ON THE A-LAPLACIAN

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We prove, for Orlicz spaces $\mathbf{L}_A(\mathbb{R}^N)$ such that A satisfies the Δ_2 condition, the nonresolvability of the A-Laplacian equation $\Delta_A u + h = 0$ on $\mathbb{R}^N$, where $\int h \neq 0$, if $\mathbb{R}^N$ is A-parabolic. For a large class of Orlicz spaces including Lebesgue spaces $\mathbf{L}^p$ $(p > 1)$, we also prove that the same equation, with any bounded measurable function h with compact support, has a solution with gradient in $\mathbf{L}_A(\mathbb{R}^N)$ if $\mathbb{R}^N$ is A-hyperbolic.

1. Introduction

An important application of the nonlinear potential theory is the resolution of some equations involving the p-Laplacian operator. In [6], Gol'dshtein and Troyanov proved that the p-Laplace equation $\Delta_p u + h = 0$ on $\mathbb{R}^N$, $N \leq p$, has no solution if h has a nonzero average. This result remains true for the same equation on any p-parabolic manifold. The proof is essentially based on a capacity argument. Later, Troyanov proved in [9] that the equation $\Delta_p u + h = 0$, on a p-hyperbolic manifold M, has a solution with p-integrable gradient for any bounded measurable function $h : M \to \mathbb{R}$ with compact support.

Since the strongly nonlinear potential theory is sufficiently developed, we propose in this paper the generalization of these two equations on $\mathbb{R}^N$ to the setting of Orlicz spaces. For this goal, we introduce, for a given $\mathcal{N}$-function A, the notion of A-parabolicity and A-hyperbolicity which reduces to p-parabolicity and p-hyperbolicity when $A(t) = p^{-1}|t|^p$. We also consider the so-called A-Laplacian Δ_A, which is the p-Laplacian Δ_p, when the Orlicz space $\mathbf{L}_A$ is the Lebesgue space $\mathbf{L}^p$. If the $\mathcal{N}$-function A satisfies the Δ_2 condition and $\mathbb{R}^N$ is A-parabolic, then the equation $\Delta_A u + h = 0$ has no weak solution for any function h having a nonzero average.

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For reflexive Orlicz spaces $\mathbf{L}_A$, with A satisfying the condition $s(A) > 0$, where

$$s(A) := \inf\left\{\frac{\log \int A \circ f\, d\lambda}{\log |||f|||_A} - 1,\ f \in \mathbf{L}_A,\ |||f|||_A > 1\right\}, \tag{1.1}$$

if the function h is in $\mathbf{L}^\infty$ and has a compact support, then the equation $\Delta_A u + h = 0$ has a weak solution when $\mathbb{R}^N$ is A-hyperbolic. We give large classes of Orlicz spaces $\mathbf{L}_A$, including Lebesgue spaces $\mathbf{L}^p$ $(p > 1)$, which satisfies $s(A) > 0$.

This paper is organized as follows. In Section 2, we list the prerequisites from the Orlicz spaces and we introduce the notion of A-hyperbolicity. Section 3 is reserved to the resolution of the equation $\Delta_A u + h = 0$ when h has a nonzero average or bounded with compact support.

2. Preliminaries

2.1. Orlicz spaces. We recall some definitions and results about Orlicz spaces. For more details, one can consult [1, 7, 8].

Let $A : \mathbb{R} \to \mathbb{R}^+$ be an $\mathcal{N}$*-function*, that is, A is continuous, convex, with $A(t) > 0$ for $t > 0$, $\lim_{t\to 0} A(t)/t = 0$, $\lim_{t\to +\infty} A(t)/t = +\infty$, and A is even.

Equivalently, A admits the representation: $A(t) = \int_0^{|t|} a(x)dx$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $\lim_{t\to +\infty} a(t) = +\infty$.

The $\mathcal{N}$-function A^* *conjugate* to A is defined by $A^*(t) = \int_0^{|t|} a^*(x)dx$, where a^* is given by $a^*(s) = \sup\{t : a(t) \le s\}$.

Let A be an $\mathcal{N}$-function, let λ be the Lebesgue measure on $\mathbb{R}^N$, and let Ω be an open set in $\mathbb{R}^N$. We denote by $\mathcal{L}_A(\Omega)$ the set, called an *Orlicz class*, of measurable functions f, on Ω, such that

$$\rho(f, A, \Omega) = \int_\Omega A(f(x))\, d\lambda(x) < \infty. \tag{2.1}$$

Let A and A^* be two conjugate $\mathcal{N}$-functions and let f be a measurable function defined almost everywhere in Ω. The *Orlicz norm of* f, $\|f\|_{A,\Omega}$, or $\|f\|_A$, if there is no confusion, is defined by

$$\|f\|_A = \sup\left\{\int_\Omega |f(x)g(x)|\, d\lambda(x) : g \in \mathcal{L}_{A^*}(\Omega),\ \rho(g, A^*, \Omega) \le 1\right\}. \tag{2.2}$$

The set $\mathbf{L}_A(\Omega)$ of measurable functions f such that $\|f\|_A < \infty$ is called an *Orlicz space*. When $\Omega = \mathbb{R}^N$, we set $\mathbf{L}_A$ in place of $\mathbf{L}_A(\mathbb{R}^N)$.

