ON THE MILD SOLUTIONS OF HIGHER-ORDER DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Received 28 January 2003

For the higher-order abstract differential equation $u^{(n)}(t) = Au(t) + f(t)$, $t \in \mathbb{R}$, we give a new definition of mild solutions. We then characterize the regular admissibility of a translation-invariant subspace $\mathcal{M}$ of $\mathrm{BUC}(\mathbb{R}, E)$ with respect to the above-mentioned equation in terms of solvability of the operator equation $AX - X\mathcal{D}^n = C$. As applications, periodicity and almost periodicity of mild solutions are also proved.

1. Introduction

The qualitative theory of mild solutions on the whole line of the differential equation of type

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed operator on a Banach space E, has been of increasing interest in the last decades. If A is a bounded operator on E, mild solutions of (1.1), which are the same as the classical solutions, are defined by

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)} f(s)ds, \quad t \in \mathbb{R}. \tag{1.2}$$

In [4], Dalec'kiĭ and Kreĭn made a systematic study on the asymptotic behavior of solutions of the form (1.2). For unbounded operator A, where the situation changes dramatically, the first question is, which solutions of (1.1) are considered as *mild solutions*? If A is the generator of a C_0-semigroup $T(t)$, $t \geq 0$, it is logical to define mild solutions of (1.1) by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau, \quad t \geq s. \tag{1.3}$$

Abstract and Applied Analysis 2003:15 (2003) 865–880
2000 Mathematics Subject Classification: 34G10, 34K06, 47D06
URL: http://dx.doi.org/10.1155/S1085337503303057

With this definition in hand, many authors investigated the qualitative behavior of (1.3) in different ways (see [10, 12, 13, 14, 17] and references therein). The second-order differential equation $u''(t) = Au(t) + f(t)$, where A is the generator of a cosine family $(C(t))$, and for which mild solutions are defined by

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)d\tau, \tag{1.4}$$

has been also studied in [3, 8, 18].

Recently, Arendt and Batty [1], Schweiker [20], and Schüler and Phóng [19] studied the first- and second-order differential equations, in which A is not the generator of a C_0-semigroup or of a cosine family, respectively. Although their definitions of mild solutions are slightly different, they all showed that the existence and uniqueness of mild solutions, which belong to a subspace $\mathcal{M}$ of $\mathrm{BUC}(\mathbb{R}, E)$, are closely related to the solvability of the operator equation of the form

$$AX - X\mathcal{D} = -\delta_0, \tag{1.5}$$

where $\mathcal{D}$ is the differential operator in $\mathcal{M}$ and δ_0 is the Dirac operator defined by $\delta_0(f) := f(0)$.

Inspired by this rapid development, in this paper, we consider the higher-order differential equation

$$u^{(n)}(t) = Au(t) + f(t), \tag{1.6}$$

where A is a closed linear operator on E and f is a continuous function from $\mathbb{R}$ to E. First, we give a general definition of mild solutions to (1.6). This definition is an extension of that introduced in [1], where $n = 1$, $n = 2$, and A generally is neither the generator of a C_0-semigroup nor of a cosine family, respectively. Several properties of mild solutions are then shown in Section 2.

In Section 3, we consider the conditions for the solvability of operator equation $AX - XB = C$, in particular, when $B = \mathcal{D}^n$, where $\mathcal{D}$ is the differential operator on a function space and $C = -\delta_0$.

Assume that $\mathcal{M}$ is a closed, translation-invariant subspace of $\mathrm{BUC}(\mathbb{R}, E)$. The subspace $\mathcal{M}$ is said to be *regularly admissible* with respect to (1.6) if for every $f \in \mathcal{M}$, (1.6) has a unique mild solution $u \in \mathcal{M}$. In Section 4, we characterize the regular admissibility of $\mathcal{M}$ in terms of solvability of the operator equation. Namely, we show that the subspace $\mathcal{M}$ is regularly admissible if and only if the operator equation of the form

$$AX - X\mathcal{D}^n = -\delta_0 \tag{1.7}$$

has a unique bounded solution. As applications, in Section 5 we show that if the admissible subspace $\mathcal{M}$ is the space of 1-periodic functions, then

$$\sup_{k\in\mathbb{Z}}\left\|k^m\left((2\pi ki)^n - A\right)^{-1}\right\| < \infty \tag{1.8}$$

is a necessary condition, that each mild solution on $\mathcal{M}$ belongs to $C^{(m)}(\mathbb{R},E)$, where $0 \le m \le n$. Finally, we prove that, under some classical condition, if $\sigma(A) \cap (i\mathbb{R})^n$ is countable, then each bounded mild solution of the higher-order equation is almost periodic provided f is almost periodic. This result, shown by a short proof, generalizes [1, Theorem 4.5].

2. Mild solutions of higher-order differential equations

First, we fix some notations. By $C^{(n)}(\mathbb{R},E)$ we denote the space of continuous functions with continuous derivatives $u', u'', \dots, u^{(n)}$ and by $\mathrm{BUC}(\mathbb{R},E)$ the space of bounded, uniformly continuous functions with values in E. The operator $I : C(\mathbb{R},E) \to C(\mathbb{R},E)$ is defined by $If(t) := \int_0^t f(s)ds$ and $I^n f := I(I^{n-1}f)$.

Definition 2.1. (a) We say that $u : \mathbb{R} \to E$ is a classical solution of (1.6) if $u \in D(A)$, $u \in C^n(\mathbb{R},E)$, and (1.6) is satisfied.

