We give an example of an unbounded, convex, and closed set \( C \) in the Hilbert space \( l^2 \) with the following two properties: (i) \( C \) has the approximate fixed-point property for nonexpansive mappings, (ii) \( C \) is not contained in a block for every orthogonal basis in \( l^2 \).

1. Introduction

In [6], Goebel and the author observed that some unbounded sets in Hilbert spaces have the approximate fixed-point property for nonexpansive mappings. Namely, they proved that every closed convex set \( C \), which is contained in a block, has the approximate fixed-point property for nonexpansive mappings (AFPP). This result was extended by Ray [14] to all linearly bounded subsets of \( l_p \), \( 1 < p < \infty \). Next, he proved that a closed convex subset \( C \) of a real Hilbert space has the fixed-point property for nonexpansive mappings if and only if it is bounded [15]. The first result of Ray [14] was generalized by Reich [16] (for other results of this type see [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19]). Reich [16] proved the following remarkable theorem: a closed, convex subset of a reflexive Banach space has the AFPP if and only if it is linearly bounded. Next, Shafrir [18] introduced the notion of a directionally bounded set. Using this concept, he proved two important theorems [18].

1. A convex subset \( C \) of a Banach space \( X \) has the AFPP if and only if \( C \) is directionally bounded.

2. For a Banach space \( X \), the following two conditions are equivalent: (i) \( X \) is reflexive; (ii) every closed, convex, and linearly bounded subset \( C \) of \( X \) is directionally bounded.

Therefore, the following statements are equivalent: (a) \( X \) is reflexive; (b) a closed, convex subset \( C \) of \( X \) has the AFPP if and only if \( C \) is linearly bounded. This result is strictly connected with the above-mentioned Reich theorem [16].

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A remark on the approximate fixed-point property

Now, it is worth to note that, recently, there is a return to study the AFPP. First, Espínola and Kirk [3] published a paper about the AFPP in the product spaces. They proved that the product space $D = (M \times C)_\infty$ has the AFPP for nonexpansive mappings whenever $M$ is a metric space which has the AFPP for such mappings and $C$ is a bounded, convex subset of a Banach space. Next, Wiśnicki wrote a paper about a common approximate fixed-point sequence for two commuting nonexpansive mappings (see [20] for details). Therefore, the author decided to publish an example of a set which is closely related to the AFPP. Namely, it is obvious that every blockable set in $\ell^2$ is linearly bounded, but there are linearly bounded sets in $\ell^2$ which are not contained in any block with respect to an arbitrary basis. This was mentioned in [6] but never published. The aim of this paper is to show the construction of such a set.

2. Preliminaries

Throughout this paper, $\ell^2$ is real, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\ell^2$, and $\{e_n\}$ is the standard basis in $\ell^2$.

For any nonempty set $K \subset \ell^2$, the closed convex hull of $K$ is denoted by $\text{conv} K$.

Let $C$ be a nonempty subset of a Banach space $X$. A mapping $T : C \to C$ is said to be nonexpansive if for each $x, y \in C$,

$$\|T(x) - T(y)\| \leq \|x - y\|. \quad (2.1)$$

A convex subset $C$ of a Banach space $X$ has the approximate fixed-point property (AFPP) if each nonexpansive $T : C \to C$ satisfies

$$\inf \{\|x - T(x)\| : x \in C\} = 0. \quad (2.2)$$

It is obvious that bounded convex sets always have the AFPP.

A set $K \subset \ell^2$ is said to be a block in the orthogonal basis $\{\tilde{e}_n\}$ if $K$ is of the form

$$K = \{x \in \ell^2 : |\langle x, \tilde{e}_n \rangle| \leq M_n, \ n = 1, 2, \ldots\}, \quad (2.3)$$

where $\{M_n\}$ is a sequence of positive reals.

The set $C \subset \ell^2$ is called a block set if there exists a block $K \subset \ell^2$ such that $C$ is a subset of $K$.

A subset $C$ of a Banach space $X$ is linearly bounded if $C$ has bounded intersections with all lines in $X$. 

