THE OPERATOR B^*L FOR THE WAVE EQUATION WITH DIRICHLET CONTROL

I. LASIECKA AND R. TRIGGIANI

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In the case of the wave equation, defined on a sufficiently smooth bounded domain of arbitrary dimension, and subject to Dirichlet boundary control, the operator B^*L from boundary to boundary is bounded in the L_2-sense. The proof combines hyperbolic differential energy methods with a microlocal elliptic component.

1. Corrigendum and addendum to [10, Section 5.2]

In this paper, we primarily make reference to [10, Section 5.2, pages 1117–1120]. At the end, in Section 3 below, we will also examine its impact on [10, Section 7.1], which is a direct consequence of [10, Section 5.2]. Section 5.2 of [10] deals with the regularity of the map $g \to B^*Lg$, where $v = Lg$ is the solution of the two-dimensional wave equation [10, equation (5.2.2)] in the half-space, with zero initial conditions and Dirichlet boundary control g. (See problem (1.9) below for the general case on a bounded domain in $\mathbb{R}^n$, $n \geq 2$.)

The claim made in [10, Section 5.2] that $B^*L \notin \mathcal{L}(L_2(0,T;U))$ is *incorrect*, due to a spurious appearance of the symbol "Re" (real part) in [10, equation (5.2.18)]—and, consequently, in [10, equation (5.2.22)]—while in view of the *correct* [10, equation (5.2.10)], *the symbol "Re" should have been omitted.*

Luckily, the same analysis given in [10, Section 5.2], once the spurious symbol "Re" is omitted from [10, equation (5.2.18)] (as it should be), *provides, in fact, a direct proof of the positive result* that

$$g \longrightarrow B^*Lg \quad \text{is continuous on } L_2(\Sigma), \quad \Sigma = (0,T) \times \Gamma. \tag{1.1}$$

This result in (1.1)—at this step, with the correction noted above, valid just for the two-dimensional wave equation on the half-space—can, in fact, be generalized to hold true for the wave equation with Dirichlet boundary control g defined on any bounded, sufficiently smooth domain in $\mathbb{R}^n$, $n \geq 2$—see Theorem 1.1 below. The positive conclusion in the case $n = 1$ was already noted in [10, Section 4.7 and equation (5.1.19)]. Thus, this note serves a two-fold purpose: (i) on the one hand, it provides a *corrigendum* to the counter-example

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in the two-dimensional half space of [10, Section 5.2]; (ii) on the other hand, it provides its replacement in the *addendum*—the positive statement of Theorem 1.1 below.

Corrigendum. Reference is made to [10, Section 5.2].

(i) In equation (5.2.18), page 1120, *suppress* the symbol "Re" (real part).

(ii) As a consequence of (i), in equation (5.2.22), page 1120, *suppress* the symbol "Re," so that the *corrected* equation becomes, for $(\sigma,\tau)\in\mathcal{B}_{\sigma,\tau}$,

$$\begin{aligned}\int_0^\infty e^{-\sqrt{\tau^2+\eta^2}x}e^{-|\eta|x}dx &= \frac{1}{\sqrt{\tau^2+\eta^2}+\eta} = \frac{1}{A+iB+\eta} \sim \frac{1}{1+i\sigma+\eta}\\ &\sim \frac{1}{1+i\eta^2+\eta} \sim \frac{1}{\eta^2} \sim \frac{1}{\sigma}.\end{aligned} \tag{1.2}$$

(iii) As a consequence of (ii), in equation (5.2.23), page 1120, *suppress* the symbol "Re," so that the *corrected* equation becomes by (1.2), with $\Sigma_\infty=(0,\infty)\times\Gamma$,

$$\iint_{\mathcal{B}_{\sigma\eta}} \sigma\left|\hat{g}(\sigma,\eta)\right|^2\int_0^\infty e^{-\sqrt{\tau^2+\eta^2}x}e^{-|\eta|x}dx \sim \iint_{\mathcal{B}_{\sigma\eta}}\frac{\sigma}{\sigma}\left|\hat{g}(\sigma,\eta)\right|^2 d\sigma\,d\eta \le C|g|^2_{L_2(\Sigma_\infty)}. \tag{1.3}$$

The very same argument with "Re" omitted, as it should be, instead of a negative result, gives the positive result in (1.1) in the half-space; in fact, for any $n\ge 2$. We will see this below.

