# COINCIDENCE THEOREMS FOR FAMILIES OF MULTIMAPS AND THEIR APPLICATIONS TO EQUILIBRIUM PROBLEMS

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We apply some continuous selection theorems to establish coincidence theorems for a family of multimaps under various conditions. Then we apply these coincidence theorems to study the equilibrium problem with $m$ families of players and $2m$ families of constraints on strategy sets. We establish the existence theorems of equilibria of this problem and existence theorem of equilibria of abstract economics with two families of players.

## 1. Introduction

For multimaps $F : X \multimap Y$ and $S : Y \multimap X$, a point $(x, y) \in X \times Y$ is called a coincidence point of $F$ and $S$ if $y \in F(x)$ and $x \in S(y)$. In 1937, Neumann [19] and in 1966 Fan [8] established the well-known coincidence theorems. In 1984, Browder [4] combined Kakutani-Fan fixed-point theorem and Fan-Browder fixed-point theorem to obtain a coincidence theorem. So many authors gave some coincidence theorems and applied them in various fields as equilibrium problem, minimax theorem, quasi-variational inequalities, game theory, mathematical economics, and so on, see [1, 5, 6, 9, 11, 18] and references therein. In 1991, Horvath [10, Theorem 3.2] obtained a continuous selection theorem. In [21, Theorem 1], Wu and Shen established another continuous selection theorem.

Let $I$ and $J$ be any index sets. For each $i \in I$ and $j \in J$, let $X_i$ and $Y_j$ be nonempty sets and $H_j : X = \prod_{i\in I} X_i \multimap Y_j$; $T_i : Y = \prod_{j\in J} Y_j \multimap X_i$ be multimaps. A point $(\bar{x}, \bar{y}) \in X \times Y$, where $\bar{x} = (\bar{x}_i)_{i\in I}$ and $\bar{y} = (\bar{y}_j)_{j\in J}$ is called a coincidence point of two families of multimaps, if $\bar{y}_j \in H_j(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$ for each $i \in I$ and $j \in J$. In this paper, we apply the continuous selection theorem of Horvath [10] and the fixed-point theorem of Park [17] to derive the coincidence theorems for two families of multimaps. Our coincidence theorems for two families of multimaps include Fan-Browder fixed-point theorem [3] and Browder coincidence theorem [4] as special cases.

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We will employ our results on coincidence theorems for two families of multimaps to consider the equilibrium problem with $m$ families of players and $2m$ families of constraints on the strategy sets introduced by Lin et al. [14]. We consider the following problem: let $I$ be any index set and for each $k \in I$, let $J_k$ be a finite index set, $X_{k_j}$ denote the strategy set of $j$th player in $k$th family, $Y_k = \prod_{j\in J_k} X_{k_j}$, $Y = \prod_{k\in I} Y_k$, $Y^k = \prod_{l\in I, l\neq k} Y_l$, and $Y = Y^k \times Y_k$. Let $F_{k_j} : X_{k_j} \times Y^k \multimap \mathbb{R}^{l_{k_j}}$ be the payoff of the $j$th player in the $k$th family, let $A_{k_j} : Y^k \multimap X_{k_j}$ be the constraint which restricts the strategy of the $j$th player in the $k$th family to the subset $A_{k_j}(Y^k)$ of $X_{k_j}$ when all players in other families have chosen their strategies $x_{ij}$, $i \in I$, $i \neq k$, and $j \in J_i$, and let $B_k : Y_k \multimap Y^k$ be the constraint which restricts the strategies of all the families except $k$th family to the subset $B_k(Y_k)$ of $Y^k$ when all the players in the $k$ family have chosen their strategies $y_k = (x_{k_j})_{j\in J_k}$, $k \in I$. Our problem is to find a strategies combination $\bar{y} = (\bar{y}_k)_{k\in I} \in \prod_{k\in I} Y_k = Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$, $\bar{y}_k \in A_k(\bar{y}^k)$, $\bar{u}^k \in B_k(\bar{y}_k)$, and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{1.1}$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in A_{k_j}(\bar{y}^k)$, and for all $k \in I$ and $j \in J_k$. In the Nash equilibrium problem, the strategy of each player is subject to no constraint. In the Debreu equilibrium problem, the strategy of each player is subject to a constraint which is a function of the strategies of the other players. For the special case of our problem, if each of the families contain one player, we find that the strategy of each player is subject a constraint which is a function of strategies of the other players, and for each $k \in I$, where $I$ is the index set of players, the strategies combination of the players other than the $k$th player is a function of strategy of $k$th player. Therefore, our problem is different from the Nash equilibrium problem and their generalizations. As we note from Remark 4.8, for each $k \in I$, if $J_k = \{k\}$ be a singleton set, then the above problem reduces to the problem which is different from the Debreu social equilibrium problem [6] and the Nash equilibrium problem [19]. Lin et al. [14] demonstrate the following example of this kind of equilibrium problem in our real life. Let $I = \{1, 2, \ldots, m\}$ denote the index set of the companies. For each $k \in I$, let $J_k = \{1, 2, \ldots, n_k\}$ denote the index set of factories in the $k$th company, $F_{k_j}$ denote the payoff function of the $j$th factory in the $k$th company. We assume that the products between the factories in the same company are different, and the financial systems and management systems are independent between the factories in the same company, while some collections of products are the same and some collections of products are different between different factories in different companies. Therefore, the strategy of the $j$th factory in the $k$th company depends on the strategies of all factories in different companies. The payoff function $F_{k_j}$ of the $j$th factory in the $k$th company depends on its strategy and the strategies of factories in other companies. We also assume that for each $k \in I$, the strategies of the $k$ company influence the strategies of all other companies. With this strategies combination, each factory can choose a collection of products, and from these collection of products, there exists a product that minimizes the loss of each factory. In this type of abstract economic problem with two families of players, the strategy and the preference correspondence of each player in family $A$ depend on the strategies combination of all players in family $B$, but does not depend on the strategies combination of the players in family $A$. The same situation occurs to each player of family $B$. The abstract economic problem studied in the literature, the strategy and preference correspondence of each player depend on the strategies combination of all the players. Therefore, the abstract economic problem, we studied in this paper, is different from the abstract economic studied in the literature. In some economic model with two companies (say $A$ and $B$), the strategy of each factory of company $A$ depend on the strategies combination of factories in company $B$. The same case occurs in company $B$. We can use this example to explain the abstract economic problem we study in this paper. We also apply the coincidence theorems for a family of multimaps to consider the abstract economic problem with two families of players. In this paper, we want to establish the existence theorems of equilibria of constrained equilibrium problems with $m$ families of players and $2m$ families of constraints and existence theorem of equilibria of abstract economic with two families of players.