If $f \in \mathbf{L}_A(\Omega)$, then

$$\|f\|_A = \inf\left\{k^{-1}\left[1 + \int_\Omega A(k|f|(x))\, d\lambda(x)\right] : k > 0\right\}. \tag{2.3}$$

The *Luxemburg norm* $|||f|||_{A,\Omega}$ or $|||f|||_A$, if there is no confusion, is defined in $\mathbf{L}_A(\Omega)$ by

$$|||f|||_A = \inf\left\{r > 0 : \int_\Omega A\left(\frac{f(x)}{r}\right)d\lambda(x) \le 1\right\}. \tag{2.4}$$

Orlicz and Luxemburg norms are equivalent. More precisely, if $f \in \mathbf{L}_A(\Omega)$, then

$$|||f|||_A \le \|f\|_A \le 2|||f|||_A. \tag{2.5}$$

It is well known that we can suppose that a and a^* are continuous and strictly increasing. Hence the $\mathcal{N}$-functions A and A^* are strictly convex and $a^* = a^{-1}$.

Let A be an $\mathcal{N}$-function. We say that A *verifies the* Δ_2 *condition* if there exists a constant $C > 0$ such that $A(2t) \le CA(t)$ for all $t \ge 0$.

Recall that A *verifies the* Δ_2 *condition* if and only if $\mathcal{L}_A = \mathbf{L}_A$. Moreover, $\mathbf{L}_A$ is reflexive if and only if A and A^* satisfy the Δ_2 condition.

Hölder inequality in Orlicz spaces is expressed in the following way:

$$\int |f \cdot g| d\lambda \le |||f|||_A \cdot \|g\|_{A^*}, \quad f \in \mathbf{L}_A,\ g \in \mathbf{L}_{A^*}. \tag{2.6}$$

We recall the following results. Let A be an $\mathcal{N}$-function and a its derivative. Then the following occurs.

(1) The $\mathcal{N}$-function A verifies the Δ_2 condition if and only if one of the following holds:
   (i) for all $r > 1$, there exists $k = k(r)$ (for all $t \ge 0$, $A(rt) \le kA(t)$) ;
   (ii) there exists $\alpha > 1$ (for all $t \ge 0$, $ta(t) \le \alpha A(t)$) ;
   (iii) there exists $\beta > 1$ (for all $t \ge 0$, $ta^*(t) \ge \beta A^*(t)$) ;
   (iv) there exists $d > 0$ (for all $t \ge 0$, $(A^*(t)/t)' \ge d(a^*(t)/t)$).
   Moreover, α in (ii) and β in (iii) can be chosen such that $\alpha^{-1} + \beta^{-1} = 1$.
   We note that $\alpha(A)$ is the smallest α such that (ii) holds.

(2) If A verifies the Δ_2 condition, then

$$\begin{aligned} A(t) &\le A(1)t^\alpha, \quad \forall t \ge 1, \qquad A(t) \ge A(1)t^\alpha, \quad \forall t \le 1, \\ A^*(t) &\ge A^*(1)t^\beta, \quad \forall t \ge 1, \qquad A^*(t) \le A^*(1)t^\beta, \quad \forall t \le 1. \end{aligned} \tag{2.7}$$

We set $\alpha^* = \alpha(A^*)$.

Recall also that if A verifies the Δ_2 condition, then

$$\int A\left(\frac{f}{|||f|||_A}\right)(x)d\lambda(x) = 1. \tag{2.8}$$

2.2. A-hyperbolicity

Definition 2.1. Let A be an $\mathcal{N}$-function and K a compact set in $\mathbb{R}^N$. The A-capacity of K is defined by

$$\Gamma_A(K) = \inf\left\{\| |\nabla u| \|_A : u \in C_0^\infty(\mathbb{R}^N),\ u = 1 \text{ in a neighborhood of } K\right\}. \tag{2.9}$$

The space $\mathbb{R}^N$ is said to be A-parabolic if $\Gamma_A(K) = 0$ for all compact subsets $K \subset \mathbb{R}^N$ and A-hyperbolic otherwise.

Remark 2.2. In the definition of Γ_A, a simple truncation argument shows that we may restrict ourselves to functions $u \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le u \le 1$.