Positive result on a half-space, $n\ge 2$. The proof is essentially contained in [10, Section 5.2], modulo the corrections as stated above. We consider the half-space wave equation problem in [10, equation (5.2.2)]. Let $u\in L_2(0,\infty;L_2(\Gamma))$. Then, the corresponding version of [10, equation (5.2.10), page 1119] is

$$\left(e^{-2\gamma t}B^*Lg,u\right)_{L_2(\Sigma_\infty)} = \frac{1}{2\pi}\int_{\mathbb{R}^n_{\sigma,\eta}}\left(\tau\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx\right)\hat{g}(\tau,\eta)\overline{\hat{u}(\tau,\eta)}d\sigma\,d\eta. \tag{1.4}$$

Let

$$H(\sigma,\eta)\equiv\sigma\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx. \tag{1.5}$$

It is immediate to show that $|H(\sigma,\eta)|$ is uniformly bounded for all $(\sigma,\eta)\in\mathbb{R}^n_{\sigma,\eta}$. Indeed, first notice that

$$\left|H(\sigma,\eta)\right| \equiv \left|\sigma\int_0^\infty e^{-\sqrt{\tau^2+|\eta|^2}x}e^{-|\eta|x}dx\right| \le c\frac{|\sigma|}{|A|+|\eta|+|B|} \equiv ch(\sigma,\eta) \tag{1.6}$$

in the notation of [10, equation (5.2.19)], where $A+iB\equiv\sqrt{\tau^2+|\eta|^2}$. On the one hand, considering the hyperbolic region $|\sigma|\ge 2|\eta|$, $A\sim 1$ (see [10, equation (5.2.20)]), $|B|\sim|\sigma|$ (see [10, equation (5.2.19b)]), and $h(\sigma,\eta)\le|\sigma|/|B|\le|\sigma|/|\sigma|=1$.

On the other hand, in the elliptic region $|\sigma|\le 2|\eta|$, we have $h(\sigma,\eta)\le|\sigma|/|\eta|\le 2$. Thus,

$$\left|H(\sigma,\eta)\right| \le C<\infty, \quad \forall\sigma\in\mathbb{R}^1,\ \eta\in\mathbb{R}^{n-1}. \tag{1.7}$$

Then, (1.4) and (1.7) yield the desired conclusion:

$$\left| \left(e^{-2\gamma t}B^* Lg, u\right)_{L_2(\Sigma_\infty)} \right| \le C\|g\|_{L_2(\Sigma_\infty)}\|u\|_{L_2(\Sigma_\infty)}, \tag{1.8}$$

and thus (1.1) holds true for the wave equation on the n-dimensional half-space $n \ge 2$.

The argument above is very transparent and shows exactly what is going on in order to gain the additional derivative on the boundary in the present case.

Addendum. We now state the general positive result.

THEOREM 1.1. *Let* Ω *be a sufficiently smooth bounded domain in* $\mathbb{R}^n$, $n \ge 2$. *Consider the* v*-problem in* [10, equation (5.1.1), page 1114], *that is,*

$$v_{tt} - \Delta v = 0 \quad \text{in } Q \equiv (0,T] \times \Omega, \qquad v|_\Sigma = g \quad \text{on } \Sigma \equiv (0,T] \times \Gamma, \tag{1.9}$$

and zero initial conditions: $v(0,\cdot) = v_t(0,\cdot) = 0$ *on* Ω. *Then, in the notation of* [10, Section 5.2], *the regularity in (1.1) holds true. This is to say, the map (see* [10, equations (5.1.8)–(5.1.10)]*)*

$$g \longrightarrow B^* Lg = D^* v_t = -\frac{\partial z}{\partial \nu} \quad \text{is bounded on } L_2(\Sigma). \tag{1.10}$$

Here, D is the Dirichlet map $\varphi = Dg \Leftrightarrow \{\Delta\varphi = 0 \text{ in } \Omega;\ \varphi|_\Gamma = g \text{ in } \Gamma\}$ as in [10, equation (5.1.6)]. Moreover, $z = \mathcal{A}^{-1}v_t$, see [10, equation (5.1.10)], where $\mathcal{A}$ is $-\Delta$ with Dirichlet boundary condition (BC) as in [10, equation (5.1.6)].