## 2. Preliminaries

In order to establish our main results, we first give some concepts and notations.

Throughout this paper, all topological spaces are assumed to be Hausdorff. Let $A$ be a nonempty subset of topological vector space (t.v.s.) $X$, we denote by $\mathrm{int}A$ the interior of $A$, by $\bar{A}$ the closure of $A$ in $X$, by $\mathrm{co}A$ the convex hull of $A$, and by $\bar{\mathrm{co}}A$ the closed convex hull of A. Let $X$, $Y$, and $Z$ be nonempty sets. A multimap (or map) $T : X \multimap Y$ is a function from $X$ into the power set of $Y$ and $T^- : Y \multimap X$ is defined by $x \in T^-(y)$ if and only if $y \in T(x)$. Let $B \subset Y$, we define $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$. Given two multimaps $F : X \multimap Y$ and $G : Y \multimap Z$, the composite $GF : X \multimap Z$ is defined by $GF(x) = G(F(x))$ for all $x \in X$.

Let $X$ and $Y$ be two topological spaces, a multimap $T : X \multimap Y$ is said to be compact if there exists a compact subset $K \subset Y$ such that $T(X) \subset K$; to be closed if its graph $Gr(T) = \{(x, y) \mid x \in X,\ y \in T(x)\}$ is closed in $X \times Y$; to have local intersection property if, for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of $x$ such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$; to be upper semicontinuous (u.s.c.) if $T^-(A)$ is closed in $X$ for each closed subset $A$ of $Y$; to be lower semicontinuous (l.s.c.) if $T^-(G)$ is open in $X$ for each open subset $G$ of $Y$, and to be continuous if it is both u.s.c. and l.s.c.

A topological space is said to be *acyclic* if all of its reduced Čech homology groups vanish. In particular, any convex set is *acyclic*.

*Definition 2.1* (see [15]). Let $X$ be a nonempty convex subset of a t.v.s. $E$. A multimap $G : X \multimap \mathbb{R}$ is said to be $\mathbb{R}^+$-*quasiconvex* if, for any $\alpha \in \mathbb{R}$, the set

$$\{x \in X : \text{there is a } y \in G(x) \text{ such that } \alpha - y \geq 0\} \tag{2.1}$$

is convex.

*Definition 2.2* (see [15]). Let $Z$ be a real t.v.s. with a convex solid cone $C$ and $A$ be a nonempty subset of $Z$. A point $\bar{y} \in A$ is called a weak vector minimal point of $A$ if, for any $y \in A$, $y - \bar{y} \notin -\operatorname{int} C$. Moreover, the set of weak vector minimal points of $A$ is denoted by $\operatorname{wMin}_C A$.

Lemma 2.3. *Let $I$ be any index set and $\{E_i\}_{i\in I}$ be a family of locally convex t.v.s. For each $i \in I$, let $X_i$ be a nonempty convex subset of $E_i$, $F_i$, $H_i$ such that $X := \prod_{i\in I} X_i \multimap X_i$ be multimaps satisfying the following conditions:*

(i) *for each $x \in X$, $\operatorname{co} F_i(x) \subset H_i(x)$;*
(ii) $X = \bigcup\{\operatorname{int} F_i^-(x_i) : x_i \in X_i\}$;
(iii) *$H_i$ is compact.*

*Then there exists a point $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x} \in H(\bar{x}) := \prod_{i\in I} H_i(\bar{x})$; that is, $\bar{x}_i \in H_i(\bar{x})$ for each $i \in I$.*

*Proof.* Lemma 2.3 follows immediately from [12, Proposition 1] and [21, Theorem 2]. □

## 3. Coincidence theorems for families of multivalued maps

Theorem 3.1. *Let $I$ and $J$ be any index sets, and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be families of locally convex t.v.s. For each $i \in I$ and $j \in J$, let $X_i$ and $Y_j$ be nonempty convex subsets, each in $U_i$ and $V_j$, respectively. Let $F_j$, $H_j : X := \prod_{i\in I} X_i \multimap Y_j$; $S_i$, $T_i : Y := \prod_{j\in J} Y_j \multimap X_i$ be multimaps satisfying the following conditions:*

(i) *for each $x \in X$, $\operatorname{co} F_j(x) \subset H_j(x)$;*
(ii) $X = \bigcup\{\operatorname{int} F_j^{-}(y_j) : y_j \in Y_j\}$;
(iii) *for each $y \in Y$, $\operatorname{co} S_i(y) \subset T_i(y)$;*
(iv) $Y = \bigcup\{\operatorname{int} S_i^{-}(x_i) : x_i \in X_i\}$;
(v) *$T_i$ is compact.*

*Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ such that $\bar{y}_j \in H_j(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$ for each $i \in I$ and $j \in J$.*

*Proof.* Since for each $i \in I$, $T_i$ is compact, there exists a compact subset $D_i \subset X_i$ such that $T_i(Y) \subset D_i$ for each $i \in I$. Let $D = \prod_{i\in I} D_i$ and $K = \operatorname{co} D$, it follows from [7, Lemma 1] that $K$ is a nonempty paracompact convex subset in $X$. For each $i \in I$, let $K_i$ be the $i$th projection of $K$. By assumption (iii), $S_i$, $T_i : Y \multimap K_i$ and $\operatorname{co} S_i(y) \subset T_i(y)$ for each $y \in Y$. By (i), for each $x \in K$, $\operatorname{co} F_j |_K (x) \subset H_j |_K (x)$. By (ii), $K = \bigcup\{\operatorname{int}_K F_j^{-}(y_j) : y_j \in Y_j\}$.