3. The construction

Let \( \{k_n\}_{n=2}^{\infty} \) and \( \{l_n\}_{n=2}^{\infty} \) be two sequences of positive reals such that

\[
\sum_{n=2}^{\infty} \frac{k_n}{l_n} < +\infty, \quad \lim_{n \to \infty} k_n = +\infty.
\]

(3.1)

For example, we may take \( k_n = n \) and \( l_n = n^3 \) for \( n = 2, 3, \ldots \). Next, we set

\[
a_n = k_n e_1 + l_n e_n, \quad b_n = -k_n e_1 + l_n e_n,
\]

(3.2)

for \( n = 2, 3, \ldots \), and finally,

\[ C = \text{conv} \{ x \in l^2 : \exists n \geq 2 (x = a_n \lor x = b_n) \}. \]

(3.3)

**Theorem 3.1.** If

\[
x = \sum_{n=1}^{\infty} c_n e_n = c_1 e_1 + \sum_{n=2}^{\infty} d_n l_n e_n = c_1 e_1 + \bar{x}
\]

is an element of the set \( C \), then

\[
d_n \geq 0 \quad \text{for} \quad n = 2, 3, \ldots,
\]

(3.5)

\[
\sum_{n=2}^{\infty} d_n \leq 1,
\]

(3.6)

and there exist sequences \( \{\alpha_n\}_{n=2}^{\infty} \) and \( \{\beta_n\}_{n=2}^{\infty} \) such that

\[
c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n), \quad \alpha_n, \beta_n \geq 0, \quad \alpha_n + \beta_n = d_n,
\]

(3.7)

for \( n = 2, 3, \ldots \). Additionally, there exists a positive constant \( M_{\bar{x}} \) such that

\[
0 \leq (\alpha_n + \beta_n) k_n = d_n k_n \leq M_{\bar{x}} \frac{k_n}{l_n}
\]

(3.8)

for \( n = 2, 3, \ldots \).

**Proof.** Set

\[
\bar{x} = \sum_{n=2}^{\infty} c_n e_n = \sum_{n=2}^{\infty} d_n l_n e_n.
\]

(3.9)
Observe that, there exists a sequence \( \{x_j\} \) such that

\[
x = \lim_{j} x_j
\]

with

\[
x_j = \sum_{n=2}^{\infty} (\alpha_{nj}a_n + \beta_{nj}b_n)
\]

\[
= \sum_{n=2}^{\infty} (\alpha_{nj}k_n - \beta_{nj}k_n)e_1 + \sum_{n=2}^{\infty} (\alpha_{nj}l_n + \beta_{nj}l_n)e_n
\]

\[
= \sum_{n=2}^{\infty} (\alpha_{nj}k_n - \beta_{nj}k_n)e_1 + \bar{x}_j \in C,
\]

where

\[
\bar{x}_j = \sum_{n=2}^{\infty} (\alpha_{nj}l_n + \beta_{nj}l_n)e_n, \quad \alpha_{nj}, \beta_{nj} \geq 0, \quad \sum_{n=2}^{\infty} (\alpha_{nj} + \beta_{nj}) = 1.
\]

Without loss of generality, we can assume that \( \{\alpha_{nj}\} \) and \( \{\beta_{nj}\} \) tend to \( \alpha_n \) and \( \beta_n \), respectively, for \( n = 2, 3, \ldots \). Hence, we have

\[
c_1 = \sum_{n=2}^{m} (\alpha_n k_n - \beta_n k_n) + \lim_{j} \sum_{n=m+1}^{\infty} (\alpha_{nj}k_n - \beta_{nj}k_n)
\]

for each \( m \geq 2 \). On the other hand,

\[
\bar{x} = \lim_{j} \bar{x}_j = \lim_{j} \sum_{n=2}^{\infty} (\alpha_{nj}l_n + \beta_{nj}l_n)e_n
\]

and, therefore, there exists a constant \( 0 < M_{\bar{x}} < +\infty \) such that

\[
\alpha_{nj}l_n + \beta_{nj}l_n \leq M_{\bar{x}}
\]

for all \( n \geq 2 \) and \( j \in \mathbb{N} \). This implies that

\[
0 \leq \alpha_{nj}k_n + \beta_{nj}k_n = (\alpha_{nj}l_n + \beta_{nj}l_n) \frac{k_n}{l_n} \leq M_{\bar{x}} \frac{k_n}{l_n},
\]