For future reference in the proof of Section 2, we recall from [10, equations (5.1.3), (5.1.10), (5.1.13)] that

$$z \in C([0,T];\mathcal{D}(\mathcal{A}^{1/2}) \equiv H_0^1(\Omega)), \quad \Delta z = -v_t \in C([0,T];H^{-1}(\Omega)); \tag{1.11}$$

$$z_t = \mathcal{A}^{-1}v_{tt} = \mathcal{A}^{-1}[-\mathcal{A}v + \mathcal{A}Dg] = -v + Dg \in L_2(0,T;L_2(\Omega)). \tag{1.12}$$

Remark 1.2. The above Theorem 1.1 was first stated in [1] (see estimate (2.7), page 121). We believe that the proof that we will give below in Section 2 is essentially self-contained and much simpler than the sketch given in [1]. The idea pursued in [1] is based on a full microlocal analysis of the fourth-order operator $\Delta(D_t^2 - \Delta)$ (where the extra Δ is used to eliminate Dg from the z-dynamics $z_{tt} = \Delta z + Dg_t$, see [10, equation (5.1.11b)], as $\Delta Dg_t \equiv 0$). The subsequent microlocal analysis of [1] considers, as usual [8], three regions: the hyperbolic region, the elliptic region, and the "glancing rays" region. The latter is the most demanding, and it is unfortunate that no details are provided in [1] for the analysis in the glancing region, except for reference to the author's Ph.D. thesis.

By contrast, our proof in Section 2 below invokes, for the most critical part, the sharp regularity of the wave equation from [5]—which is obtained via differential, rather than pseudodifferential/microlocal analysis methods. In addition, standard elliptic (interior and) trace regularity of the Dirichlet map D is used. Thus, by simply invoking these results in (1.12) above for z_t, we obtain—by purely differential methods—the critical result on $\partial z_t/\partial\nu$ of Step 1, (2.3). This then provides automatically the desired regularity of $\partial z/\partial\nu$ microlocally outside the elliptic sector of the D'Alambertian $\Box = D_t^2 - \Delta$, where the time variable dominates the tangential space variable in the Fourier space, see (2.11) below.

Thus, the rest of the proof follows from pseudodifferential operator (PDO) elliptic regularity of the localized problem.

2. Proof of Theorem 1.1

Step 1. Let $g \in L_2(\Sigma)$. Then, the following interior and boundary sharp regularity for the v-problem (1.9) is known [5, Theorem 2.3, page 153; or else Theorem 3.3, page 176 (interior regularity) plus Theorem 3.7, page 178 (boundary regularity)]:

$$\{v, v_t\} \in C([0,T]; L_2(\Omega) \times H^{-1}(\Omega)), \qquad \frac{\partial}{\partial \nu} v \Big|_{\Sigma} \in H^{-1}(\Sigma) \tag{2.1}$$

continuously in g (as noted in [10, equation (5.1.3)]). Moreover, elliptic regularity of the Dirichlet map gives $Dg \in L_2(0,T; H^{1/2}(\Omega))$, and thus [2]

$$\frac{\partial}{\partial \nu} Dg \in L_2(0,T; H^{-1}(\Gamma)). \tag{2.2}$$

Next, using (2.1) and (2.2) in (1.12) yields

$$\frac{\partial}{\partial \nu} z_t = -\frac{\partial}{\partial \nu} v + \frac{\partial}{\partial \nu} Dg \in H^{-1}(\Sigma). \tag{2.3}$$

The above relation provides us with the desired regularity of $\partial z / \partial \nu$ microlocally outside the elliptic sector of the D'Alambertian $\Box = D_t^2 - \Delta$; that is, when the dual Fourier variable σ (corresponding to time) dominates the dual Fourier variable $|\eta|$ (corresponding to the space tangential variable). A quantitative statement of this is given in (2.11) below.