(b) A continuous function $u(t) \in C(\mathbb{R},E)$ is called a mild solution of (1.6) if $I^{(n)}u(t) \in D(A)$ for all $t \in \mathbb{R}$ and there exist n points $v_0, v_1, \dots, v_{n-1}$ in E such that

$$u(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!} v_i + AI^n u(t) + I^n f(t) \tag{2.1}$$

for all $t \in \mathbb{R}$.

Remark 2.2. Using the standard argument, we can prove the following statements:

(i) if a mild solution u is m-times differentiable, $0 \le m < n$, then v_i, $i = 0, 1, \dots, m$, are the initial values, that is, $u(0) = v_0$, $u'(0) = v_1, \dots,$ and $u^{(m)}(0) = v_m$;

(ii) if $n = 1$ and A is the generator of a C_0-semigroup $T(t)$, then a continuous function $u : \mathbb{R} \to E$ is a mild solution of (1.6) if and only if it has the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr; \tag{2.2}$$

(iii) similarly, if $n = 2$ and A is a generator of a cosine family $(C(t))$ on E, any continuously differentiable function u on E of the form

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)d\tau, \tag{2.3}$$

where $(S(t))$ is the associated sine family, is a mild solution of (1.6);

(iv) if u is a bounded mild solution of (1.6) corresponding to a bounded inhomogeneity f and $\phi \in L^1(\mathbb{R}, E)$, then $u * \phi$ is a mild solution of (1.6) corresponding to $f * \phi$.

Directly from their definitions, we can collect some properties of mild solutions of (1.6).

Lemma 2.3. *Let u be a mild solution of the higher-order differentiable equation (1.6). If*

(i) *u is in $C^{(n)}(\mathbb{R}, E)$; or*

(ii) *$u(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Au(\cdot) \in C(\mathbb{R}, E)$,*

then u is a classical solution.

Proof. (i) Since u is a mild solution, we have

$$AI^n u(t) = u(t) - \sum_{0}^{n-1} \frac{t^i}{i!} v_i - I^n f(t). \tag{2.4}$$

The right-hand side of (2.4) is n-time differentiable so is the left-hand side. Hence,

$$\begin{aligned} \lim_{h\to 0} A \frac{1}{h} \int_t^{t+h} I^{n-1} u(s) ds &= \lim_{h\to 0} \frac{1}{h} \left(A \int_0^{t+h} I^{n-1} u(s) ds - A \int_0^t I^{n-1} u(s) ds \right) \\ &= \frac{d}{dt} (AI^n(t)) \end{aligned} \tag{2.5}$$

exists. Since

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} I^{n-1} u(s) ds = I^{n-1} u(t) \tag{2.6}$$

and A is closed, we obtain that $I^{n-1}u(t) \in D(A)$ and

$$\frac{d}{dt}(AI^n u(t)) = AI^{n-1} u(t). \tag{2.7}$$

By taking the derivative on both sides of (2.4), we obtain

$$AI^{n-1}(t) = u'(t) - \sum_{0}^{n-2} \frac{t^i}{i!} v_{i+1} - I^{n-1} f(t) \tag{2.8}$$

for all $t \in \mathbb{R}$. Repeating this procedure $(n-1)$ times, we obtain that u is n-times differentiable and $u^{(n)}(t) = Au(t) + f(t)$, that is, u is a classical solution.

(ii) If $u(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Au(\cdot) \in C(\mathbb{R}, E)$, then $AI^n u(t) = I^n Au(t)$. Taking the nth derivative of the right-hand side of

$$u(t) = \sum_{0}^{n-1} \frac{t^i}{i!} v_i + I^n Au(t) + I^n f(t), \tag{2.9}$$

we have that u is n-times continuously differentiable and $u^{(n)}(t) = Au(t) + f(t)$, that is, u is a classical solution. □

In what follows we consider the spectrum of mild solutions of (1.6). For a bounded function $u \in L^\infty(\mathbb{R}, E)$, the *Carleman transform* $\hat{u}$ of u is defined by

$$\hat{u}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} u(t)\,dt, & \operatorname{Re}(\lambda) > 0, \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} u(t)\,dt, & \operatorname{Re}(\lambda) < 0. \end{cases} \tag{2.10}$$

It is clear that $\hat{u}$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$. A point $\mu \in \mathbb{R}$ is called a *regular point* if $\hat{u}$ has a holomorphic extension in a neighborhood of $i\mu$. The spectrum of u is defined as follows:

$$\operatorname{sp}(u) = \{\mu \in \mathbb{R} : \mu \text{ is not regular}\}. \tag{2.11}$$

The following lemma, whose proof can be found in [7, 15], will be needed later.

Lemma 2.4. *Let f, g be in $\mathrm{BUC}(\mathbb{R}, E)$ and $\phi \in L^1(\mathbb{R}, E)$. Then*

(i) $\operatorname{sp}(f)$ *is closed and* $\operatorname{sp}(f) = \emptyset$ *if and only if* $f = 0$;
(ii) $\operatorname{sp}(f + g) \subset \operatorname{sp}(f) \cup \operatorname{sp}(g)$;
(iii) $\operatorname{sp}(f * \phi) \subset \operatorname{sp}(f) \cap \operatorname{supp} \mathcal{F}\phi$, *where $\mathcal{F}\phi$ is the Fourier transform of ϕ.*

The following lemma is the first result about the spectrum of mild solutions of (1.6).