By [12, Proposition 1] and [21, Theorem 1], $H_j |_K (x)$ has a continuous selection, that is, for each $j \in J$, there exists a continuous function $f_j : K \to Y_j$ such that $f_j(x) \in H_j(x)$ for all $x \in K$. Let $f : K \to Y$ be defined by $f(x) = \prod_{j\in J} f_j(x)$ and $P_i$, $W_i : K \multimap K_i$ be defined by $W_i(x) = S_i(f(x))$ and $P_i(x) = T_i(f(x))$ for all $x \in K$. It is easy to see that $W_i^-(x_i) = f^{-1}(S_i^-(x_i))$ for all $x_i \in K_i$ and for all $i \in I$. By assumption (iii), for all $i \in I$ and for all $x \in X$, $\operatorname{co} W_i(x) = \operatorname{co} S_i(f(x)) \subset T_i(f(x)) = P_i(x)$. Since $S_i(Y) \subset T_i(Y) \subset K_i$, it follows from assumption (iv) and the continuity of $f$ that

$$\begin{aligned} K = f^{-1}(Y) &= f^{-1}\Big[\bigcup\{\operatorname{int} S_i^-(x_i) : x_i \in X_i\}\Big] \\ &= f^{-1}\Big[\bigcup\{\operatorname{int} S_i^-(x_i) : x_i \in K_i\}\Big] \\ &\subset \bigcup\{\operatorname{int} f^{-1}(S_i^-(x_i)) : x_i \in K_i\} \\ &= \bigcup\{\operatorname{int} W_i^-(x_i) : x_i \in K_i\} \subset K. \end{aligned} \tag{3.1}$$

Hence,

$$K = \bigcup\{\operatorname{int} W_i^-(x_i) : x_i \in K_i\}. \tag{3.2}$$

Then by Lemma 2.3, there exists a point $\bar{x} = (\bar{x}_i)_{i\in I} \in K \subset X$ such that $\bar{x}_i \in P_i(\bar{x}) = T_i(f(\bar{x}))$ for each $i \in I$. Let $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ such that $\bar{y} = f(\bar{x})$, then, for each $i \in I$ and $j \in J$, $\bar{y}_j = f_j(\bar{x}) \in H_j(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$. The proof is complete. □

Corollary 3.2 [22, Theorem 8]. *In Theorem 3.1, if the condition (v) is replaced by (v′), then the conclusion is still true, where (v′) $X_i$ is compact.*

*Proof.* Since $X_i$ is compact, $T_i$ is compact and the conclusion of Corollary 3.2 follows from Theorem 3.1. □

*Remark 3.3.* Theorem 3.1 improves [15, Theorem 8].

Theorem 3.4. *Let $I$ and $J$ be any index sets, let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be families of locally convex t.v.s. For each $i \in I$ and $j \in J$, let $X_i$ and $Y_j$ be nonempty convex subsets of $U_i$ and $V_j$, respectively, and let $D_j$ be a nonempty compact metrizable subset of $Y_j$. For each $i \in I$ and $j \in J$, let $S_j, T_j : X := \prod_{i\in I} X_i \multimap D_j$; $F_i, H_i : Y := \prod_{j\in J} Y_j \multimap X_i$ be multimaps satisfying the following conditions:*

(i) *for each $x \in X$, $\overline{\operatorname{co}} S_j(x) \subset T_j(x)$ and $S_j(x) \neq \emptyset$;*
(ii) *$S_j$ is l.s.c.;*
(iii) *for each $y \in Y$, $\operatorname{co} F_i(y) \subset H_i(y)$;*
(iv) *$Y = \bigcup\{\operatorname{int} F_i^-(x_i) : x_i \in X_i\}$;*
(v) *$H_i$ is compact.*

*Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in D := \prod_{j\in J} D_j$ such that $\bar{y}_j \in T_j(\bar{x})$ and $\bar{x}_i \in H_i(\bar{y})$ for each $i \in I$ and $j \in J$.*

*Proof.* Since for each $i \in I$, $H_i$ is compact, there exists a compact subset $C_i \subset X_i$ such that $H_i(Y) \subset C_i$ for each $i \in I$. Let $C = \prod_{i\in I} C_i$ and $D := \prod_{j\in J} D_j$, then $\operatorname{co} C$ and $K = \operatorname{co} D$ are nonempty paracompact convex subsets each in $X$ and $Y$, respectively by [7, Lemma 1]. By assumptions (i), (ii), and following the same argument as in the proof of [20, Theorem 1], there exists an u.s.c. multimap $P_j : \mathrm{co}\, C \multimap D_j$ with nonempty compact convex values such that $P_j(x) \subset T_j(x)$ for all $x \in \mathrm{co}\, C$. Define $P : \mathrm{co}\, C \multimap D$ by $P(x) = \prod_{j\in J} P_j(x)$ for all $x \in \mathrm{co}\, C$. Then it follows from [8, Lemma 3] that $P$ is an u.s.c. multimap with nonempty compact convex values.

By [12, Proposition 1] and [21, Theorem 1], for each $i \in I$, $H_i\,|_K\,(y)$ has a continuous selection $f_i : K \to C_i$ such that $f_i(y) \in H_i(y)$ for all $y \in K$. Let $f : K \to C$ be defined by $f(y) = \prod_{i\in I} f_i(y)$ for all $y \in K$, and let $W : K \multimap D$ be defined by $W(y) = P\,|_C\,(f(y))$ for all $y \in K$. It is easy to see that $W$ is an u.s.c. multimap with nonempty closed convex values. Then, by [17, Theorem 7], there exists $\bar{y} \in D$ such that $\bar{y} \in W(\bar{y}) = P\,|_C\,(f(\bar{y}))$. Let $\bar{x} \in C$ such that $\bar{x} = f(\bar{y})$ and $\bar{y} \in P\,|_C\,(\bar{x})$, then for each $i \in I$ and $j \in J$, $\bar{y}_j \in P_j(\bar{x}) \subset T_j(\bar{x})$ and $\bar{x}_i = f_i(\bar{y}) \in H_i(\bar{y})$. □

Theorem 3.5. *Let $I$ and $J$ be any index sets. For each $i \in I$ and $j \in J$, let $X_i$ and $Y_j$ be nonempty convex subsets in locally convex t.v.s. $U_i$ and $V_j$, respectively, let $D_j$ be a nonempty compact metrizable subset of $Y_j$, and let $C_i$ be a nonempty compact metrizable subset of $X_i$. For each $i \in I$ and $j \in J$, let $S_j$, $T_j : X := \prod_{i\in I} X_i \multimap D_j$; $F_i$, $H_i : Y := \prod_{j\in J} Y_j \multimap C_i$ be multimaps satisfying the following conditions:*

(i) *for each $x \in X$, $\overline{\mathrm{co}} S_j(x) \subset T_j(x)$ and $S_j(x) \neq \emptyset$;*

(ii) *$S_j$ is l.s.c.;*

(iii) *for each $y \in Y$, $\overline{\mathrm{co}} F_i(y) \subset H_i(y)$ and $F_i(y) \neq \emptyset$;*

