_2 \geq 0$ with $\mu_1 + \mu_2 = 1$, then* (c) *either f has a Φ-fixed point in C or there exist $c \in C$ and $p \in \Gamma$ such that $0 < p[\Phi(c, f(c))] = \mathrm{Min}\{p[\Phi(c, w)] : w \in f(C)\}$;* (d) *if, for each $c \in C$, there is some $v \in f(C)$ with $\Phi(c, v) = 0$, then f has a Φ-fixed point in C.*

(iii) *If $f : C \to E$ is continuous on C and if $f(C)$ is convex, then we have* (e) *either f has a fixed point in C or there exist $c \in C$ and $p \in \Gamma$, such that $0 < p[c - f(c)] = \mathrm{Min}\{p(c - w) : w \in f(C)\}$;* (f) *if, for each $c \in C$, there exists $\lambda \in \mathbb{K}$ such that $|\lambda| < 1$ and $\lambda f(c) + (1-\lambda)c \in f(C)$, then f has a fixed point in C.*

Proof. We prove only (b). Assume that the assertion does not hold, that is, $g(c) \neq f(c)$ for any $c \in C$. By (a), there exist a point $c \in C$ and some $p \in \Gamma$, such that

$$0 < p[g(c) - f(c)] = \mathrm{Min}\{p[g(c) - w] : w \in f(C)\}. \tag{7.16}$$

Obviously, by the assumption, there is a number $\lambda \in \mathbb{K}$ such that $|\lambda| < 1$ and $w = \lambda f(c) + (1-\lambda)g(c) \in f(C)$. But then $w - g(c) = \lambda[f(c) - g(c)]$ and, by (7.16), we get $0 < p[g(c) - f(c)] \leq p[g(c) - w] = |\lambda| p[g(c) - f(c)]$ which is impossible because $|\lambda| < 1$. □

Example 7.5. Let $E = \mathbb{C}$,

$$C = \left\{c \in E : \left|\operatorname{Arg}(c) - \pi\right| \leq \frac{\pi}{4}, \frac{1}{2} \leq |c| \leq 1\right\} \cup \left\{c \in E : \left|\operatorname{Arg}(c)\right| \leq \frac{\pi}{4}, |c| \leq 2\right\}, \tag{7.17}$$

$g(c) = c$ and $f(c) = -(\bar{c})^2 + a$, $c \in C$, where $a = 2^{1/2} + 14^{1/2}$. Then C is compact but nonconvex, f is not injective on C, $f(C) = \{c \in E : |c - a| \leq 4, \pi/2 \leq \operatorname{Arg}(c - a) \leq 3\pi/2\}$, f is neither expansive nor inner and $\operatorname{Fix}(f) = \{[-1 + (1+4a)^{1/2}]/2\}$. For all $c \in C$, the segment $[c, f(c)]$ contains at least two points of $f(C)$. Moreover, the set ∂C contains an infinite subset C_0 such that, for all $c \in C_0$, the segment $[c, f(c)]$ contains only one point of C. Thus, the assumptions of Theorem 7.4(b) are satisfied, but the assumptions of [19, Theorems 1 and 3] are not.

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