Lemma 2.5. *Let f be a bounded continuous function and let u be a bounded mild solution of (1.6). Then*

$$\operatorname{sp}(u) \subseteq \{\mu \in \mathbb{R} : (i\mu)^n \in \sigma(A)\} \cup \operatorname{sp}(f). \tag{2.12}$$

Proof. It is easy to see that $\widehat{Iu}(\lambda) = (1/\lambda)\hat{u}(\lambda)$, hence $\widehat{I^n u}(\lambda) = (1/\lambda^n)\hat{u}(\lambda)$. Taking the Carleman transform on both sides of (2.1), we have

$$\hat{u}(\lambda) = Q(\lambda) + \frac{1}{\lambda^n} A\hat{u}(\lambda) + \frac{1}{\lambda^n}\hat{f}(\lambda), \tag{2.13}$$

where

$$Q(\lambda) = \int_0^\infty e^{-\lambda t} \left(\sum_{i=0}^{n-1} \frac{t^i}{i!} v_i \right) dt = \sum_{i=0}^{n-1} \frac{u_i}{\lambda^i}. \tag{2.14}$$

From (2.13) we obtain

$$(\lambda^n - A)\hat{u}(\lambda) = \lambda^n Q(\lambda) + \hat{f}(\lambda) \tag{2.15}$$

for $\lambda \notin i\mathbb{R}$. Hence, for $\lambda^n \in \varrho(A)$ we have

$$\hat{u}(\lambda) = (\lambda^n - A)^{-1}(\lambda^n Q(\lambda) + \hat{f}(\lambda)). \tag{2.16}$$

Note that $\lambda^n Q(\lambda)$ is a holomorphic function in terms of λ. It implies that if $\mu \in \mathbb{R}$ is a regular point of f and $(i\mu)^n \in \varrho(A)$, then $\hat{u}$ has holomorphic extension in a neighborhood of $i\mu$, that is, μ is a regular point of u. Hence, we have the inclusive relation. □

From Lemma 2.5, we directly have the following corollary.

Corollary 2.6. *If u is a bounded mild solution of (1.6) corresponding to $f \equiv 0$, then* $\operatorname{sp}(u) \subseteq \{\mu \in \mathbb{R} : (i\mu)^n \in \sigma(A)\}$.

Corollary 2.7. *If $(i\mathbb{R})^n \cap \sigma(A) = \emptyset$, then (1.6) has at most one bounded mild solution.*

3. The equation $AX - XD^n = C$

Let A and B be closed, generally unbounded, linear operators on Banach spaces E and F with dense domains $D(A)$ and $D(B)$, respectively, and let C be a bounded linear operator from E to F. A bounded operator $X : F \to E$ is called a *solution* of the operator equation

$$AX - XB = C \tag{3.1}$$

if for every $f \in D(B)$ we have $Xf \in D(A)$ and $AXf - XBf = Cf$. Equation (3.1) has been considered by many authors. It was first studied intensively for bounded operators by Dalec'kiĭ and Kreĭn [4], Rosenblum [16]. For unbounded case, (3.1) was studied in [2, 11, 12, 13], when A and B are generators of C_0-semigroups, and in [17, 19] when A and B are closed operators. We cite here some main results which will be used in the sequel.

Theorem 3.1. (i) *Let A and B be generators of C_0-semigroups on E and F, one of which is analytic such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then for every bounded operator C, (3.1) has a unique bounded solution (see* [11, Theorem 15]*).*

(ii) *Let A be a closed operator and let B be a bounded operator such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then for every bounded operator C, (3.1) has a unique bounded solution X which has the following integral form:*

$$X = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} C(\lambda - B)^{-1} d\lambda, \tag{3.2}$$

where Γ is a closed Cauchy contour around $\sigma(B)$ and is separated from $\sigma(A)$ (see [17, Theorem 3.1]*).*

(iii) *If (3.1) has a unique bounded solution for every bounded operator C, then $\sigma(A) \cap \sigma(B) = \emptyset$ (see* [2, Theorem 2.1]*).*

We now consider the situation when $F = \mathcal{M}$, a translation-invariant subspace of $\mathrm{BUC}(\mathbb{R}, E)$, and $B = \mathfrak{D}^n_{\mathcal{M}}$, the restriction of $\mathfrak{D}^n$ to $\mathcal{M}$, where $\mathfrak{D} := d/dt$ on $\mathrm{BUC}(\mathbb{R}, E)$. It is well known that $\sigma(\mathfrak{D}) = i\mathbb{R}$ and $\sigma(\mathfrak{D}^n) = (\sigma(\mathfrak{D}))^n$.

Let now $\mathcal{M}_k := \{f \in \mathcal{M} : \mathrm{sp}(f) \subset [-ik, ik]\}$, $k \geq 1$. Then the following properties hold (see [5, 19]):

(i) $\mathcal{M}_k$ are translation-invariant subspaces,

(ii) $\mathcal{M}_k \subset \mathcal{M}_{k+1}$,

(iii) $\mathfrak{D}_{\mathcal{M}_k}$ is bounded.

We first need the following lemma which was proved in [19].

Lemma 3.2. *Let $\mathfrak{D}_{\mathcal{M}}$ and $\mathfrak{D}_{\mathcal{M}_k}$ be as above, then*

$$\sigma(\mathfrak{D}_{\mathcal{M}}) = \cup_{k=1}^{\infty} \sigma(\mathfrak{D}_{\mathcal{M}_k}). \tag{3.3}$$

From Lemma 3.2 we obtain the following lemma.