(iv) *$F_i$ is l.s.c.*

*Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in \mathrm{co}\, C := \mathrm{co} \prod_{i\in I} C_i$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in \mathrm{co}\, D := \mathrm{co} \prod_{j\in J} D_j$ such that $\bar{y}_j \in T_j(\bar{x})$ and $\bar{x}_i \in H_i(\bar{y})$ for each $i \in I$ and $j \in J$.*

*Proof.* Following the same argument as in the proof of [20, Theorem 1], for each $i \in I$ and $j \in J$, there are two u.s.c. multimaps $P_j : \mathrm{co}\, C \multimap D_j$ and $Q_i : \mathrm{co}\, D \multimap C_i$ with nonempty closed convex values such that $P_j(x) \subset T_j(x)$ for all $x \in \mathrm{co}\, C$, and $Q_i(y) \subset H_i(y)$ for all $y \in \mathrm{co}\, D$. Define $P : \mathrm{co}\, C \multimap D$, $Q : \mathrm{co}\, D \multimap C$ by $P(x) = \prod_{j\in J} P_j(x)$ for all $x \in \mathrm{co}\, C$, and $Q(y) = \prod_{i\in I} Q_i(y)$ for all $y \in \mathrm{co}\, D$. By [8, Lemma 3], $P$ and $Q$ both are u.s.c. multimaps with nonempty closed convex values. Let $W : \mathrm{co}\, C \times \mathrm{co}\, D \multimap C \times D$ be defined by $W(x, y) = (Q(y), P(x))$ for $(x, y) \in (\mathrm{co}\, C) \times (\mathrm{co}\, D)$. It is easy to see that $W$ is an u.s.c. multimap with nonempty closed convex values. Therefore, by [17, Theorem 7], there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in \mathrm{co}\, C$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in \mathrm{co}\, D$ such that $(\bar{x}, \bar{y}) \in W(\bar{x}, \bar{y})$. Then for each $i \in I$ and $j \in J$, $\bar{y}_j \in P_j(\bar{x}) \subset T_j(\bar{x})$ and $\bar{x}_i \in Q_i(\bar{y}) \subset H_i(\bar{y})$. □

*Remark 3.6.* In Theorem 3.1, we do not assume that $\{X_i\}$ and $\{Y_j\}$ are metrizable for each $i \in I$ and $j \in J$.

Theorem 3.7. *Let $I$ and $J$ be finite index sets, let $\{U_i\}_{i\in I}$ be a family of t.v.s., and let $\{V_j\}_{j\in J}$ be a family of locally convex t.v.s. For each $i \in I$ and $j \in J$, let $X_i$ be a nonempty convex subset of $U_i$, let $Y_j$ be a nonempty compact convex subset of $V_j$,* *and let* $F_j : X := \prod_{i\in I} X_i \multimap Y_j$ *and let* $G_i : Y := \prod_{j\in J} Y_j \multimap X_i$ *be two multimaps satisfying the following conditions:*

(i) $F_j$ *is an u.s.c. multimap with nonempty closed acyclic values;*
(ii) $Y = \bigcup\{\operatorname{int} G_i^{-}(x_i) : x_i \in X_i\}$ *and* $G_i(y)$ *is convex for all* $y \in Y$.

*Then there exist* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *and* $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ *such that* $\bar{y}_j \in F_j(\bar{x})$ *and* $\bar{x}_i \in G_i(\bar{y})$ *for each* $i \in I$ *and* $j \in J$.

*Proof.* By [12, Proposition 1] and [21, Theorem 1], $G_i$ has a continuous selection $g_i : Y \to X_i$ such that $g_i(y) \in G_i(y)$ for all $y \in Y$. Let $g : Y \to X$ be defined by $g(y) = \prod_{i\in I} g_i(y)$ for all $y \in Y$, then $g$ is also continuous. Define $P_j : Y \multimap Y_j$ and $P : Y \multimap Y$ by $P_j(y) = F_j(g(y))$, and $P(y) = \prod_{j\in J} P_j(y)$ for all $y \in Y$, then $P_j : Y \multimap Y_j$ is an u.s.c. multimap with nonempty closed acyclic values. By Kunneth formula (see [16] and [8, Lemma 3]), $P : Y \multimap Y$ is also an u.s.c. multimap with nonempty closed acyclic values. Therefore, by [17, Theorem 7], there exists $\bar{y} \in Y$ such that $\bar{y} \in P(\bar{y}) = \prod_{j\in J} P_j(\bar{y}) = \prod_{j\in J} F_j(g(\bar{y}))$. Let $\bar{x} = g(\bar{y})$ such that $\bar{y}_i \in F_i(\bar{x})$, then $\bar{x}_i = g_i(\bar{y}) \in G_i(\bar{y})$ and $\bar{y}_j \in F_j(\bar{x})$ for all $i \in I$ and $j \in J$. □

*Remark 3.8.* (i) In particular, if $I = J$ is a singleton, $X = Y$, $E = V$, and $F = I_X$, the identity mapping on $X$ in the above theorem, then we can obtain well-known Browder fixed-point theorem [3].

(ii) If $I = J$ is a singleton, $F$ is an u.s.c. multimap with nonempty closed convex values, $G(y)$ is nonempty for all $y \in Y$, and $G^{-}(x)$ is open for all $x \in X$, then Theorem 3.7 reduces to Browder coincidence theorem [4].

THEOREM 3.9. *Let* $I$ *and* $J$ *be finite index sets,* $\{U_i\}_{i\in I}$ *and* $\{V_j\}_{j\in J}$ *be families of locally convex t.v.s. For each* $i \in I$ *and* $j \in J$*, let* $X_i$ *be a nonempty convex metrizable compact subsets of* $U_i$ *and let* $Y_j$ *be a nonempty compact convex subsets of* $V_j$*. For each* $i \in I$ *and* $j \in J$*, let* $F_j : X := \prod_{i\in I} X_i \multimap Y_j$ *and* $G_i : Y := \prod_{j\in J} Y_j \multimap X_i$ *be two multimaps satisfying the following conditions:*

(i) $F_j$ *is an u.s.c. multimap with nonempty closed acyclic values;*
(ii) $G_i$ *is a l.s.c. multimap with nonempty closed convex values.*

*Then there exist* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *and* $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ *such that* $\bar{y}_j \in F_j(\bar{x})$ *and* $\bar{x}_i \in G_i(\bar{y})$ *for each* $i \in I$ *and* $j \in J$.