For $m < N$, the Riesz kernel is defined on $\mathbb{R}^N$ by $R_m(x) = |x|^{m-N}$.
For $X \subset \mathbb{R}^N$, we define $R_{m,A}(X)$ by

$$R_{m,A}(X) = \inf\left\{\| |f| \|_A : f \in \mathbf{L}_A,\ f \ge 0,\ R_m * f \ge 1 \text{ on } X\right\}. \tag{2.10}$$

The following lemma is proved in [3, Lemma 3.6].

Lemma 2.3. *Let $\mathbf{L}_A$ be a reflexive Orlicz space. Then there is a positive constant C such that*

$$C^{-1}R_{1,A}(K) \le \Gamma_A(K) \le CR_{1,A}(K), \tag{2.11}$$

for all compact K, C independent of K.

We recall the following result proved in [4, Theorem 3.1].

Lemma 2.4. *Let A be an $\mathcal{N}$-function such that $\| |R_m| \|_{A^*,\{|x|>1\}} = \infty$. Then for all X, $R_{m,A}(X) = 0$.*

We will need the following lemma in the sequel.

Lemma 2.5. *Let A be any $\mathcal{N}$-function such that A^* verifies the Δ_2 condition and let m be a positive integer such that $m < N$ and $\alpha^* \le N/(N-m)$. Then $R_{m,A}(X) = 0$ for all X.*

Proof. From Lemma 2.4, it suffices to prove that $\| |R_m| \|_{A^*,\{|x|>1\}} = \infty$. Since A^* verifies the Δ_2 condition, we must establish that

$$\int_{\{|x|>1\}} A^*\left(|x|^{m-N}\right) d\lambda(x) = \infty. \tag{2.12}$$

By a change of variable, there is a positive constant C such that

$$\int_{\{|x|>1\}} A^*\left(|x|^{m-N}\right) d\lambda(x) = C\int_1^\infty A^*\left(t^{m-N}\right) \cdot t^{N-1}\, dt. \tag{2.13}$$

From the inequality $A^*(t^{m-N}) \geq A^*(1) \cdot t^{\alpha^*(m-N)}$, we get

$$\int_{\{|x|>1\}} A^*(|x|^{m-N})\,d\lambda(x) \geq CA^*(1) \cdot \int_1^\infty t^{\alpha^*(m-N)+N-1}\,dt. \tag{2.14}$$

Now, the inequality

$$\alpha^*(m-N)+N-1 \geq \frac{N}{N-m}(m-N)+N-1 = -1 \tag{2.15}$$

gives the desired result. □

3. On the A-Laplacian

The Orlicz-Sobolev space $W^1\mathbf{L}_A(\mathbb{R}^N)$ is defined as the space of functions u such that u and its derivatives, in a distributional sense, of order less or equal to one are in $\mathbf{L}_A$. The space $W^1\mathbf{L}_A(\mathbb{R}^N)$ is a Banach space when equipped with the norm

$$|||u|||_{1,A} = \sum_{|\gamma|\leq 1} |||D^\gamma u|||_A. \tag{3.1}$$

Recall that $W^1\mathbf{L}_A(\mathbb{R}^N)$ is reflexive if and only if A and A^* satisfy the Δ_2 condition.

The A-Dirichlet space $\mathbf{L}_A^1(\mathbb{R}^N)$ is the space of functions $u \in W^1_{A,\mathrm{loc}}(\mathbb{R}^N)$ (i.e., u is locally in $W^1\mathbf{L}_A(\mathbb{R}^N)$) admitting a weak gradient such that $|||\nabla u|||_A < \infty$.

Let A be any $\mathcal{N}$-function and let a be its derivative. For $x \in \mathbb{R}^N$, we define

$$M_A(x) = \frac{a(|x|)}{|x|} \cdot x \quad \text{if } x \neq 0,\ M_A(0) = 0. \tag{3.2}$$

The A-Laplacian of a function f on $\mathbb{R}^N$ is defined by $\Delta_A f = \operatorname{div} M_A(\nabla f)$.

A function $u \in W^1_{A,\mathrm{loc}}(\mathbb{R}^N)$ is said to be a weak solution to the equation

$$\Delta_A u + h = 0 \tag{3.3}$$

if, for all $\varphi \in C_0^1(\mathbb{R}^N)$, we have

$$\int \langle M_A(\nabla u), \nabla\varphi\rangle\,d\lambda = \int h\varphi\,d\lambda. \tag{3.4}$$

Let $D \subset \mathbb{R}^N$ be a nonempty bounded domain. The Banach space $\mathscr{E}_A(D)$ is the space of functions $u \in W^1_{A,\mathrm{loc}}(\mathbb{R}^N)$ such that

$$|||u|||_A^D := |||u|||_{A,D} + |||\nabla u|||_A < \infty. \tag{3.5}$$

We denote by $\mathscr{E}_A^0(D)$ the closure of $C_0^1(\mathbb{R}^N)$ in $\mathscr{E}_A(D)$.