Lemma 3.3. *For any positive integer $n \geq 1$, the following equality holds:*

$$\sigma(\mathfrak{D}^n_{\mathcal{M}}) = \cup_{k=1}^{\infty} \sigma(\mathfrak{D}^n_{\mathcal{M}_k}). \tag{3.4}$$

Proof. We show that

$$\sigma(\mathfrak{D}^n_{\mathcal{M}}) \subseteq \cup_{k=1}^{\infty} \sigma(\mathfrak{D}^n_{\mathcal{M}_k}). \tag{3.5}$$

Note that $\sigma(\mathfrak{D}^n) = (i\mathbb{R})^n$, hence $\sigma(\mathfrak{D}^n_{\mathcal{M}}) \subseteq (i\mathbb{R})^n$. Assume that $(i\lambda)^n \in \sigma(\mathfrak{D}^n_{\mathcal{M}})$, $\lambda \in \mathbb{R}$. Then there is a sequence of vectors $(f_k)_k \subset \mathcal{M}$ such that $f_k \in D(\mathfrak{D}^n_{\mathcal{M}})$, $\|f_k\| = 1$, and

$$\lim_{k \to \infty} \left|\left| \left((i\lambda)^n - \mathfrak{D}^n_{\mathcal{M}}\right) f_k \right|\right| = 0. \tag{3.6}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n complex roots of the equation $x^n = (i\lambda)^n$. Then we have

$$\left((i\lambda)^n - \mathfrak{D}^n_{\mathcal{M}}\right) f_k = \prod_{j=1}^{n} \left(\lambda_j - \mathfrak{D}_{\mathcal{M}}\right) f_k. \tag{3.7}$$

We show that there is at least one λ_j belonging to the spectrum of $\mathfrak{D}_{\mathcal{M}}$. Assume contrarily that all λ_j belong to $\varrho(\mathfrak{D}_{\mathcal{M}})$, then

$$f_k = \prod_{j=1}^{n} \left(\lambda_j - \mathfrak{D}_{\mathcal{M}}\right)^{-1} \left((i\lambda)^n - \mathfrak{D}^n_{\mathcal{M}}\right) f_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.8}$$

which is contradictory to $\|f_k\| = 1$. Hence, there is a λ_j which belongs to $\sigma(\mathfrak{D}_{\mathcal{M}})$. By Lemma 3.2, there is a number k such that $i\lambda_j \in \sigma(\mathfrak{D}_{\mathcal{M}_k})$. Since $\mathfrak{D}_{\mathcal{M}_k}$ is bounded, $(i\lambda)^n = (i\lambda_j)^n \in \sigma(\mathfrak{D}^n_{\mathcal{M}_k})$, and hence the inclusion (3.5) follows. Since the inverse of (3.5) is obvious, the lemma is proved. □

From Lemmas 3.2 and 3.3 we have the following lemma.

Lemma 3.4. *For any positive integer $n \geq 1$ the following equality holds:*

$$\sigma(\mathcal{D}^n_{\mathcal{M}}) = \{\lambda^n : \lambda \in \sigma(\mathcal{D}_{\mathcal{M}})\}. \tag{3.9}$$

We now return to the operator equation

$$AX - X\mathcal{D}^n_{\mathcal{M}} = \delta_0^{\mathcal{M}}, \tag{3.10}$$

where $\delta_0^{\mathcal{M}}$ is the restriction of the Dirac operator to $\mathcal{M}$. Assume that

$$\sigma(A) \cap \{\lambda^n : \lambda \in \sigma(\mathcal{D}_{\mathcal{M}})\} = \emptyset. \tag{3.11}$$

Then, by Lemma 3.4, it is equivalent to

$$\sigma(A) \cap \sigma(\mathcal{D}^n_{\mathcal{M}}) = \emptyset. \tag{3.12}$$

Therefore, for $k = 1, 2, \ldots$, we have

$$\sigma(A) \cap \sigma(\mathcal{D}^n_{\mathcal{M}_k}) = \emptyset. \tag{3.13}$$

By Theorem 3.1, the operator equation

$$AX - X\mathcal{D}^n_{\mathcal{M}_k} = \delta_0^{\mathcal{M}_k} \tag{3.14}$$

has a unique bounded solution X_k which is of the form

$$X_k = -\frac{1}{2\pi i}\int_{\Gamma_k} (\lambda - A)^{-1}\delta_0^{\mathcal{M}_n}(\lambda - \mathcal{D}^n_{\mathcal{M}_k})^{-1} d\lambda, \tag{3.15}$$

where Γ_k is a contour around $\sigma(\mathcal{D}^n_{\mathcal{M}_k})$ and is separated from $\sigma(A)$. Moreover, the uniqueness of X_k implies

$$X_k | \mathcal{M}_l = X_l \quad \text{for } l < k. \tag{3.16}$$

We state a result about the existence and uniqueness of bounded solutions of (3.10), whose proof is similar to that of [19, Theorem 7] (for $n = 2$) and is omitted.

Theorem 3.5. *Assume that condition (3.11) holds. Then the operator equation (3.10) has a unique bounded solution if and only if*

$$\sup_{n\geq 1} ||X_k|| < \infty, \tag{3.17}$$

where X_k are defined by (3.15).

4. Admissible subspaces

Let $\mathcal{M}$ be a closed translation-invariant subspace of $\mathrm{BUC}(\mathbb{R}, E)$, which is regularly admissible with respect to (1.6). Define the linear operator G on $\mathcal{M}$ such that for each $f \in \mathcal{M}$, Gf is the unique mild solution of (1.6) in $\mathcal{M}$, we have the following lemma.

