GEOMETRIC PROPERTIES OF BANACH SPACES AND THE EXISTENCE OF NEAREST AND FARTHEST POINTS

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1. Introduction

The aim of this paper is to present some existence results for nearest-point and farthest-point problems, in connection with some geometric properties of Banach spaces. The idea goes back to Efimov and Stečkin who, in a series of papers (see [28, 29, 30, 31]), realized for the first time that some geometric properties of Banach spaces, such as strict convexity, uniform convexity, reflexivity, and the Kadec-Klee property, and so forth, characterize some generic results concerning the uniqueness and existence of nearest points.

Let X be a real normed linear space and Z a nonempty subset of X. For $x \in X$, put

$$d(x,Z) = \inf\{\|x-z\| : z \in Z\},$$

$$P(x) = P_Z(x) = \{z \in Z : \|x-z\| = d(x,Z)\},$$

$$E(Z) = \{x \in X : P(x) \neq \varnothing\}, \tag{1.1}$$

$$U(Z) = \{x \in X : \operatorname{card} P(x) \leq 1\},$$

$$EU(Z) = \{x \in X : \operatorname{card} P(x) = 1\}.$$

The elements (if any) of the set $P_Z(x)$ are called *nearest points* to x in Z. The set Z is called *proximinal* if $E(Z) = X$, *antiproximinal* if $E(Z) = Z$, *a uniqueness set* if $U(Z) = X$, and *Chebyshev* if $EU(Z) = X$. If these sets only contain a subset which is G_δ and dense in X (or, equivalently, if their complements are of the first Baire category in X), then the set Z is called *almost proximinal, an almost uniqueness set, almost Chebyshev*, respectively.

Suppose further that the set Z is bounded and let

$$q(x,Z) = \sup \{\|x - z\| : z \in Z\},$$

$$Q(x) = Q_Z(x) = \{z \in Z : \|x - z\| = h(x,Z)\},$$

$$e(Z) = \{x \in X : Q(x) \neq \varnothing\}, \qquad (1.2)$$

$$u(Z) = \{x \in X : \operatorname{card} Q(x) \leq 1\},$$

$$eu(Z) = \{x \in X : \operatorname{card} Q(x) = 1\}.$$

The elements of the (possibly empty) set $Q_Z(x)$ are called *farthest points* from x in Z. The set Z is called *remotal* if $Q_Z(x) \neq \varnothing$ for every $x \in X$, and *uniquely remotal* if $\operatorname{card} Q_Z(x) = 1$ for every $x \in X$. As in the case of nearest points, the terms *almost remotal and almost uniquely remotal* are used to designate the fact that the set $X \setminus e(Z)$, (resp., $X \setminus eu(Z)$) is of the first Baire category.

In the following proposition, we collect some simple but useful properties of the distance function and of the projection.

PROPOSITION 1.1. *Let Z be a nonempty set of a normed space X, and x, y elements of X. Then*
(1) $d(x,Z) = d(x,\overline{Z})$ and $d(x,Z) = 0 \Leftrightarrow x \in \overline{Z}$;
(2) *the function $d(\cdot,Z)$ is nonexpansive, that is,*

$$\forall x_1, x_2 \in X, \quad |d(x_1,Z) - d(x_2,Z)| \leq \|x_1 - x_2\|, \qquad (1.3)$$

hence uniformly continuous on X;
(3) $d(x + y, y + Z) = d(x,Z)$ and $P_{y+Z}(x + y) = P_Z(x)$, $d(x,\alpha Z) = |\alpha| d(\alpha^{-1}x, Z)$, and $P_{\alpha Z}(x) = P_Z(\alpha^{-1}x)$, for $\alpha \neq 0$.

For the convenience of the reader, we will recall some geometric properties of normed spaces. If X is a normed space, then, for $x \in X$ and $r > 0$, denote by $B(x,r) = \{y \in X : \|y - x\| \leq r\}$ (resp., $B'(x,r) = \{y \in X : \|y - x\| < r\}$) the closed (resp., open) balls in X, and let $S(x,r) = \{y \in X : \|y - x\| = r\}$ be the sphere of center x and radius r. Let also $B_X = B(x,1)$, $B'_X = B'(o,1)$, and $S_X = S(0,1)$ be the closed unit ball, the open unit ball, and the unit sphere of X, respectively.

For $x, y \in X$, denote by $[x; y] = \{x + t(y - x) : t \in [0;1]\}$ the closed interval joining x and y. The intervals $[x; y)$, $(x; y]$, and $(x; y)$ are defined in an obvious way. The ray $R(x, y)$ is defined by $R(x, y) = \{x + t(y - x) : t \in [0; \infty)\}$ and the line passing through x and y by $D(x, y) = \{x + t(y - x) : t \in \mathbb{R}\}$.

Strict convexity. The normed space X is called *strictly convex* (or *rotund*) if its unit sphere S_X does not contain nontrivial segments, that is, for every $x, y \in S_X$, $x \neq y$, and $t \in (0;1)$, we have $\|(1 - t)x + ty\| < 1$, or, equivalently, if the equality $\|(1 - t)x + ty\| = 1$ holds for some $x, y \in S_X$ and some $t \in (0;1)$, then $x = y$.