*Proof.* Since $Y_j$ is compact for each $j \in J$, $Y = \prod_{j\in J} Y_j$ is compact. By assumption (ii) and following the same argument as in the proof of [20, Theorem 1], for each $i \in I$, there exists an u.s.c. multimap $\varphi_i : Y \multimap X_i$ with nonempty closed convex values such that $\varphi_i(y) \subset G_i(y)$ for all $y \in Y$. Let $\varphi : Y \multimap X$ defined by $\varphi(y) = \prod_{i\in I} \varphi_i(y)$ for all $y \in Y$, then $\varphi$ is u.s.c. with nonempty convex compact values. Define a multimap $F : X \multimap Y$ by $F(x) = \prod_{j\in J} F_j(x)$ for all $x \in X$, then $F : X \multimap Y$ is also an u.s.c. multimap with nonempty closed acyclic values.

Define a multimap $W : X \times Y \multimap X \times Y$ by $W(x, y) = (\varphi(y), F(x))$ for all $x \in X$ and for all $y \in Y$. It is easy to see that $W$ is a compact u.s.c. multimap with nonempty closed acyclic values. It follows from [17, Theorem 7] that there exists

$(\bar{x}, \bar{y}) \in X \times Y$ such that $x \in \varphi(\bar{y})$ and $\bar{y} \in F(\bar{x})$. Therefore, $\bar{x}_i \in G_i(\bar{y})$ and $\bar{y}_j \in F_j(\bar{x})$ for each $i \in I$ and $j \in J$. □

*Remark 3.10.* In Theorem 3.5, if $F_j$ is an u.s.c. multimap with nonempty closed convex values for each $j \in J$, then $J$ may be any index set.

Let $I$ be any index set and, for each $i \in I$, let $\{U_i\}_{i\in I}$ be a family of locally convex t.v.s. For each $i \in I$, let $X_i$ be a nonempty convex subset in t.v.s. $E_i$ for each $i \in I$. Let $X = \prod_{i\in I} X_i$, $X^i = \prod_{j\in I, j\neq i} X_j$ and we write $X = X^i \times X_i$. For each $x \in X$, $x_i \in X_i$ denotes the $i$th coordinate and $x^i \in X^i$ the projection of $x$ onto $X^i$, and we also write $x = (x^i, x_i)$.

Theorem 3.11. *Let $I$ be a finite index set and let $\{U_i\}_{i\in I}$ be a family of locally convex t.v.s. For each $i \in I$, let $X_i$ be a nonempty compact convex subset of $U_i$ and let $F_i : X^i \multimap X_i$ and $G_i : X_i \multimap X^i$ be multimaps satisfying the following conditions:*

(i) *$F_i$ is an u.s.c. multimap with nonempty closed acyclic values;*
(ii) *$X_i = \bigcup\{\mathrm{int}_{X_i} G_i^-(x^i) : x^i \in X^i\}$ and $G_i(x_i)$ is convex for all $x_i \in X_i$.*

*Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{u} = (\bar{u}_i)_{i\in I} \in X$ such that $\bar{x}_i \in F_i(\bar{u}^i)$ and $\bar{u}^i \in G_i(\bar{x}_i)$ for all $i \in I$.*

*Proof.* Since $X_i = \bigcup\{\mathrm{int}_{X_i} G_i^-(x^i) : x^i \in X^i\}$, by [12, Proposition 1] and [21, Theorem 1], $G_i$ has a continuous selection $g_i : X_i \to X^i$.

Define multimaps $P_i : X \multimap X_i$ and $P : X \multimap X$ by $P_i(x) = F_i(g_i(x_i))$ and $P(x) = \prod_{i\in I} P_i(x)$ for all $x = (x_i)_{i\in I} \in X$, then $P_i$ is an u.s.c. multimap with nonempty closed acyclic values for all $i \in I$. Therefore, $P : X \multimap X$ is also an u.s.c. multimap with nonempty closed acyclic values. Since $X$ is a nonempty compact convex subset in a locally convex t.v.s. $E = \prod_{i\in I} E_i$, it follows from [17, Theorem 7] that there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x} \in P(\bar{x}) = \prod_{i\in I} P_i(\bar{x}) = \prod_{i\in I} F_i(g_i(\bar{x}_i))$, that is, for all $i \in I$, $\bar{x}_i \in F_i(g_i(\bar{x}_i))$. For all $i \in I$, let $\bar{u}^i = g_i(\bar{x}_i)$, then $\bar{u}^i \in X^i$. Hence, $\bar{u}^i = g_i(\bar{x}_i) \in G_i(\bar{x}_i)$ and $\bar{x}_i \in F_i(\bar{u}^i)$ for all $i \in I$. □

*Remarks 3.12.* (i) In Theorem 3.11, if $F_i$ is an u.s.c. multimap with nonempty closed convex values for each $i \in I$, then $I$ may be any index set.

(ii) The proofs and conditions between Theorem 3.7 and Theorem 3.11 are somewhat different.

## 4. Applications of coincidence theorem for families of multimaps to equilibrium problems

In this section, we establish the existence theorem of equilibrium problem with $m$ families of players and $2m$ families of constraints on strategy sets which has been introduced by Lin et al. [14].

Let $I$ be a finite index set and for each $k \in I$ and $j \in J_k$, let $X_{k_j}$, $Y_k$, $Y^k$, $Y$, $F_{k_j}$, and $A_{k_j}$ be the same as in introduction. For each $k \in I$ and $j \in J_k$, let $W_{k_j} \in \mathbb{R}_+^{l_{k_j}} \setminus \{0\}$ and $W_k = \prod_{j\in J_k} W_{k_j}$. For each $k \in I$, let $A_k : Y^k \multimap Y_k$ be defined by

$A_k(y^k) = \prod_{j\in J_k} A_{k_j}(y^k)$ where $y_k = (x_{k_j})_{j\in J_k} \in Y_k$ and $y^k = (y_l)_{l\in I, l\neq k} \in Y^k$. Let $S^{W_k} : Y_k \times Y^k \multimap \mathbb{R}$ be defined by

$$\begin{aligned} S^{W_k}(y_k, y^k) &= \sum_{j\in J_k} W_{k_j} \cdot F_{k_j}(x_{k_j}, y^k) \\ &= \left\{ u : u = \sum_{j\in J_k} W_{k_j} \cdot z_{k_j},\ \text{for } z_{k_j} \in F_{k_j}(x_{k_j}, y^k) \right\}. \end{aligned} \tag{4.1}$$

For each $k \in I$, let $M^{W_k} : Y^k \multimap Y_k$ be defined by

$$\begin{aligned} M^{W_k}(y^k) &= \{y_k \in A_k(y^k) : \inf S^{W_k}(y_k, y^k) = \inf S^{W_k}(A_k(y^k), y^k)\}, \\ M(y) &= \prod_{k\in I} M^{W_k}(y^k). \end{aligned} \tag{4.2}$$

Throughout the paper, we will use the above mentioned notations, unless otherwise specified.