Lemma 4.1. *The operator G is a linear, bounded operator on $\mathcal{M}$.*

Proof. We define operator $\tilde{G} : \mathcal{M} \to \mathcal{M} \otimes E^n$ by

$$\tilde{G}f := (u, v_0, v_1, \ldots, v_{n-1}), \tag{4.1}$$

where u is the unique mild solution of (1.6) corresponding to f and $v_0, v_1, \ldots, v_{n-1}$ are contained in the mild solution

$$u(t) = \sum_0^{n-1} \frac{t^i}{i!} v_i + AI^n u(t) + I^n f(t). \tag{4.2}$$

We will show that $\tilde{G}$ is closed. Let $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ with $\lim_k f_k = f$ and $\tilde{G}f_k = (u_k, v_{0,k}, \ldots, v_{n-1,k})$ with $\lim_{k\to\infty} \tilde{G}f_k = (u, v_0, \ldots, v_{n-1})$, that is, $\lim_{k\to\infty} u_k = u$ and $\lim_{k\to\infty} v_{j,k} = v_k$ for $j = 0, 1, \ldots, n-1$. Then we have $\lim_{k\to\infty} I^n u_k(t) = I^n u(t)$ and, by (4.2),

$$\begin{aligned} AI^n u_k(t) &= u_k(t) - \sum_0^{n-1} \frac{t^i}{i!} v_{i,k} - I^n f_k(t) \\ &\longrightarrow u(t) - \sum_0^{n-1} \frac{t^i}{i!} v_k - I^n f(t) \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{4.3}$$

Since A is closed we obtain that $I^n u(t) \in D(A)$ and

$$AI^n u(t) = u(t) - \sum_0^{n-1} \frac{t^i}{i!} v_i - I^n f(t). \tag{4.4}$$

That means that $\tilde{G}f = (u, v_0, v_1, \ldots, v_{n-1})$. Hence, $\tilde{G}$ is closed and thus bounded. Since $G = \tilde{G} \circ P$, where $P : \mathcal{M} \otimes E^n \to \mathcal{M}$ is the projection on the first coordinate and thus is a bounded operator, we obtain that G is bounded. □

The operator G is called the *solution operator* of (1.6) and is commuting with the translation and hence is commuting with the differential operator, as the following lemma shows.

Lemma 4.2. *Let A be a closed operator on E with nonempty resolvent set and let $\mathcal{M}$ be an admissible subspace of* $\mathrm{BUC}(\mathbb{R}, E)$. *Then the following conditions hold:*

(i) $S_h \cdot G = G \cdot S_h$, *where S_h is the translation operator on $\mathcal{M}$;*

(ii) $\mathcal{D}_{\mathcal{M}} \cdot G = G \cdot \mathcal{D}_{\mathcal{M}}$.

Proof. (i) Let $u = Gf$ be the unique mild solution of the higher-order differential equation (1.6). If u is a classical solution, then $(Gf)^{(n)}(t+h) = A(Gf)(t+h) + f(t+h)$, and hence $S_h \cdot Gf = G \cdot S_h f$. For the case that u is not a classical solution, let $\lambda \in \varrho(A)$. Since

$$R(\lambda,A)u(t) = \sum_{0}^{n-1} \frac{t^i}{i!} R(\lambda,A)u_i + AI^n R(\lambda,A)u(t) + I^n R(\lambda,A)f(t), \tag{4.5}$$

it is easy to see that $\tilde{u}(t) = R(\lambda,A)u(t)$ is the unique solution of (1.6) corresponding to $\tilde{f} = R(\lambda,A)f$. But $\tilde{u}(t) \in D(A)$ for all $t \in \mathbb{R}$. Hence, by Lemma 2.3(ii), $\tilde{u}$ is a classical solution. From the above result for a classical solution and the fact that S_h and $R(\lambda,A)$ commute, we have

$$\begin{aligned} R(\lambda,A)S_h Gf &= S_h R(\lambda,A)Gf = S_h GR(\lambda,A)f \\ &= GS_h R(\lambda,A)f = GR(\lambda,A)S_h f = R(\lambda,A)GS_h f, \end{aligned} \tag{4.6}$$

from which it follows that $S_h Gf = GS_h f$ for all $f \in \mathcal{M}$. Part (ii) is a direct consequence of (i), and the lemma is proved. □

Corollary 4.3. *Assume that A is a closed operator with nonempty resolvent set. Let $\mathcal{M}$ be a regularly admissible subspace of* BUC$(\mathbb{R},E)$ *and let u be the unique mild solution corresponding to f in $\mathcal{M}$. If $f \in C^n(\mathbb{R},E)$ such that $f', f'', \dots, f^{(n)}$ belong to $\mathcal{M}$, then u is a classical solution.*

In what follows, we assume that $\mathcal{M}$ satisfies the following additional assumption.

Assumption 4.4. For all $C \in \mathcal{L}(\mathcal{M},E)$ and $f \in \mathcal{M}$, the function $\Phi(t) = CS(t)f$ belongs to $\mathcal{M}$.

The regular admissibility of a space is closely related to the solvability of operator equation (3.1). This relation was shown in [13], when $n = 1$, and in [19, 20], when $n = 2$. The following theorem is a generalization of those results.