Step 2. It remains to show that the L_2 regularity of $\partial z / \partial \nu$ holds also in the elliptic sector. This is done by standard arguments using localization of the PDO symbols. We use standard partition of unity procedure and local change of coordinates by which Ω and Γ can be identified (locally) with $\tilde{\Omega} \equiv \{(x,y) \in \mathbb{R}^n,\ x \geq 0,\ y \in \mathbb{R}^{n-1}\}$, $\tilde{\Gamma} \equiv \{(x,y) \in \mathbb{R}^n,\ x = 0,\ y \in \mathbb{R}^{n-1}\}$. The second-order elliptic operator Δ is identified in local coordinates (Melrose-Sjostrand) with $\tilde{\Delta} = D_x^2 + r(x,y) D_y^2 + \text{lot}$, where lot (which result from commutators) are first-order differential operators and $r(x,y) D_y^2$ stands for the second-order tangential (in the y variable) strongly elliptic operator. Since solutions v satisfy zero initial data, we can also extend $v(t)$ by zero for $t < 0$. For $t > T$ we multiply the solution by a smooth cutoff function $\phi(t) = 0$, $t \geq (3/2)T$, $\phi(t) = 1$, $t \leq T$. Thus, in order to obtain the desired solution, it amounts to consider the following problem:

$$\begin{aligned} w_{tt} = \tilde{\Delta} w = \Delta_0 w + \text{lot}(v) \quad &\text{in } \tilde{Q},\ w|_{\tilde{\Gamma}} = g, \\ w(0,\cdot) = w_t(0,\cdot) = 0 \quad &\text{in } \tilde{\Omega},\ \operatorname{supp} w \in [0, 2T], \end{aligned} \tag{2.4a}$$

where $\Delta_0 = D_x^2 + r(x,y) D_y^2$ is the principal part of $\tilde{\Delta}$ and v is the original solution $v = Lg$ of problem (1.9). Below, we will write $w = u + y$, where u, y satisfy (2.5) and (2.6), respectively. As a consequence, we will obtain

$$\{w, w_t\} \in C([0,T]; L_2(\tilde{\Omega}) \times H^{-1}(\tilde{\Omega})) \quad \text{continuously in } g \in L_2(\tilde{\Sigma}). \tag{2.4b}$$

Below we will denote by u the solution of

$$u_{tt} = \Delta_0 u \quad \text{in } \tilde{Q},\ u|_{\tilde{\Sigma}} = g; \qquad u(0,\cdot) = u_t(0,\cdot) = 0 \quad \text{in } \tilde{\Omega}, \tag{2.5a}$$

$$\{u, u_t\} \in C([0,T]; L_2(\tilde{\Omega}) \times H^{-1}(\tilde{\Omega})) \quad \text{continuously in } g \in L_2(\tilde{\Sigma}), \tag{2.5b}$$

the counterpart regularity statement of (2.1) for v in Ω. Likewise, we introduce the following nonhomogenous problem:

$$y_{tt} = \Delta_0 y + f \quad \text{in } \tilde{Q},\ y|_{\tilde{\Sigma}} = 0, \qquad y(0,\cdot) = y_t(0,\cdot) = 0 \quad \text{in } \tilde{\Omega}, \tag{2.6}$$

where $f = \text{lot}(v)$ results from the presence of the lower-order terms applied to the original variable v in (2.4), that is, in (1.9). Thus, recalling that $v \in C([0,T]; L_2(\Omega))$ by (2.1), we obtain

$$f \in C([0,T]; H^{-1}(\tilde{\Omega})), \quad \text{hence } \{y, y_t\} \in C([0,T]; L_2(\tilde{\Omega}) \times H^{-1}(\tilde{\Omega})) \tag{2.7}$$

[5, Theorem 2.3, page 153] continuously in $g \in L_2(\Sigma)$.

By the principle of superposition, we have $w = u + y$, as announced above.

Step 3. In this step, we handle the y-problem (2.6). We first recall from (1.10) that our original objective is showing that $D^* v_t \in L_2(\Sigma)$ continuously in $g \in L_2(\Sigma)$. Moreover, we recall that v in Ω is transferred into $w = u + y$, on the half-space $\tilde{\Omega}$ (locally). Thus, by (2.6), (2.7), what suffices to show for y is the following regularity property:

$$f \longrightarrow D^* y_t : \text{continuous } L_2(0,T; H^{-1}(\tilde{\Omega})) \longrightarrow L_2(0,T; L_2(\tilde{\Gamma})), \tag{2.8}$$