3.1. A nonresolvability result

Theorem 3.1. *Let A be an $\mathcal{N}$-function satisfying the Δ_2 condition. Suppose that $\mathbb{R}^N$ is A-parabolic and let $h \in \mathbf{L}_1(\mathbb{R}^N)$ be such that $\int h\,d\lambda \neq 0$. Then the equation*

$$\Delta_A u + h = 0 \tag{3.6}$$

has no weak solution on $\mathbf{L}_A^1(\mathbb{R}^N)$.

Proof. We may suppose that $\int h\,d\lambda > 0$. Hence there is a bounded set $D \subset \mathbb{R}^N$ such that $\lambda(D) > 0$, $s := \inf_D h > 0$, and $\int_D h\,d\lambda > |\int h^- d\lambda|$.

Let $0 < c < 1$ be such that $0 \leq -\int h^- d\lambda < c\int_D h\,d\lambda$.

By the definition of $\Gamma_{1,A}(D)$, for $\varepsilon > 0$, we can find a function $v \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq v \leq 1$, $v = 1$, on D and

$$\| |\nabla v| \|_A \leq \Gamma_A(D) + \varepsilon. \tag{3.7}$$

On the other hand, we have $-c\int_D vh\,d\lambda < \int vh^- d\lambda \leq 0$. Hence

$$\begin{aligned}(1-c)\int_D vh\,d\lambda &< \int_D vh\,d\lambda + \int vh^- d\lambda \\ &< \int_D vh\,d\lambda + \int vh^- d\lambda + \int_{^cD} vh^+ d\lambda \\ &\leq \int vh\,d\lambda.\end{aligned} \tag{3.8}$$

But $s \cdot \lambda(D) \leq \int_D vh\,d\lambda$. Thus

$$(1-c)\cdot s \cdot \lambda(D) \leq \int vh\,d\lambda. \tag{3.9}$$

Now suppose that $u \in \mathbf{L}_A^1(\mathbb{R}^N)$ is a weak solution of (3.6) and let $\xi := -(a(|\nabla u|)/|\nabla u|)\cdot \nabla u$. Then $\operatorname{div}(\xi) = -\Delta_A u = h$, and since A satisfies the Δ_2 condition, $|\xi| \in \mathbf{L}_{A^*}(\mathbb{R}^N)$.

An integration by part and Hölder inequality in Orlicz spaces applied to inequality (3.9) imply that

$$\begin{aligned}(1-c)\cdot s\cdot\lambda(D) &\leq \int v\cdot \operatorname{div}(\xi)\,d\lambda \\ &= -\int \langle \nabla v, \xi\rangle\,d\lambda \leq \|\xi\|_{A^*}\| |\nabla v| \|_A.\end{aligned} \tag{3.10}$$

From (3.7), and since ε is arbitrary, we get

$$0 < \lambda(D) \leq \frac{\|\xi\|_{A^*}}{(1-c)\cdot s}\cdot \Gamma_A(D). \tag{3.11}$$

This is impossible, and the theorem is proved. □

Corollary 3.2. *Let $\mathbf{L}_A$ be a reflexive Orlicz space such that $\alpha^* \le N/(N-1)$. Let $h \in \mathbf{L}_1(\mathbb{R}^N)$ be such that $\int h\,d\lambda \neq 0$. Then (3.6) has no weak solution on $\mathbf{L}_A^1(\mathbb{R}^N)$.*

Proof. By Lemmas 2.5 and 2.3, $\mathbb{R}^N$ is then A-parabolic. We apply Theorem 3.1 to get the result. □

Remark 3.3. When $A(t) = p^{-1}|t|^p$, $\mathbf{L}_A = \mathbf{L}^p$ is the usual Lebesgue space and $\alpha^* = p/(p-1)$. Hence the condition $\alpha^* \le N/(N-1)$ is exactly the condition $N \le p$. Thus our result recovers the one in [6, Théorème 1].

3.2. A resolvability result. In this section, we resolve the equation $\Delta_A u + h = 0$ under some assumptions on the $\mathcal{N}$-function A and on the function h.

We begin by recalling the following Poincaré inequality for Orlicz-Sobolev functions, which is a combination of [5, Theorem 3.3] and [5, Proposition 3.9].