Theorem 4.5. *Let A be a closed operator on E with nonempty resolvent set and let $\mathcal{M}$ be a translation-invariant subspace in* BUC$(\mathbb{R},E)$, *which satisfies Assumption 4.4. Then the following statements are equivalent:*

(i) *$\mathcal{M}$ is a regularly admissible subspace;*
(ii) *the operator equation*

$$AX - X\mathcal{D}_{\mathcal{M}}^{(n)} = -\delta_0 \tag{4.7}$$

has a unique solution;

(iii) *for every bounded operator $C : \mathcal{M} \to E$, the operator equation*

$$AX - X\mathcal{D}_{\mathcal{M}}^{(n)} = C \tag{4.8}$$

has a unique solution.

Proof. (i)⇒(ii). Let $G : \mathcal{M} \to \mathcal{M}$ be the bounded operator defined by $Gf = u$, where u is the unique mild solution in $\mathcal{M}$. We define the operator $X : \mathcal{M} \mapsto E$ by

$$Xf := (Gf)(0). \tag{4.9}$$

Then X is a bounded operator. Now let $f \in \mathcal{D}_{\mathcal{M}}^n$. By Lemma 4.2, $u = Gf$ is a classical solution of (1.6), that is,

$$(Gf)^{(n)}(t) = A(Gf)(t) + f(t). \tag{4.10}$$

Note that, by Lemma 4.2, $(Gf)^{(n)} = Gf^{(n)}$. Taking $t = 0$ from (4.10) and using this fact, we have $AXf - X\mathcal{D}^n f = -\delta_0 f$ for $f \in \mathcal{D}_{\mathcal{M}}^n$, that is, X is a bounded solution of (4.7).

To show the uniqueness, we assume that X_0 is a solution of (4.7). Then for every $f \in \mathcal{D}_{\mathcal{M}}^n$, the function $u \in \mathcal{M}$, defined by $u(t) = X_0 S(t) f$, is a classical solution of (1.6). Indeed,

$$u^{(n)}(t) = X_0 \mathcal{D}^n S(t) f = (AX_0 + \delta_0) S(t) f = Au(t) + f(t) \tag{4.11}$$

for all $t \in \mathbb{R}$. We will show that $u(t) = X_0 S(t) f$ is a mild solution of (1.6) for every $f \in \mathcal{M}$. To this end, let $f \in \mathcal{M}$ and $(f_k)_{k\in\mathbb{N}} \subseteq D(\mathcal{D}_{\mathcal{M}}^n)$ with $\lim_k f_k = f$. Then $Gf = \lim_k Gf_k = \lim_k X_0 S(\cdot) f_k = X_0 S(\cdot) f$. Hence, $Gf = X_0 S(\cdot) f$, that is, $u = X_0 S(\cdot) f$ is a mild solution of (1.6).

Assume now that X_1 and X_2 are two solutions of (4.7). Then, for every $f \in \mathcal{M}$, $u = (X_1 - X_2) S(\cdot) f$ is a mild solution of the higher-order equation $u^{(n)}(t) = Au(t)$. By the uniqueness of the mild solution we have $u \equiv 0$, which implies $X_1 = X_2$.

(ii)⇒(iii). Let X be the unique solution of (4.7). Define the bounded operator $Y : \mathcal{M} \to E$ by $Yf := X\tilde{f}$, where $\tilde{f}(t) = -CS(t)f$. Let $f \in D(\mathcal{D}_{\mathcal{M}}^n)$, then $(\mathcal{D}_{\mathcal{M}}^n f)\tilde{}(t) = -CS(t)\mathcal{D}_{\mathcal{M}}^n f = \mathcal{D}_{\mathcal{M}}^n \tilde{f}(t)$. Hence, we have

$$AYf = AX\tilde{f} = X\mathcal{D}_{\mathcal{M}}^n \tilde{f} + \delta_0 \tilde{f} = X(\mathcal{D}_{\mathcal{M}}^n f)\tilde{} + Cf = Y\mathcal{D}_{\mathcal{M}}^n f + Cf, \tag{4.12}$$

that is, Y is a bounded solution of (4.8).

The uniqueness of the solution of operator equation $AX - X\mathcal{D}_{\mathcal{M}}^n = C$ follows directly from the uniqueness of the solution of $AX - X\mathcal{D}_{\mathcal{M}}^n = -\delta_0$.

(iii)⇒(i). We have shown above that if X is a bounded solution of (4.7), then $u(t) := XS(t)f$ is a mild solution of the higher-order equation (1.6). It remains to show that this solution is unique. In order to do it, assume that u is a mild solution of the homogeneous equation $u^{(n)}(t) = Au(t)$, $t \in \mathbb{R}$. By Corollary 2.6,

$(i\mathrm{sp}(u))^n \subseteq \sigma(A)$. On the other hand, since $u \in \mathcal{M}$, $i\mathrm{sp}(u) \subseteq \sigma(\mathcal{D}_{\mathcal{M}})$, which implies $(i\mathrm{sp}(u))^n \subseteq \sigma(\mathcal{D}^n_{\mathcal{M}})$. By Theorem 3.1(iii), it follows from (iii) that $\sigma(A) \cap \sigma(\mathcal{D}^n_{\mathcal{M}}) = \emptyset$. Hence, $\mathrm{sp}(u) = \emptyset$, so $u \equiv 0$ and the theorem is proved. □

5. Applications

In this section, we will apply the results of Section 4 to the space of periodic and of almost periodic functions. Let $P(\omega)$ be the space of periodic functions from $\mathbb{R}$ to E with the period ω. For the sake of simplicity, we assume the period $\omega = 1$. We begin with the case in which $n = 2$ and A is the generator of a cosine family $(C(t))$. It is well known that

(1) A is the generator of an analytic C_0-semigroup given by

$$e^{Az}x = \frac{1}{\sqrt{(\pi z)}} \int_0^\infty e^{-t^2/4z} C(t)x\,dt, \qquad \mathrm{Re}(z) > 0; \tag{5.1}$$

(2) $\mathcal{D}^2$ is the generator of a cosine family given by

$$C(t) = \frac{1}{2}\left(\mathcal{S}(t) + \mathcal{S}(-t)\right) \tag{5.2}$$

and hence is the generator of an (analytic) C_0-semigroup in $P(1)$.