whereby $D^* y_t$ is ultimately continuous in $g \in L_2(\Sigma)$. However, the above property (2.8) is known from [5, Theorem 3.11, page 182] and has been used in the past several times. In fact, set $A = -\Delta_0$, with $\mathcal{D}(A) = H^2(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega})$ and rewrite (2.6) abstractly as $y_{tt} = -Ay + f$. Apply A^{-1} throughout and set $\Psi = A^{-1} y \in C([0,T]; \mathcal{D}(A))$ via (2.7). Moreover, $A^{-1} f \in L_2(0,T; H_0^1(\tilde{\Omega}))$, again by (2.7). Thus, Ψ solves the problem

$$\Psi_{tt} = \Delta_0 \Psi + A^{-1} f \quad \text{in } \tilde{Q},\ \Psi|_{\tilde{\Sigma}} = 0, \qquad \Psi(0,\cdot) = \Psi_t(0,\cdot) = 0 \quad \text{in } \tilde{\Omega}. \tag{2.9}$$

We further have that $A^{-1} y_t \in C([0,T]; H_0^1(\tilde{\Omega}))$, again by (2.7). Finally we recall that $D^* A A^{-1} y_t = -(\partial/\partial\nu)\Psi_t$ (see [9], [10, equation (5.1.9)]). One can simply quote [5, Theorem 3.11, page 182] or [9, equation (10.5.5.11), page 952] to obtain the desired regularity (2.8):

$$D^* y_t = -\frac{\partial}{\partial \nu} \Psi_t \in L_2(\tilde{\Sigma}) \quad \text{continuously in } g \in L_2(\Sigma). \tag{2.10}$$

Step 4. Having accounted for the lot(v) in Step 3—which are responsible for the y-problem—we may in this step set $y \equiv 0$ and thus identify w with $u : w \equiv u$. Thus it remains to consider problem (2.5) in u, involving only the principal part of the D'Alambertian. Let $\mathscr{X} \in S^0(\tilde{Q})$ denote the PDO operator $\mathscr{X}(x,y,t)$ with smooth symbol of localization $\chi(x,y,t,\sigma,\eta)$ supported in the elliptic sector of $\Box \equiv D_t^2 - D_x^2 - r(x,y)D_y^2$, where the principal part of the D'Alambertian is written in local coordinates. The dual variables $\sigma \in \mathbb{R}^1$, $\eta \in \mathbb{R}^{n-1}$ correspond to the Fourier's variables of $t \to i\sigma$, $\eta \to i\eta$. Thus, $\operatorname{supp}\chi \in \{(x,y,t,\sigma,\eta) \in \tilde{Q} \times \mathbb{R}^1 \times \mathbb{R}^{n-1},\ \sigma^2 - r(0,y)|\eta|^2 < 0\}$. The established regularity (2.3) and the fact that $|\sigma| \geq c|\eta|$ on $\operatorname{supp}\chi$ imply that

$$(I - \mathscr{X})\frac{\partial}{\partial \nu} z \in L_2(\Sigma), \tag{2.11}$$

a statement that $|\sigma|(\partial z/\partial \nu)$, and thus a fortiori $|\eta|(\partial z/\partial \nu)$, are in L_2 in time and space in the (hyperbolic) sector $|\sigma| \geq c|\eta|$. On the other hand, returning to problem (2.5) for u, rewritten as $\Box u = 0$ and applying $\mathscr{X}$, we see that the variable $\mathscr{X}u$ satisfies