Lemma 3.4. *Let A be an $\mathcal{N}$-function such that A and A^* satisfy the Δ_2 condition. Let E be any measurable set in $\mathbb{R}^N$ such that $0 < \lambda(E) < \infty$. Then there exists a positive constant C such that*

$$|||u - u_E|||_{A,E} \le C\||\nabla u|\|_{A,E}, \tag{3.12}$$

for all $u \in W^1_{A,\mathrm{loc}}(\mathbb{R}^N)$, where $u_E = (1/\lambda(E))\int_E u\,d\lambda$ is the mean value of u on E.

An application of Hölder inequality in Orlicz spaces gives

$$\int_E |u - u_E|\,d\lambda \le \|\chi_E\|_{A^*}|||u - u_E|||_{A,E}, \tag{3.13}$$

where χ_E is the characteristic function of E.

Recall that

$$\begin{gathered} \|\chi_E\|_{A^*} = \lambda(E)\cdot A^{-1}\left(\frac{1}{\lambda(E)}\right), \\ \||1|\|_{A,E} = |||\chi_E|||_A = \frac{1}{A^{-1}(1/\lambda(E))}. \end{gathered} \tag{3.14}$$

Hence we obtain the following proposition.

Proposition 3.5. *Let A be an $\mathcal{N}$-function such that A and A^* satisfy the Δ_2 condition. Let E be any measurable set in $\mathbb{R}^N$ such that $0 < \lambda(E) < \infty$. Then there exists a positive constant C such that*

$$\int_E |u - u_E|\,d\lambda \le C\||\nabla u|\|_{A,E}, \tag{3.15}$$

for all $u \in W^1_{A,\mathrm{loc}}(\mathbb{R}^N)$.

We will need the following proposition in what follows.

Proposition 3.6. *Let A be an $\mathcal{N}$-function such that A and A^* satisfy the Δ_2 condition. Suppose that $\mathbb{R}^N$ is A-hyperbolic. Let E be any nonempty bounded domain in $\mathbb{R}^N$. Then there exists a positive constant C such that, for all $u \in \mathscr{E}_A^0(E)$,*

$$\int_E |u| d\lambda \leq C \| |\nabla u| \|_A. \tag{3.16}$$

Proof. Suppose that such constant does not exist. Then for all $\varepsilon > 0$, we can find a function $u \in \mathscr{E}_A^0(E)$ such that

$$\int_E |u| d\lambda = \lambda(E), \qquad \| |\nabla u| \|_A \leq \varepsilon. \tag{3.17}$$

We may assume that $u \geq 0$. Proposition 3.5 implies that

$$\int_E |u| d\lambda \leq C\varepsilon. \tag{3.18}$$

We now choose a ball $B \Subset E$ and a function $\varphi \in C_0^1$ such that $0 \leq \varphi \leq 2^{-1}$, $\operatorname{supp}(\varphi) \subset E$, and $\varphi = 2^{-1}$ on B. Define the function $v \in \mathscr{E}_A^0(E)$ by $v = 2\max(u, \varphi)$. Then $v \geq 1$ on B. Now, define the sets

$$S = \{x \in E : \varphi(x) \geq u(x)\}, \qquad S' = \{x \in E : |u(x) - 1| \geq 2^{-1}\}. \tag{3.19}$$

We have $S \subset S'$ and, by (3.18), $2^{-1}\lambda(S') \leq C\varepsilon$. Thus

$$\lambda(S) \leq 2C\varepsilon. \tag{3.20}$$

On the other hand, we have almost everywhere

$$\nabla v = \begin{cases} 2\nabla u & \text{on } {}^cS, \\ 2\nabla \varphi & \text{on } S. \end{cases} \tag{3.21}$$

This implies that

$$|\nabla v| \leq 2|\nabla u| + 2\chi_S |\nabla \varphi| \quad \text{a.e.} \tag{3.22}$$

Since $v \geq 1$ on B and ε is arbitrary, we deduce that $\Gamma_A(B) = 0$. This contradicts the fact that $\mathbb{R}^N$ is A-hyperbolic. The proof is complete. □

Lemma 3.7. *Let A be an $\mathcal{N}$-function. If $\mathbb{R}^N$ is A-parabolic, then $1 \in \mathscr{E}_A^0(D)$ for any nonempty bounded domain D.*

Proof. Since $\mathbb{R}^N$ is A-parabolic, $\Gamma_A(\overline{D}) = 0$. Hence for all $\varepsilon > 0$, there exists a function $u \in C_0^1$ such that $u = 1$ on D and $\| |\nabla u| \|_A \leq \varepsilon$. Thus

$$\| |1 - u| \|_A = \| |1 - u| \|_{A,D} + \| |\nabla u| \|_A = \| |\nabla u| \|_A \leq \varepsilon. \tag{3.23}$$

This means that $1 \in \mathscr{E}_A^0(D)$. □

Theorem 3.8. *Let A be an $\mathcal{N}$-function such that A and A^* satisfy the Δ_2 condition. Let D be nonempty bounded domain in $\mathbb{R}^N$. Then the following assertions are equivalent*

(i) $\mathbb{R}^N$ *is A-hyperbolic;*
(ii) *there exists a constant C such that, for all $u \in \mathscr{E}_A^0(D)$,*

$$\|\,|u|\,\|_{A,D} \le C\|\,|\nabla u|\,\|_A; \tag{3.24}$$

(iii) $1 \notin \mathscr{E}_A^0(D)$.