The following proposition contains some equivalent conditions for strict convexity.

PROPOSITION 1.2. *For a normed space X, the following conditions are equivalent:*

(1) *X is strictly convex;*

(2) *for very $x, y \in S_X$ with $x \neq y$, $\|x + y\| < 2$;*

(3) *for every $x, y \in X \setminus \{0\}$, the equality $\|x + y\| = \|x\| + \|y\|$ implies $y = \alpha x$ for some $\alpha > 0$.*

Proof. Condition (2) follows from (1) taking $t = 1/2$ in the definition of strict convexity. The implication (2)⇒(1) follows from the convexity of the function $\varphi(t) = \|x + t(y - x)\|$, $t \in [0, 1]$.

(1)⇒(3) Suppose that the equality $\|x + y\| = \|x\| + \|y\|$ holds for some nonzero $x, y \in X$. It can be written in the equivalent form

$$\left\| \frac{\|x\|}{\|x\| + \|y\|} \cdot \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \cdot \frac{y}{\|y\|} \right\| = 1, \tag{1.4}$$

which, by the strict convexity of X, implies $y = \alpha x$ with $\alpha = \|y\|/\|x\|$.

(3)⇒(2) If $\|x + y\| = 2 = \|x\| + \|y\|$ for some $x, y \in S_X$, then $y = \alpha x$ for some $\alpha > 0$. Since $\|y\| = \|x\| = 1$, it follows $\alpha = 1$ and $y = x$.   □

Uniform convexity. A normed space X is called *uniformly convex* (or *uniformly rotund*) if for every ϵ, $0 < \epsilon \leq 2$, there exists $\delta = \delta(\epsilon) > 0$ such that for every $x, y \in B_X$,

$$\left\| \frac{x + y}{2} \right\| > 1 - \delta \Longrightarrow \|x - y\| < \epsilon; \tag{1.5}$$

or, equivalently,

$$\|x - y\| \geq \epsilon \Longrightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{1.6}$$

The *modulus of convexity* of a normed space X is the function $\delta_X : [0; 2] \to [0; 1]$, defined by one of the following equivalent conditions:

$$\begin{aligned} \delta_X(\epsilon) &= \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| \geq \epsilon \right\} \\ &= \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in B_X, \|x - y\| \geq \epsilon \right\}. \end{aligned} \tag{1.7}$$

For $0 \leq \epsilon < 2$, equalities of the above type hold also with the condition $\|x - y\| > \epsilon$. see, for instance, [58, pages 442–446].

Obviously the space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for every ϵ, $0 < \epsilon \leq 2$.

The modulus of convexity satisfies $\delta_X(\epsilon) \leq \epsilon/2$ and $\delta_X(\epsilon) \leq \delta_X(\epsilon')$ if $0 \leq \epsilon \leq \epsilon'$. If X is uniformly convex, then the mutually equivalent conditions (1.5) and (1.6) hold with $\delta = \delta_X(\epsilon)$.

By the monotonicity of the modulus δ_X, conditions (1.5) and (1.6) are also equivalent with

$$\|x + y\| \leq 2\left(1 - \delta_X\left(\|x - y\|\right)\right) \tag{1.8}$$

for all $x, y \in B_X$, $x \neq y$.

Uniform convexity can be characterized in terms of some sequences in the unit ball of X.

PROPOSITION 1.3 [58, page 447]. *For a normed space X, the following conditions are equivalent:*

(1) *X is uniformly convex;*
(2) *if (x_n) and (y_n) are two sequences in the unit sphere S_X of X such that $\|x_n + y_n\| \to 2$ for $n \to \infty$, then $\|x_n - y_n\| \to 0$;*
(3) *if (x_n) and (y_n) are two sequences in the unit ball B_X of X such that $\|x_n + y_n\| \to 2$ for $n \to \infty$, then $\|x_n - y_n\| \to 0$;*
(4) *if (x_n) and (y_n) are two sequences in X such that $(\|x_n\|)$, $(\|y_n\|)$, $(\|2^{-1}(x_n + y_n)\|)$ all tend to some $d > 0$ for $n \to \infty$, then $\|x_n - y_n\| \to 0$.*

Local uniform convexity. If for fixed $x \in S_X$ and every ϵ, $0 < \epsilon \leq 2$, there exists $\delta = \delta(\epsilon, x) > 0$ such that one of the equivalent conditions (1.5) or (1.6) holds, then the space X is called *locally uniformly convex* (or *locally uniformly rotund*). Obviously uniform convexity implies local uniform convexity and, in its turn, this implies strict convexity. A modulus of local uniform convexity can be defined as a localized version of the modulus of uniform convexity. For $x \in S_X$ and $0 < \epsilon \leq 2$, put

$$\begin{aligned} \delta_X(\epsilon, x) &= \inf\left\{y \in S_X : \|x - y\| \geq \epsilon\right\} \\ &= \inf\left\{y \in B_X : \|x - y\| \geq \epsilon\right\}. \end{aligned} \tag{1.9}$$

Obviously the space X is locally uniformly convex if and only if $\delta_X(\epsilon, x) > 0$ for every $x \in S_X$ and every ϵ, $0 < \epsilon \leq 2$. In this case, conditions (1.5) and (1.6) hold with $\delta = \delta_X(\epsilon, x)$. Condition (1.8) holds also with $\delta_X(\|x - y\|, x)$, $x \in S_x$, $y \in B_X$, $x \neq y$, instead of $\delta_X(\|x - y\|)$.