$$\Box \mathscr{X} u = -[\mathscr{X},\Box]u \in H^{-1}(\tilde{Q}), \tag{2.12}$$

where henceforth we take for $\tilde{Q}$ an extended cylinder based on $\tilde{\Omega} \times [-T,2T]$. Indeed, this last inclusion follows from $[\mathscr{X},\Box] \in S^1(\tilde{Q})$ and the priori regularity (2.5b) for u implying $u \in L_2(\tilde{Q})$, which jointly lead to $[\mathscr{X},\Box]u \in H^{-1}(\tilde{Q})$. Moreover, $\mathscr{X}u|_\Gamma = \mathscr{X}g \in L_2(\tilde{\Sigma})$. Furthermore, still by (2.5b) and the fact that $\operatorname{supp} u \in [0,(3/2)T]$, we have, by the pseudolocal property of pseudodifferential operators, that $(\mathscr{X}u)(2T) \in C^\infty(\tilde{\Omega})$, $(\mathscr{X}u)(-T) \in C^\infty(\tilde{\Omega})$. We conclude that $\mathscr{X}u|_{\partial\tilde{Q}} \in L_2(\partial\tilde{Q})$, a boundary condition to be associated to (2.12). Since $\Box\mathscr{X}$ is a pseudodifferential elliptic operator, classical elliptic theory, applied to $\Box\mathscr{X}u \in H^{-1}(\tilde{Q})$, $\mathscr{X}u|_{\partial\tilde{Q}} \in L_2(\partial\tilde{Q})$—the elliptic problem obtained above—yields

$$\mathscr{X}u \in H^{1/2}(\tilde{Q}) + H^1(\tilde{Q}) \subset H^{1/2}(\tilde{Q}), \tag{2.13}$$

where the first containment on the right-hand side of (2.13) is due to the boundary term, and the second to the interior term. Next, we return to the elliptic problem $\Delta z = -v_t$ in Q, $z|_\Sigma = 0$ from (1.11), with a priori regularity noted in (1.11). The counterpart of the above elliptic problem in the half-space $\tilde{Q}$ (locally) is $\tilde{\Delta} z = -u_t$ in $\tilde{Q}$, $z|_{\tilde{\Sigma}} = 0$ (we retain the symbol z in $\tilde{Q}$), as we are identifying w with u in the present Step 4 (due to the results of Step 3). Applying $\mathscr{X}$ throughout yields

$$\tilde{\Delta}\mathscr{X}z = -\mathscr{X}u_t + [\tilde{\Delta},\mathscr{X}]z = -\frac{d}{dt}\mathscr{X}u + \left[\frac{d}{dt},\mathscr{X}\right]u + [\tilde{\Delta},\mathscr{X}]z. \tag{2.14}$$

Note $[\tilde{\Delta},\mathscr{X}] \in S^1(\tilde{Q})$ and $[d/dt,\mathscr{X}] \in S^0(\tilde{Q})$. Hence, by the a priori regularity in (2.5b) for u and in (1.11) for z, we conclude

$$\left[\frac{d}{dt},\mathscr{X}\right]u + [\tilde{\Delta},\mathscr{X}]z \in L_2(\tilde{Q}). \tag{2.15}$$

Moreover, by virtue of (2.13), $(d/dt)\mathscr{K}u \in H_{(0,-1/2)}(\tilde{Q})$ where we have used the anisotropic Hörmander's spaces [3, Volume III, page 477], $H_{(m,s)}(\tilde{Q})$, where m is the order in the normal direction to the plane $x = 0$ (which plays a distinguished role) and $(m+s)$ is the order in the tangential direction in t and y. Via (2.15), we are thus led to solving the problem

$$\tilde{\Delta}\mathscr{K}z \in H_{(0,-1/2)}(\tilde{Q}) + L_2(\tilde{Q}), \quad (\mathscr{K}z)|_{\tilde{\Gamma}} = 0. \tag{2.16}$$

By elliptic regularity (note that $\tilde{\Delta}\mathscr{K}$ is elliptic in $\tilde{Q}$), we obtain again

$$\mathscr{K}z \in H^{3/2}(\tilde{Q}), \quad \frac{\partial}{\partial\nu}\mathscr{K}z \in L_2(\tilde{\Sigma}). \tag{2.17}$$

Combining (2.17) and (2.11) yields the final conclusion

$$\frac{\partial}{\partial\nu}z = (I-\mathscr{K})\frac{\partial}{\partial\nu}z + \mathscr{K}\frac{\partial}{\partial\nu}z \in L_2(\tilde{\Sigma}), \tag{2.18}$$

and Theorem 1.1 is proved.

3. Impact on [10, Section 7.1]

Theorem 1.1 and the decomposition argument in [10, Section 7.1, page 1129] allow one to deduce the analogous positive result valid for the Kirchhoff plate with moment controls. Indeed, with reference to the model in [10, equations (7.1.1)], we have the following theorem.