Proof. It is easy to verify that (ii) implies (iii). The implication (iii)⇒(i) is Lemma 3.7. It remains to prove that (i) implies (ii).

Write $u = (u - u_D) + u_D$. Proposition 3.6 and Lemma 3.4 give

$$\begin{aligned}
\|\,|u|\,\|_{A,D} &\le \big\|\,\big|u - u_D\big|\,\big\|_{A,D} + \big\|\,\big|u_D\big|\,\big\|_{A,D} \\
&\le C\|\,|\nabla u|\,\|_{A,D} + \big|u_D\big| \cdot \|\,|1|\,\|_{A,D} \\
&\le C\|\,|\nabla u|\,\|_{A,D} + \frac{1}{A^{-1}(1/\lambda(D))} \cdot \lambda(D)^{-1} \int_D |u|\, d\lambda \\
&\le C\|\,|\nabla u|\,\|_{A,D} + \frac{1}{A^{-1}(1/\lambda(D))} \cdot \lambda(D)^{-1} C' \|\,|\nabla u|\,\|_A \\
&\le C'' \|\,|\nabla u|\,\|_A.
\end{aligned} \tag{3.25}$$

The proof is complete. □

Recall that for all $f \in \mathbf{L}_A$ such that $\|\,|f|\,\|_A > 1$, we have $\int A \circ f \, d\lambda > \|\,|f|\,\|_A$. We set

$$s(A) = \inf\left\{\frac{\log \int A \circ f \, d\lambda}{\log \|\,|f|\,\|_A} - 1,\ f \in \mathbf{L}_A,\ \|\,|f|\,\|_A > 1\right\}. \tag{3.26}$$

Hence $s(A) \ge 0$.

Now we are ready to solve the A-Laplace equation.

Theorem 3.9. *Let $\mathbf{L}_A$ be a reflexive Orlicz space such that $s(A) > 0$. Let $h \in \mathbf{L}^\infty(\mathbb{R}^N)$ have compact support. Then the equation $\Delta_A u + h = 0$ has a weak solution $u \in \mathbf{L}_A^1(\mathbb{R}^N)$ if $\mathbb{R}^N$ is A-hyperbolic.*

Proof. Let D be a bounded domain such that $\operatorname{supp}(h) \subset D$. Define the functional $\mathscr{F} : \mathscr{E}_A^0(D) \to \overline{\mathbb{R}}$ by

$$\mathscr{F}(u) = \int A(|\nabla u|)\, d\lambda - \int h u \, d\lambda. \tag{3.27}$$

Hence

$$\begin{aligned}\mathcal{F}(u) &\geq \int A(|\nabla u|)\,d\lambda - \left|\int hu\,d\lambda\right| \\ &\geq \int A(|\nabla u|)\,d\lambda - \|h\|_\infty \cdot \|u\|_{L^1(D)}.\end{aligned} \tag{3.28}$$

Since $\mathbb{R}^N$ is A-hyperbolic, by Proposition 3.6, we get

$$\mathcal{F}(u) \geq \int A(|\nabla u|)\,d\lambda - C\|h\|_\infty \cdot \||\nabla u|\|_A. \tag{3.29}$$

Hence there is a constant C_1 such that

$$\mathcal{F}(u) \geq \int A(|\nabla u|)\,d\lambda - C_1 \cdot \|\nabla u|\|_A. \tag{3.30}$$

By (2.3) and (2.5), there is a constant C_2 such that, for all $k > 0$,

$$\mathcal{F}(u) \geq \int A(|\nabla u|)\,d\lambda - \frac{C_2}{k}\int A(k|\nabla u|)\,d\lambda - \frac{C_2}{k}. \tag{3.31}$$

Now, let $t > 0$ and consider the continuous function ψ_t defined on $\mathbb{R}^+$ by $\psi_t(k) = (C_2/k)A(kt) - A(t)$. Since

$$\begin{gathered} xa(x) \geq A(x), \quad \forall x \geq 0, \\ \lim_{t\to 0}\frac{A(t)}{t} = 0, \qquad \lim_{t\to+\infty}\frac{A(t)}{t} = +\infty, \end{gathered} \tag{3.32}$$

the function ψ_t increases from $-A(t)$ to $+\infty$. Hence there is a k_0 such that $\psi_t(k_0) = 0$. Thus

$$\mathcal{F}(u) \geq -\frac{C_2}{k_0}. \tag{3.33}$$

We conclude that the functional $\mathcal{F}$ is bounded below on the space $\mathcal{E}_A^0(D)$.