Theorem 4.1. *For each $k \in I$ and $j \in J_k$, let $X_{k_j}$ be a nonempty compact convex subset of a locally convex t.v.s. $E_{k_j}$ satisfying the following conditions:*

(i) *the multimap $S^{W_k}$ is continuous with compact values;*

(ii) *$A_{k_j} : Y^k \multimap X_{k_j}$ is a continuous multimap with nonempty closed convex values;*

(iii) *for all $y^k \in Y^k$, $M^{W_k}(y^k)$ is an acyclic set; and*

(iv) *for each $k \in I$, let $B_k : Y_k \multimap Y^k$ be a multimap with convex values and $Y_k = \bigcup\{\operatorname{int}_{Y_k} B_k^{-1}(y^k) : y^k \in Y^k\}$.*

*Then there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k \in A_k(\bar{u}^k)$, $\bar{u}^k \in B_k(\bar{y}_k)$, and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.3}$$

*for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in A_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$.*

*Proof.* By assumptions (i), (ii), (iii), and following the same argument as in the proof of [13, Theorem 3.2], we show that $M^{W_k} : Y^k \multimap Y_k$ is an u.s.c. multimap with nonempty compact acyclic values. Since $B_k(y_k)$ is convex for each $k \in I$, $y_k \in Y_k$, and $Y_k = \bigcup\{\operatorname{int}_{Y_k} B_k^{-1}(y^k) : y^k \in Y^k\}$, it follows from Theorem 3.11 that there exist $(\bar{u}_k)_{k\in I} \in Y$ and $(\bar{y}_k)_{k\in I} \in Y$ such that $\bar{y}_k \in M^{W_k}(\bar{u}^k)$ and $\bar{u}^k \in B_k(\bar{y}_k)$ for all $k \in I$. Therefore, for each $k \in I$, $\bar{y}_k \in A_k(\bar{u}^k)$, and $\min S^{W_k}(\bar{y}_k, \bar{u}^k) = \min S^{W_k}(A_k(\bar{u}^k), \bar{u}^k) \leq \min S^{W_k}(p_k, \bar{u}^k)$ for all $p_k \in A_k(\bar{u}^k)$. Let $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in A_k(\bar{u}^k)$. Following the same argument as in the proof of [13, Theorem 3.1], then we show Theorem 4.1. □

*Remark 4.2.* It is easy to see that the conclusion of Theorem 4.1 remains true if condition (iii) is replaced by (iii′) for each $k \in I$ and for each fixed $y^k \in Y^k$, the multimap $u_k \multimap S^{W_k}(u_k, y^k)$ is $\mathbb{R}_+$-quasiconvex.

As a simple consequence of Theorem 4.1, we give a simple proof of the following corollaries.

Corollary 4.3 ([14]). *For each $k \in I$ and $j \in J_k$, let $X_{k_j}$ be a nonempty compact convex subset of a locally convex t.v.s. $E_{k_j}$, let $A_{k_j} : Y^k \multimap X_{k_j}$ be a multimap with nonempty convex values such that for each $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ is open in $Y^k$ and $\bar{A}_{k_j}$ is an u.s.c. multimap. For each $k \in I$, let $B_k : Y_k \multimap Y^k$ be a multimap with convex values satisfying the following conditions:*

(i) $Y_k = \bigcup\{\mathrm{int}_{Y_k} B_k^{-1}(y^k) : y^k \in Y^k\}$;
(ii) $S^{W_k} : Y_k \times Y^k \multimap \mathbb{R}$ *is a continuous multimap with compact values;*
(iii) *for any $y^k \in Y^k$, $u_k \multimap S^{W_k}(u_k, y^k)$ is $\mathbb{R}_+$-quasiconvex.*

*Then there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k \in \bar{A}_k(\bar{u}^k)$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k}$, $\bar{u}^k \in B_k(\bar{y}_k)$, and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\mathrm{int}\,\mathbb{R}_+^{l_{k_j}}, \tag{4.4}$$

*for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$.*

*Proof.* Since $A_{k_j}^{-1}(x_{k_j})$ is open in $Y^k$ for each $x_{k_j} \in X_{k_j}$, $A_{k_j}$ is a l.s.c. multimap. It is easy to see that $\bar{A}_{k_j}$ is also a l.s.c. multimap. By assumption, $\bar{A}_{k_j}$ is an u.s.c. multimap. Therefore, $\bar{A}_{k_j}$ is a continuous multimap with nonempty convex closed values. Let $\bar{A}_k = \prod_{j\in J_k} \bar{A}_{k_j}$. Applying assumption (iii) and following the same argument as in [13, Theorem 3.3], it is easy to see that $H^{W_k}(y^k)$ is an convex set for all $y^k \in Y^k$, where

$$H^{W_k}(y^k) = \{y_k \in \bar{A}_k(y^k) : \inf S^{W_k}(y_k, y^k) = \inf S^{W_k}(\bar{A}_k(y^k), y^k)\}. \tag{4.5}$$

By Theorem 4.1, there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u}=(\bar{u}_k)_{k\in I} \in Y$ with $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in \bar{A}_k(\bar{u}^k)$, and $\bar{u}^k \in B_k(\bar{y}_k)$, $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\mathrm{int}\,\mathbb{R}_+^{l_{k_j}} \tag{4.6}$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$. □

Corollary 4.4. *For each $k \in I$ and $j \in J_k$, let $X_{k_j}$ be a nonempty compact convex subset of a locally convex t.v.s. $E_{k_j}$ satisfying the following conditions:*

(i) *$S^{W_k}$ is a continuous multimap with compact values;*
(ii) *for each $j \in J_k$, $A_{k_j} : Y^k \multimap X_{k_j}$ is a continuous multimap with nonempty closed convex values;*
(iii) *for all $y^k \in Y^k$, $M^{W_k}(y^k)$ is an acyclic set.*

*Then there exist* $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in A_k(\bar{u}^k)$, *and* $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ *such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.7}$$

*for all* $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ *and for all* $k \in I$, $j \in J_k$.