By Theorems 3.1(i) and 4.5, $P(1)$ is regularly admissible if and only if $\sigma(A) \cap \sigma(\mathcal{D}^2_{P(1)}) = \emptyset$. On the other hand, $\sigma(\mathcal{D}^2_{P(1)}) = \{(2k\pi i)^2 : k \in \mathbb{Z}\} = \{-k^2\pi^2 : k \in \mathbb{Z}\}$. Hence, we have the following theorem.

Theorem 5.1. *Let A be the generator of a strongly continuous cosine family. Then $P(1)$ is regularly admissible with respect to $u''(t) = Au(t) + f(t)$ if and only if $\{-4k^2\pi^2 : k \in \mathbb{Z}\} \subset \varrho(A)$.*

In general, however, the condition of the form $\sigma(A) \cap \sigma(\mathcal{D}^n_{\mathcal{M}}) = \emptyset$ does not imply the regular admissibility of subspace $\mathcal{M}$. At least the operator A must satisfy some conditions, as the following theorem shows.

Theorem 5.2. *Let A be a closed operator on a Banach space E with nonempty resolvent set and suppose that $P(1)$ is regularly admissible with respect to the equation*

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}. \tag{5.3}$$

Then

(1) $(2\pi ki)^n \in \varrho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|((2\pi ki)^n - A)^{-1}\| < \infty$,
(2) *if each mild solution on $P(1)$ belongs to $C^{(m)}(\mathbb{R}, E)$, $0 \le m \le n$, then* $(2\pi ki)^n \in \varrho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|k^m((2\pi ki)^n - A)^{-1}\| < \infty$.

Proof. By assumption, $P(1)$ is a regularly admissible function space, so, by Theorem 4.5, the equation $AX - X\mathcal{D}^n_{P(1)} = C$ has a unique solution for every bounded operator C. Hence, by Theorem 3.1(iii), $\sigma(A) \cap \sigma(\mathcal{D}^n_{P(1)}) = \emptyset$. On the other hand, it is not hard to see that $\sigma(\mathcal{D}^n_{P(1)}) = \{(2k\pi i)^n : k \in \mathbb{Z}\}$. It follows that $\sigma(A) \cap \{(2k\pi i)^n : k \in \mathbb{Z}\} = \emptyset$ or, in other words, $\{(2k\pi i)^n : k \in \mathbb{Z}\} \subset \varrho(A)$.

To prove (1), let $G : P(1) \to P(1)$ be the solution operator and take $f(t) = e^{2k\pi it}x_0$, $x_0 \in E$, as a 1-periodic function. It is not too hard to check that $Gf(t) = e^{2k\pi it} \cdot ((2k\pi i)^n - A)^{-1}x_0$ is the (unique) mild solution of (5.3). Hence,

$$\left\|\left((2k\pi i)^n - A\right)^{-1} x_0\right\| = \|Gf\| \leq \|G\| \cdot \|f\| = \|G\| \cdot \|x_0\| \tag{5.4}$$

for all $x_0 \in E$ and $k \in \mathbb{Z}$. Hence, $\sup_{k\in\mathbb{Z}} \|((2k\pi i)^n - A)^{-1}\| \leq \|G\| < \infty$.

To prove (2) observe that since each mild solution on $P(1)$ belongs to $C^{(m)}(\mathbb{R}, E)$, the composite operator $\mathcal{D}^m_{P(1)}G$ is everywhere defined and closed. Hence, it is a bounded operator. Thus,

$$\begin{aligned} \|\mathcal{D}^m_{P(1)}Gf\| = \left\|(2k\pi)^m\left((2k\pi i)^n - A\right)^{-1} x_0\right\| &\leq \|\mathcal{D}^m_{P(1)}G\| \cdot \|f\| \\ &= \|\mathcal{D}^m_{P(1)}G\| \cdot \|x_0\| \end{aligned} \tag{5.5}$$

for all $x_0 \in E$ and $k \in \mathbb{Z}$. Hence, $\sup_{k\in\mathbb{Z}} \|k^m((2k\pi i)^n - A)^{-1}\| \leq C \cdot \|\mathcal{D}^m_{P(1)}G\|$ for a certain constant C, and that completes the proof. □

The converse of Theorem 5.2 generally does not hold (see [6] for a counterexample). However, we have the affirmative answer in certain special cases. If E is a Hilbert space, $n = 1$, and A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, we have the following theorem whose proof of (b)⇒(a) can be found in [14].

Theorem 5.3. *Let A be the generator of a C_0-semigroup on a Hilbert space E. Then the following conditions are equivalent:*

(a) *for each 1-periodic function f, the equation*

$$u'(t) = Au(t) + f(t) \tag{5.6}$$

has a unique 1-periodic mild solution;

(b) $\{2\pi ki : k \in \mathbb{Z}\} \subset \varrho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|(2\pi ki - A)^{-1}\| < \infty$.