Now $\mathcal{E}_A(D)$ is a reflexive Banach space and $\mathcal{E}_A^0(D)$ is a closed convex subspace of $\mathcal{E}_A(D)$. We first prove that $\mathcal{F}$ is lower semicontinuous. Let $t \in \mathbb{R}$, and consider the set $\mathcal{F}_t = \{u \in \mathcal{E}_A^0(D) : \mathcal{F}(u) \leq t\}$. Let $(u_i)_i \subset \mathcal{E}_A^0(D)$ be such that $\mathcal{F}(u_i) \leq t$, for all i, and $(u_i)_i$ converges to u in $\mathcal{E}_A^0(D)$. By the compactness of the imbedding $\mathcal{E}_A^0(D) \subset \mathbf{L}^1(D)$, we may assume that $(u_i)_i$ converges strongly in $\mathbf{L}^1(D)$. Hence

$$\int_D hu_i\,d\lambda \longrightarrow \int_D hu\,d\lambda. \tag{3.34}$$

Theorem 3.8 implies that $u \to \||\nabla u|\|_A$ is an equivalent norm on $\mathcal{E}_A^0(D)$.

Hence $\||\nabla u - \nabla u_i|\|_A \to 0$. Since A verifies the Δ_2 condition, $\int A(|\nabla u - \nabla u_i|)\,d\lambda \to 0$. Hence there is a subsequence of the sequence $(A(|\nabla u - \nabla u_i|))_i$, still denoted by $(A(|\nabla u - \nabla u_i|))_i$, which converges λ-almost everywhere to 0.

Thus $(|\nabla u_i|)_i$ converges λ-almost everywhere to $|\nabla u|$. By the continuity of A, Fatou's lemma, and (3.34), we get

$$\begin{aligned}\mathcal{F}(u) &= \int \lim_{i\to\infty} A(|\nabla u_i|)\,d\lambda - \lim_{i\to\infty}\int hu_i\,d\lambda \\ &\le \liminf_{i\to\infty}\int A(|\nabla u_i|)\,d\lambda - \lim_{i\to\infty}\int hu_i\,d\lambda \le t.\end{aligned} \tag{3.35}$$

Hence $\mathcal{F}$ is lower semicontinuous.

Now, $s(A) > 0$ implies that $\int A(|\nabla u|)d\lambda \ge \||\nabla u|\|_A^{s(A)+1}$ for $\||\nabla u|\|_A > 1$. Hence

$$\mathcal{F}(u) \ge \||\nabla u|\|_A^{s(A)+1} - C_1 \cdot \||\nabla u|\|_A \quad \text{for } \||\nabla u|\|_A > 1. \tag{3.36}$$

This proves that $\mathcal{F}$ is coercive.

Thus $\mathcal{F}$ attains its minimum on $\mathcal{E}_A^0(D)$; that is, there is $u^* \in \mathcal{E}_A^0(D)$ such that $\mathcal{F}(u^*) = \min\{\mathcal{F}(u) : u \in \mathcal{E}_A^0(D)\}$. By the usual arguments from variational calculus, we deduce that u^* is a weak solution to the equation $\Delta_A u + h = 0$. The proof is complete. □

Remark 3.10. We have in fact solved the equation in the space $\mathcal{E}_A^0(D) \subset \mathbf{L}_A^1(\mathbb{R}^N)$.

Remark 3.11. When $A(t) = p^{-1}|t|^p$, $p > 1$, and $\mathbf{L}_A = \mathbf{L}^p$ is the usual Lebesgue space, we have $s(A) = p - 1 > 0$. Thus we recover the result in [9, Theorem 2] when the manifold M is $\mathbb{R}^N$.

Recall the following result in [2, Lemma 3].