*Proof.* For each $k \in I$, let $B_k : Y_k \multimap Y^k$ be defined by $B_k(y_k) = Y^k$, then all the conditions of Theorem 4.1 are satisfied. Therefore, there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in \bar{A}_k(\bar{u}^k)$, $\bar{u}^k \in B_k(\bar{y}_k)$, and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.8}$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$. □

Theorem 4.5. *For each* $k \in I$ *and* $j \in J_k$, *let* $X_{k_j}$ *be a nonempty compact convex subset of a locally convex t.v.s.* $E_{k_j}$ *satisfying the following conditions:*

(i) $F_{k_j}$ *is a continuous multimap with nonempty closed values;*
(ii) $A_{k_j} : Y^k \multimap X_{k_j}$ *is a continuous multimap with nonempty closed values;*
(iii) *for all* $y^k \in Y^k$, $\{x_{k_j} \in A_{k_j}(y^k) : F_{k_j}(x_{k_j}, y^k) \cap \operatorname{wMin} F_{k_j}(A_{k_j}(y^k), y^k) \neq \varnothing\}$ *is an acyclic set;*
(iv) *for each* $k \in I$, $B_k : Y_k \multimap Y^k$ *is a multimap with convex values and* $Y_k = \bigcup\{\operatorname{int}_{Y_k} B_k^{-1}(y^k) : y^k \in Y^k\}$.

*Then there exist* $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in \bar{A}_k(\bar{u}^k)$, $\bar{u}^k \in B_k(\bar{y}_k)$, *and* $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ *such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.9}$$

*for all* $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in A_{k_j}(\bar{u}^k)$ *and for all* $k \in I$, $j \in J_k$.

*Proof.* Define a multimap $M_{k_j} : Y^k \multimap X_{k_j}$ by $M_{k_j}(y^k) = \{x_{k_j} \in A_{k_j}(y^k) : F_{k_j}(x_{k_j}, y^k) \cap \operatorname{wMin} F_{k_j}(A_{k_j}(y^k), y^k) \neq \varnothing\}$ for all $y^k \in Y^k$ and for each $k \in I$ and $j \in J_k$. Since $F_{k_j}$ is a continuous multimap and $X_{k_j}$ is compact for each $k \in I$ and $j \in J_k$, it follows from [2, Proposition 3, page 42] that $F_{k_j}$ is a compact continuous multimap with closed values. By [2, Proposition 2, page 41], $M_{k_j}$ is a closed compact u.s.c. multimap for each $k \in I$ and $j \in J_k$. Moreover, for each $k \in I$ and $j \in J_k$, $M_{k_j}$ is an u.s.c. multimap with compact acyclic values. Define a multimap $M_k : Y^k \multimap Y_k$ by $M_k(y^k) = \prod_{j\in J_k} M_{k_j}(y^k)$ for all $y^k \in Y^k$ and for each $k \in I$. Then $M_k$ is also an u.s.c. multimap with compact acyclic values for each $k \in I$.

By Theorem 3.11, there exist $(\bar{u}_k)_{k\in I} \in Y$ and $(\bar{y}_k)_{k\in I} \in Y$ such that $\bar{y}_k \in M_k(\bar{u}^k)$ and $\bar{u}^k \in B_k(\bar{y}_k)$ for all $k \in I$. Let $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in M_k(\bar{u}^k) = \prod_{j\in J_k} M_{k_j}(\bar{u}^k)$, then $\bar{x}_{k_j} \in M_{k_j}(\bar{u}^k)$ for each $k \in I$ and $j \in J_k$. This implies, there exists $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.10}$$

for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in A_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$. □

Applying Theorem 4.5 and following the same argument as in the proof of Corollary 4.3, we have the following corollary.

Corollary 4.6. *For each $k \in I$ and $j \in J_k$, let $X_{k_j}$ be a nonempty compact convex subset of a locally convex t.v.s. $E_{k_j}$ satisfying the following conditions:*

(i) *$F_{k_j}$ is a continuous multimap with nonempty closed values;*
(ii) *$A_{k_j} : Y^k \multimap X_{k_j}$ is a multimap with nonempty convex values such that for each $x_{k_j} \in X_{k_j}$, $A_{k_j}^{-1}(x_{k_j})$ is open in $Y^k$ and $\bar{A}_{k_j}$ is an u.s.c. multimap;*
(iii) *for all $y^k \in Y^k$, $\{x_{k_j} \in A_{k_j}(y^k) : F_{k_j}(x_{k_j}, y^k) \cap \operatorname{wMin} F_{k_j}(A_{k_j}(y^k), y^k) \neq \emptyset\}$ is an acyclic set; and*
(iv) *for each $k \in I$, let $B_k : Y_k \multimap Y^k$ be a multimap with convex values and $Y_k = \bigcup\{\operatorname{int}_{Y_k} B_k^{-1}(y^k) : y^k \in Y^k\}$.*

*Then there exist $\bar{y} = (\bar{y}_k)_{k\in I} \in Y$, $\bar{u} = (\bar{u}_k)_{k\in I} \in Y$, $\bar{y}_k = (\bar{x}_{k_j})_{j\in J_k} \in \bar{A}_k(\bar{u}^k)$, $\bar{u}^k \in B_k(\bar{y}_k)$, and $\bar{z}_{k_j} \in F_{k_j}(\bar{x}_{k_j}, \bar{u}^k)$ such that*

$$z_{k_j} - \bar{z}_{k_j} \notin -\operatorname{int}\mathbb{R}_+^{l_{k_j}}, \tag{4.11}$$

*for all $z_{k_j} \in F_{k_j}(x_{k_j}, \bar{u}^k)$, $x_{k_j} \in \bar{A}_{k_j}(\bar{u}^k)$ and for all $k \in I$, $j \in J_k$.*

Corollary 4.7. *Let $I$ be a finite index set. For each $k \in I$, let $X_k$ be a nonempty compact convex subset of a locally convex t.v.s. and let $f_k : X \to \mathbb{R}$ be a continuous function, and for each $x^k \in X^k$, the function $x_k \to f_k(x_k, x^k)$ is quasiconvex. Then there exist $\bar{x} = (\bar{x}_k)_{k\in I} \in X$ and $\bar{y} = (\bar{y}_k)_{k\in I} \in X$,*

$$f_k(x_k, \bar{y}^k) \geq f_k(\bar{x}_k, \bar{y}^k), \tag{4.12}$$

*for all $x_k \in X_k$ and for all $k \in I$.*

*Proof.* For each $k \in I$, $J_k$ is a singleton. By assumption, $\{x_k \in X_k : f_k(x_k, y^k) = \operatorname{Min} f_k(X_k, y^k)\}$ is a convex set. The conclusion of Corollary 4.7 follows from Corollary 4.6 by taking $A_k(x^k) = X_k$ and $B_k(x_k) = X^k$ for all $k \in I$. □

*Remark 4.8.* (i) The index $I$ in Corollary 4.7 can be any index set.