Also, if $n = 2$, $m = 1$, and A is the generator of a cosine family $(C(t))$ on a Hilbert space E, we have a positive answer. Namely, we have the following theorem whose proof of the converse part (b)⇒(a) can be found in [8].

Theorem 5.4. *If A is the generator of a cosine family on a Hilbert space E, then the following statements are equivalent:*

(a) *for each 1-periodic function f, the equation*

$$u''(t) = Au(t) + f(t) \tag{5.7}$$

has a unique 1-periodic mild solution which belongs to $C^1(\mathbb{R}, E)$;

(b) $\{-4\pi^2k^2 : k \in \mathbb{Z}\} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|k(4\pi^2k^2 + A)^{-1}\| < \infty$.

We now apply the results of Section 4 to $AP(\mathbb{R},E)$, the space of almost periodic functions from $\mathbb{R}$ to E. As a preparation, we recall some basic concepts and results about almost periodic functions. (For more details, readers are referred to [1, 9].) A point $\lambda \in \mathbb{R}$ is called a point of almost periodicity of the function u if there is a neighborhood $\mathcal{U}$ of λ such that for every $\phi \in L^1(\mathbb{R})$ with $\operatorname{supp}\mathcal{F}\phi \subset \mathcal{U}$, where $\mathcal{F}\phi$ is the Fourier transform of ϕ, the function $\phi * u$ is almost periodic. The complement in $\mathbb{R}$ of the set of points of almost periodicity of u is called the *almost periodic spectrum of f* and is denoted by $\operatorname{sp}_{AP}(u)$.

We say that $u \in BUC(\mathbb{R},E)$ is *totally ergodic* if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\nu s}u(s)ds \tag{5.8}$$

exists for all $\nu \in \mathbb{R}$. The following theorem can be found in [9] (parts (a) and (b)) and [17] (part (c)).

Theorem 5.5. *Let* $u \in BUC(\mathbb{R},E)$ *such that* $\operatorname{sp}_{AP}(u)$ *is countable. Assume that*

(a) $E \not\supset c_0$; *or*
(b) *the range of* $u(t)$ *is weakly relatively compact; or*
(c) u *is totally ergodic.*

Then u is almost periodic.

We now return to our higher-order equation. Let Γ be a compact set in $\mathbb{R}$ and let $\mathcal{M} = X(\Gamma)$ be the subspace of $BUC(\mathbb{R},E)$ consisting of all functions f with $\operatorname{sp}(f) \subset \Gamma$. It is easy to see that $\mathcal{M}$ satisfies Assumption 4.4. Moreover, $\mathcal{D}_{\mathcal{M}}$ is bounded, $\sigma(\mathcal{D}_{\mathcal{M}}) = i\Gamma$, and thus $\sigma(\mathcal{D}^n_{\mathcal{M}}) = (i\Gamma)^n$. Assume now that $\sigma(A) \cap (i\Gamma)^n = \emptyset$; then, by Theorem 3.1(ii), the equation $AX - X\mathcal{D}^n_{\mathcal{M}} = -\delta_0$ has a unique solution. By Theorem 4.5, $\mathcal{M}$ is regularly admissible and for any almost periodic function f, the mild solution $u(t) = XS(t)f$ is also almost periodic. Using these facts, we have the following theorem.

Theorem 5.6. *For the equation*

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{5.9}$$

assume that f is almost periodic and $\sigma(A) \cap (i\mathbb{R})^n$ is countable. Let $u \in BUC(\mathbb{R}, E)$ *be a mild solution of (5.9). Then u is almost periodic if one of the following conditions is satisfied:*

(a) $E \not\supset c_0$; *or*
(b) *the range of* $u(t)$ *is weakly relatively compact; or*
(c) u *is totally ergodic.*

Proof. In view of Theorem 5.5, we only have to show that $\operatorname{sp}_{AP}(u)$ is countable. Since $\sigma(A) \cap (i\mathbb{R})^n$ is countable, it suffices to prove that $(i\operatorname{sp}_{AP}(u))^n \subset \sigma(A)$.

Let λ be any point in $\mathbb{R}$ such that $(i\lambda)^n \in \varrho(A)$; we will show that $\lambda \notin \mathrm{sp}_{\mathrm{AP}}(u)$. Since $\varrho(A)$ is an open set, there exists $\epsilon > 0$ such that $(i\Gamma)^n \subset \varrho(A)$, where $\Gamma = [\lambda - \epsilon, \lambda + \epsilon]$. Since Γ is compact and $\sigma(A) \cap (i\Gamma)^n = \emptyset$, $X(\Gamma)$ is regularly admissible with respect to (5.9).

Let ϕ be a function in $L^1(\mathbb{R}, E)$ with $\operatorname{supp} \mathcal{F}\phi \subset \Gamma$ and define $\tilde{u} := u * \phi$ and $\tilde{f} := f * \phi$. Then $\tilde{u}$ and $\tilde{f}$ are in $X(\Gamma)$ (Lemma 2.4(iii)) and $\tilde{f}$ is an almost periodic function. Moreover, $\tilde{u}$ is the unique mild solution of (5.9) corresponding to $\tilde{f}$ in $X(\Gamma)$ (Remark 2.2). By the reasoning preceding this theorem, $\tilde{u}$ is also almost periodic. So, λ is a point of almost periodicity of u, that is, $\lambda \notin \mathrm{sp}_{\mathrm{AP}}(u)$, and the theorem is proved. □

Acknowledgments

This paper was written while the author was visiting the Department of Mathematics, Ohio University. The author thanks Dr. Vu Quoc Phóng for many valuable discussions and suggestions.

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