Lemma 3.12. *Let A be an $\mathcal{N}$-function satisfying the Δ_2 condition. If $\alpha < N$, then $R_{1,A}(B(x,r)) > 0$, where $B(x,r)$ is the open ball of radius $r > 0$, with center at x.*

Corollary 3.13. *Let $\mathbf{L}_A$ be a reflexive Orlicz space such that $s(A) > 0$ and $\alpha < N$. Suppose that $h \in \mathbf{L}^\infty(\mathbb{R}^N)$ has compact support. Then the equation $\Delta_A u + h = 0$ has a weak solution $u \in \mathbf{L}_A^1(\mathbb{R}^N)$.*

Proof. By Lemmas 3.12 and 2.3, we deduce that $\mathbb{R}^N$ is A-hyperbolic, and we apply Theorem 3.9 to get the result. □

3.3. Some examples. In addition to the $\mathbf{L}^p$ Lebesgue case corresponding to $A(t) = p^{-1}|t|^p$, $p > 1$, we consider the following $\mathcal{N}$-functions:

(1)

$$A_1(t) = \begin{cases} t^p & \text{for } 0 \le |t| \le 1, \\ t^q & \text{for } 1 < |t|, \end{cases} \qquad 1 < p \le q < \infty, \tag{3.37}$$

(2) $A_2(t) = |t|^p \log(1 + |t|)$, $p > 1$,
(3) $A_3(t) = |t|^p \log(1 + |t|^p)$, $p > 1$,

(4) $A_4(t) = |t|^p \log^p(1+|t|)$, $p > 1$,

(5) $A_{p,q,r}(t) = |t|^p \log^q(1+|t|^r)$, $p > 1$, $q > 0$, and $r > 0$.

All these $\mathcal{N}$-functions and their conjugates satisfy the Δ_2 condition. We show that $s(A_i) > 0$, $i = 1,2,3,4$, and $s(A_{p,q,r}) > 0$.

First remark that $A_2 = A_{p,1,1}$ and $A_3 = A_{p,1,p}$. Thus it suffices to show that $s(A_{p,q,r}) > 0$ and for all $p > 1$, $q > 0$, $r > 0$.

(1) Let $f \in \mathbf{L}_{A_1}$ be such that $|||f|||_{A_1} > 1$. Then, by (2.8),

$$\begin{aligned}
1 &= \int A_1\left(\frac{f}{|||f|||_{A_1}}\right)(x)\,d\lambda(x) \\
&\le \frac{1}{|||f|||_{A_1}^p}\int_{\{|f|\le |||f|||_{A_1}\}} |f|^p\,d\lambda \\
&\quad + \frac{1}{|||f|||_{A_1}^q}\int_{\{|f|> |||f|||_{A_1}\}} |f|^q\,d\lambda \\
&\le \frac{1}{|||f|||_{A_1}^p}\left[\int_{\{|f|\le |||f|||_{A_1}\}} |f|^p\,d\lambda + \int_{\{|f|> |||f|||_{A_1}\}} |f|^q\,d\lambda\right] \\
&\le \frac{1}{|||f|||_{A_1}^p}\left[\int_{\{|f|\le 1\}} |f|^p\,d\lambda + \int_{\{1<|f|\le |||f|||_{A_1}\}} |f|^p\,d\lambda \right.\\
&\qquad\qquad \left. + \int_{\{|f|> |||f|||_{A_1}\}} |f|^q\,d\lambda\right] \\
&\le \frac{1}{|||f|||_{A_1}^p}\left[\int_{\{|f|\le 1\}} |f|^p\,d\lambda + \int_{\{1<|f|\le |||f|||_{A_1}\}} |f|^q\,d\lambda \right.\\
&\qquad\qquad \left. + \int_{\{|f|> |||f|||_{A_1}\}} |f|^q\,d\lambda\right] \\
&\le \frac{1}{|||f|||_{A_1}^p}\int A_1(f)(x)\,d\lambda(x).
\end{aligned} \tag{3.38}$$

Hence $|||f|||_{A_1}^p \le \int A_1(f)(x)\,d\lambda(x)$. This implies that $s(A_1) > 0$.

(2) Let $p > 1$, $q > 0$, and $r > 0$ and set $A = A_{p,q,r}$. Let $f \in \mathbf{L}_A$ be such that $|||f|||_A > 1$. Then by (2.8),

$$\begin{aligned}
1 &= \int A\left(\frac{f}{|||f|||_A}\right)(x)\,d\lambda(x) \\
&\le \frac{1}{|||f|||_A^p}\int |f|^p \log^q\left(1+\frac{|f|^r}{|||f|||_A^r}\right) d\lambda \\
&\le \frac{1}{|||f|||_A^p}\int |f|^p \log^q\left(1+|f|^r\right) d\lambda \\
&\le \frac{1}{|||f|||_A^p}\int A(f)(x)\,d\lambda(x).
\end{aligned} \tag{3.39}$$

Thus $|||f|||_A^p \le \int A(f)(x)\,d\lambda(x)$ and hence $s(A) > 0$.

Remark 3.14. Although Theorem 3.9 gives a solution for large classes of Orlicz spaces $\mathbf{L}_A$, including $\mathbf{L}^p$ Lebesgue spaces, $p > 1$, it would be sharp if we can drop the condition $s(A) > 0$. This question is open.

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