(ii) The conclusion between Corollary 4.7 and Nash equilibrium theorem [16] is somewhat different.

## 5. Abstract economics with two families of players

In this section, we consider the following abstract economics with two families of players.

Let $I$ and $J$ be any index sets and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be families of locally convex t.v.s. For each $i \in I$ and $j \in J$, let $X_i$ and $Y_j$ be nonempty convex subsets each in $U_i$ and $V_j$, respectively. Two families of abstract economy $\Gamma = (X_i, A_i, B_i, P_i, Y_j, C_j, D_j, Q_j)$, where $A_i, B_i : Y := \prod_{j\in J} Y_j \multimap X_i$, and $C_j, D_j : X := \prod_{i\in I} X_i \multimap Y_j$ are constraint correspondences, $P_i : Y \multimap X_i$ and $Q_j : X \multimap Y_j$ are preference correspondences. An equilibrium for $\Gamma$ is to find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ such that for each $i \in I$ and $j \in J$, $\bar{x}_i \in B_i(\bar{y})$, $\bar{y}_j \in D_j(\bar{x})$, $A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset$, and $C_j(\bar{x}) \cap Q_j(\bar{x}) = \emptyset$.

With the above notation, we have the following theorem.

Theorem 5.1. *Let $\Gamma = (X_i, A_i, B_i, P_i, Y_j, C_j, D_j, Q_j)_{i\in I, j\in J}$ be two families of abstract economics satisfying the following conditions:*

(i) *for each $i \in I$ and $y \in Y$, $\operatorname{co}(A_i(y)) \subset B_i(y)$ and $A_i(y)$ is nonempty;*
(ii) *for each $j \in J$ and $x \in X$, $\operatorname{co}(C_j(x)) \subset D_j(x)$ and $C_j(x)$ is nonempty;*
(iii) *for each $i \in I$, $Y = \bigcup_{x_i\in X_i} \operatorname{int}_Y[\{(\operatorname{co} P_i)^-(x_i) \cup (Y \setminus H_i)\} \cap A_i^-(x_i)]$, where $H_i = \{y \in Y : A_i(y) \cap P_i(y) \neq \emptyset\}$;*
(iv) *for each $j \in J$, $X = \bigcup_{y_j\in Y_j} \operatorname{int}_X[\{(\operatorname{co} Q_j)^-(y_j) \cup (X \setminus M_j)\} \cap C_j^-(y_j)]$, where $M_j = \{x \in X : C_j(x) \cap Q_j(x) \neq \emptyset\}$;*
(v) *for each $i \in I$, $j \in J$ and each $x = (x_i)_{i\in I} \in X$, $y = (y_j)_{j\in J} \in Y$, $x_i \notin \operatorname{co}(P_i(y))$, and $y_i \notin \operatorname{co} Q_j(x)$;*
(vi) *$D_j$ is compact.*

*Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_j)_{j\in J} \in Y$ such that $\bar{x}_i \in B_i(\bar{y})$, $\bar{y}_j \in D_j(\bar{x})$, $A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset$, and $C_j(\bar{x}) \cap Q_j(\bar{x}) = \emptyset$ for all $i \in I$ and $j \in J$.*

*Proof.* For each $i \in I$ and $j \in J$, we define multivalued maps $S_i, T_i : Y \multimap X_i$ by

$$S_i(y) = \begin{cases} \operatorname{co} P_i(y) \cap A_i(y), & \text{for } y \in H_i, \\ A_i(y), & \text{for } y \in Y \setminus H_i, \end{cases}$$
$$T_i(y) = \begin{cases} \operatorname{co} P_i(y) \cap B_i(y), & \text{for } y \in H_i, \\ B_i(y), & \text{for } y \in Y \setminus H_i, \end{cases} \tag{5.1}$$

and $F_j, G_j : X \multimap Y_i$ by

$$F_j(x) = \begin{cases} \operatorname{co} Q_j(x) \cap C_j(x), & \text{for } x \in M_j, \\ C_j(x), & \text{for } x \in X \setminus M_j, \end{cases}$$
$$G_j(x) = \begin{cases} \operatorname{co} Q_j(x) \cap D_j(x), & \text{for } x \in M_j, \\ D_j(x), & \text{for } x \in X \setminus M_j. \end{cases} \tag{5.2}$$

Then for each $i \in I$, $j \in J$, $x \in X$, $y \in Y$, $\operatorname{co} S_i(y) \subseteq T_i(y)$, $S_i(y)$ is nonempty, and $\operatorname{co} F_j(x) \subseteq G_j(x)$, $F_j(x)$ is nonempty.

For each $i \in I$, $j \in J$, $x \in X$, and $y \in Y$, it is easy to see that

$$S_i^-(x_i) = \left[\left\{(\operatorname{co} P_i)^-(x_i) \cup (Y \setminus H_i)\right\}\right] \cap A_i^-(x_i) \tag{5.3}$$

and $F_j^-(y_j) = [\{(\operatorname{co} Q_j)^-(y_j) \cup (X \setminus M_j)\}] \cap C_j^-(y_j)$. From (iii),

$$Y = \bigcup_{x_i \in X_i} \operatorname{int}_Y \left[\left\{(\operatorname{co} P_i)^-(x_i) \cup (Y \setminus H_i)\right\} \cap A_i^-(x_i)\right] = \bigcup_{x_i \in X_i} \operatorname{int}_Y S_i^-(x_i). \tag{5.4}$$

From (iv),

$$X = \bigcup_{y_j \in Y_j} \operatorname{int}_X \left[\left\{(\operatorname{co} Q_j)^-(y_j) \cup (X \setminus X_j)\right\} \cap C_j^-(y_j)\right] = \bigcup_{y_j \in Y_j} \operatorname{int}_X F_j^-(y_j). \tag{5.5}$$

Then all conditions of Theorem 3.1 are satisfied. It follows from Theorem 3.1, there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} = (\bar{y}_j)_{j \in J} \in Y$ such that $\bar{x}_i \in T_i(\bar{y})$ and $\bar{y}_j \in G_j(\bar{x})$ for all $i \in I$ and $j \in J$. By (v), we have $\bar{x}_i \in B_i(\bar{y})$, $\bar{y}_j \in D_j(\bar{x})$, $A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset$, and $C_j(\bar{x}) \cap Q_j(\bar{x}) = \emptyset$. $\square$

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