Theorem 3.1. *Let Ω be as in Theorem 1.1, and let v be a solution to* [10, equations (7.1.1)], *that is,*

$$v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v = 0 \quad \text{in } Q,\ v = 0, \qquad \Delta v = g \quad \text{on } \Sigma, \qquad v(0) = v_t(0) \quad \text{in } \Omega. \tag{3.1}$$

Then the map $g \to (\partial/\partial\nu)v_t$ is continuous on $L_2(\Sigma)$.

This positive result replaces [10, Section 7.1].

4. From the regularity of B^*L to the regularity of L [10, Appendix]

Consider the system [10, equation (1.1)]: $y_t = Ay + Bu \in [\mathscr{D}(A^*)]'$, $y(0) = y_0 \in Y$, under the preliminary assumption stated in [10, page 1069]: (i) $A : Y \supset D(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At} in the Hilbert space Y, $t \geq 0$; (ii) $B \in \mathscr{L}(U;[\mathscr{D}(A^{*1/2})]')$, where U is another Hilbert space. Define as in [10, equation (1.2b)]

$$(Lu)(t) = \int_0^t e^{A(t-s)}Bu(s)ds, \qquad L_T u = \int_0^T e^{A(T-s)}Bu(s)ds. \tag{4.1}$$

The following result was stated in [10, Appendix, Proposition A.1, page 1132].

PROPOSITION 4.1. *In addition to the standing hypotheses (i) and (ii) above, assume that (a) A is skew adjoint: $A^* = -A$, so that $e^{A^*t} = e^{-At}$, $t \in \mathbb{R}$, and (b)*

$$B^*L \in \mathcal{L}(L_2(0,T;U)). \tag{4.2}$$

Then, in fact,

$$L \text{ is continuous}: L_2(0,T;U) \longrightarrow C([0,T];Y). \tag{4.3}$$

Proof. Let $u \in L_2(0,T;U)$. Set

$$x(t) = \int_0^t e^{-As} Bu(s)ds. \tag{4.4}$$

By (a) and (b), we then estimate via (4.4), (4.1),

$$\begin{aligned} C_T\|u\|^2_{L_2(0,T;U)} &\geq \int_0^T \big((B^*Lu)(t),u(t)\big)_U dt = \int_0^T \left(\int_0^t e^{A(t-s)}Bu(s)ds, Bu(t)\right)_Y dt \\ &= \int_0^T \left(\int_0^t e^{-As}Bu(s)ds, e^{-At}Bu(t)\right)_Y dt = \int_0^T \left(x(t), \frac{d}{dt}x(t)\right)_Y dt \\ &= \frac{1}{2}\int_0^T \frac{d}{dt}\big(x(t),x(t)\big)_Y dt = \frac{1}{2}\|x(T)\|^2_Y = \frac{1}{2}\left\|\int_0^T e^{-As}Bu(s)ds\right\|^2_Y \\ &= \frac{1}{2}\left\|e^{-AT}\int_0^T e^{A(T-s)}Bu(s)ds\right\|^2_Y \sim \left\|\int_0^T e^{A(T-s)}Bu(s)ds\right\|^2_Y \\ &= c_T\|L_Tu\|^2_Y. \end{aligned} \tag{4.5}$$

Then (4.5) says that

$$L_T : \text{continuous } L_2(0,T;U) \longrightarrow Y \tag{4.6}$$

and (4.6) is well known [6, 9] to yield (4.3). □

Remark 4.2. Proposition 4.1 can be extended to A of the form $A = iS + kI$, with S a self-adjoint operator on Y and $k \in \mathbb{R}$, so that $A^* = -A + 2kI$ and $e^{A^*t} = e^{-At}e^{2kt}$. In this case, we start with $(B^*Lu, u_1)_U$, with $u_1 = e^{-2kt}u(t)$, $u \in L_2(0,T;U)$.

Remark 4.3. The above direct proof replaces the one given in [10, Appendix], which while driven by the same idea is less direct, and, moreover, has however a computational flaw in [10, equation (A.7), page 1133].