\[
0 \leq (\alpha_n + \beta_n)k_n = \bar{d}_nk_n \leq M_{\bar{x}} \frac{k_n}{l_n}
\]

for all \( j, n \), and finally,

\[
\sup_{j} \left| \sum_{n=m+1}^{\infty} (\alpha_{nj}k_n - \beta_{nj}k_n) \right| \leq \sup_{j} \sum_{n=m+1}^{\infty} (\alpha_{nj}k_n + \beta_{nj}k_n)
\]

\[
\leq \sum_{n=m+1}^{\infty} M_{\bar{x}} \frac{k_n}{l_n} = M_{\bar{x}} \sum_{n=m+1}^{\infty} \frac{k_n}{l_n} \rightarrow 0, \quad m \rightarrow \infty.
\]
Combining (3.13) with (3.17), we conclude that
\[ c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n). \] (3.18)

This completes the proof. \(\square\)

**Theorem 3.2.** The set \( C \) is linearly bounded but is not a block set in any orthogonal basis in \( l^2 \).

**Proof.** First, we show that \( C \) is not a block set in any orthogonal basis, \[ \left\{ \tilde{e}_i \right\}_{i=1}^{\infty} = \left\{ \sum_{n=1}^{\infty} c_{in} e_n \right\}_{i=1}^{\infty} \] (3.19)
in \( l^2 \). Indeed, there exists \( i_0 \) such that \( c_{i_0 1} \neq 0 \). Since we have
\[ \max \left( \left| \langle a_n, \tilde{e}_{i_0} \rangle \right|, \left| \langle b_n, \tilde{e}_{i_0} \rangle \right| \right) = k_n \left| c_{i_0 1} \right| + l_n \left| c_{i_0 n} \right| \] (3.20)
for every \( n \geq 2 \), these two facts imply that
\[ \sup \{ \left| \langle x, \tilde{e}_{i_0} \rangle \right| : x \in C \} = +\infty. \] (3.21)

Therefore, \( C \) is not a block set in \( \{ \tilde{e}_i \}_{i=1}^{\infty} \).

Now, we prove that the set \( C \) is linearly bounded. We begin with the following simple observation:
\[ \sup \{ \left| \langle x, e_n \rangle \right| : x \in C \} \leq l_n \] (3.22)
for \( n = 2, 3, \ldots \). Next, if \( x \in C \) is of the form
\[ x = \sum_{n=1}^{\infty} c_n e_n = c_1 e_1 + \sum_{n=2}^{\infty} d_n l_n e_n = c_1 e_1 + \bar{x}, \] (3.23)
then, by Theorem 3.1, we see that
\[ d_n \geq 0 \] (3.24)
for \( n = 2, 3, \ldots \),
\[ \sum_{n=2}^{\infty} d_n \leq 1, \] (3.25)
A remark on the approximate fixed-point property

and there exist sequences \( \{ \alpha_n \}_{n=2}^{\infty} \) and \( \{ \beta_n \}_{n=2}^{\infty} \) such that

\[
c_1 = \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n), \quad \alpha_n, \beta_n \geq 0, \quad \alpha_n + \beta_n = d_n,
\]

for \( n = 2, 3, \ldots \). Additionally, there exists a positive constant \( M_\bar{X} \) such that

\[
0 \leq (\alpha_n + \beta_n) k_n = d_n k_n \leq M_\bar{X} \sum_{n=2}^{\infty} \frac{k_n}{l_n}
\]

for \( n = 2, 3, \ldots \). Hence, we obtain

\[
|c_1| = \left| \sum_{n=2}^{\infty} (\alpha_n k_n - \beta_n k_n) \right| \leq \sum_{n=2}^{\infty} (\alpha_n + \beta_n) k_n \leq M_\bar{X} \sum_{n=2}^{\infty} \frac{k_n}{l_n}.
\]

Then, it follows from (3.22) and (3.28) that an intersection of \( C \) with any line \( \{ y + tv : t \in \mathbb{R} \} \), where \( y, v \in l^2 \) and \( v \neq 0 \), is either empty or bounded which completes the proof.

\[\square\]

References


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