Remark 4.4. An alternative, perhaps more insightful, proof of Proposition 4.1 is as follows. For u smooth, we have, via (4.1) for L and its adjoint L^* [9],

$$\begin{aligned} (B^*Lu)(t) &= \int_0^t B^*e^{A(t-\tau)}Bu(\tau)d\tau, \\ (L^*Bu)(t) &= \int_t^T B^*e^{A^*(\tau-t)}Bu(\tau)d\tau = \int_t^T B^*e^{A(t-\tau)}Bu(\tau)d\tau, \end{aligned} \tag{4.7}$$

using the assumption $A^* = -A$. Thus, adding up (4.7) yields, using again skew-adjointness,

$$\begin{aligned}(B^*Lu)(t)+(L^*Bu)(t) &= \int_0^T B^*e^{A(t-\tau)}Bu(\tau)d\tau \\ &= B^*e^{A(t-T)}\int_0^T e^{A(T-\tau)}Bu(\tau)d\tau \\ &= B^*e^{A^*(T-t)}\int_0^T e^{A(T-\tau)}Bu(\tau)d\tau.\end{aligned} \tag{4.8}$$

Finally, recalling L_T in (4.1) and its adjoint L_T^* [9], we rewrite (4.8) in the following attractive form:

$$B^*Lu+L^*Bu = L_T^*L_Tu, \quad u \text{ smooth}, \tag{4.9}$$

(from which (4.5) follows, by taking the $L_2(0,T;U)$-inner product with u). Equation (4.9) shows the implication (4.5)⇒(4.3).

Remark 4.5. Another negative example where uniform stabilization is known, yet the operator $B^*L \notin \mathscr{L}(L_2(0,T;U)$, is given by the Euler-Bernoulli plate equation with boundary control only on the "moment" $\Delta w|_\Sigma$, as considered in [7, 4]. Here the class of controls is $L_2(0,T;H^{1/2}(\Gamma))$, and the space of exact controllability and uniform stabilization is $Y = [H^2(\Omega)\cap H_0^1(\Omega)]\times L_2(\Omega)$. Exact controllability (without geometrical conditions) is established in [7], while uniform stabilization is proved in [7] (under geometrical conditions) and in [4] (without geometrical conditions). Optimal regularity of L is given in [9, page 1023 and page 1029]: it shows that it would take the class $H^{1/2,1/2}(\Sigma)$ of controls—thus with an extra 1/2-derivative in time—to obtain L continuous into $C([0,T];[H^2(\Omega)\cap H_0^1(\Omega)]\times L_2(\Omega))$. Thus, by Proposition 4.1, $B^*L \notin \mathscr{L}(L_2(0,T; H^{1/2}(\Gamma))$.

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References

[1] K. Ammari, *Dirichlet boundary stabilization of the wave equation*, Asymptot. Anal. **30** (2002), no. 2, 117–130.

[2] B. Kellogg, *Properties of elliptic B.V.P.*, Mathematical Foundations of the Finite Elements Methods, Academic Press, New York, 1972, Chapter 3.

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III*, Grundlehren der Mathematischen Wissenschaften, vol. 274, Springer-Verlag, Berlin, 1985.

[4] M. A. Horn and I. Lasiecka, *Asymptotic behavior with respect to thickness of boundary stabilizing feedback for the Kirchhoff plate*, J. Differential Equations **114** (1994), no. 2, 396–433.

[5] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl. (9) **65** (1986), no. 2, 149–192.

[6] I. Lasiecka and R. Triggiani, *A lifting theorem for the time regularity of solutions to abstract equations with unbounded operators and applications to hyperbolic equations*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 745–755.

[7] ———, *Exact controllability and uniform stabilization of Euler-Bernoulli equations with boundary control only in* $\Delta w|_\Sigma$, Boll. Un. Mat. Ital. B (7) **5** (1991), no. 3, 665–702.

[8] ———, *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim. **25** (1992), no. 2, 189–224.

[9] ———, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II*, Encyclopedia of Mathematics and Its Applications, vol. 75, Cambridge University Press, Cambridge, 2000.

[10] ———, $L_2(\Sigma)$*-regularity of the boundary to boundary operator* B^*L *for hyperbolic and Petrowski PDEs*, Abstr. Appl. Anal. **2003** (2003), no. 19, 1061–1139.

I. Lasiecka: Department of Mathematics, University of Virginia, Kerchof Hall, Charlottesville, VA 22904, USA

E-mail address: il2v@virginia.edu

R. Triggiani: Department of Mathematics, University of Virginia, Kerchof Hall, Charlottesville, VA 22904, USA

E-mail address: rt7u@virginia.edu