A theorem similar to Proposition 1.3 but localized, that is, with $x_n = x \in S_X$, holds for local uniform convexity too.

Since the unit ball of a normed space X determines the shape of all balls in X, the conditions of uniform, and of local uniform convexity, can be written for arbitrary balls.

Let $B(x_0, r)$ be an arbitrary closed ball in a normed space X.

PROPOSITION 1.4. (1) *Suppose that the space X is uniformly convex with modulus of uniform convexity δ_X. Then for every $z, z' \in B(x_0, r)$, $z \neq z'$, the following inequality holds:*

$$\|z + z' - 2x_0\| \leq 2r(1 - \delta), \tag{1.10}$$

where

$$\delta = \delta_X\left(\frac{\|z - z'\|}{r}\right). \tag{1.11}$$

If $z, z' \in B(x_0, r)$ *satisfy* $\|z - z'\| \geq \epsilon$ *for some* $\epsilon \in (0; 2]$, *then inequality (1.10) holds with* $\delta = \delta_X(\epsilon/r)$.

(2) *If* X *is locally uniformly convex with modulus of local uniform convexity* $\delta_X(\cdot, \cdot)$ *and* $z \in S(x_0, r)$, $z' \in B(x_0, r)$, $z \neq z'$, *then inequality (1.10) holds with*

$$\delta = \delta_X\left(\frac{\|z - z'\|}{r}, \frac{z - x_0}{r}\right). \tag{1.12}$$

If $z \in S(x_0, r)$ *and* $z' \in B(x_0, r)$ *are such that* $\|z - z'\| \geq \epsilon$, *for some* $\epsilon \in (0; 2]$, *then (1.10) holds with*

$$\delta = \delta_X\left(\frac{\epsilon}{r}, \frac{z - x_0}{r}\right). \tag{1.13}$$

Proof. Applying (1.8) to $(z - x_0)/r$ and $(z' - x_0)/r$, one obtains

$$\|z + z' - 2x_0\| \leq 2r(1 - \delta), \tag{1.14}$$

with $\delta = \delta_X(\|z - z'\|/r)$ if X is uniformly convex, (resp., $\delta = \delta_X(\|z - z'\|/r, (z - x_0)/r)$ if X is locally uniformly convex).

The assertions involving ϵ follow from the monotonicity of the moduli of uniform and of local uniform convexity. $\qquad\square$

A normed space X is said to have the *Kadec-Klee property* provided (x_n) tends weakly to x and $\|x_n\| \to \|x\|$ implies $\|x_n - x\| \to 0$, that is, (x_n) tends strongly to x, for every sequence (x_n) in X and $x \in X$. The space $\ell^1(\Gamma)$ and any locally uniformly convex space have the Kadec-Klee property. The Kadec-Klee property is called in [58] the Radon-Riesz property, motivated by the fact that J. Radon and F. Riesz proved that the space $L^p(\Omega, \mathcal{A}, \mu)$ has this property for $1 < p < \infty$.

2. The problem of nearest points

The geometric properties we considered at the end of the preceding section are closely related to the approximation properties of the space X. First we will examine strict convexity.

PROPOSITION 2.1. *For a normed space X, the following conditions are equivalent:*
  (1) *the space X is strictly convex;*
  (2) *every nonempty convex subset of X is a uniqueness set for best approximation;*
  (3) *every nonempty closed convex subset of X is a uniqueness set for best approximation.*

Proof. (1)$\Rightarrow$(2) Let Z be a nonempty convex subset of X and suppose that for some $x \in X$, there are $z_1, z_2 \in Z$ such that $\|x - z_1\| = \|x - z_2\| = d(x, Z) = d > 0$. Then $(z_1 + z_2)/2 \in Z$ and

$$d \leq \left\|x - \frac{z_1 + z_2}{2}\right\| = \frac{1}{2}(\|x - z_1 + x - z_2\|) \leq \frac{1}{2}(\|x - z_1\| + \|x - z_2\|) = d, \tag{2.1}$$

implying $\|x - z_1 + x - z_2\| = 2d = \|x - z_1\| + \|x - z_2\|$. By the strict convexity of X, there is $\alpha > 0$ such that $x - z_2 = \alpha(x - z_1)$. Because $\|x - z_1\| = \|x - z_2\| = d$, it follows that $\alpha = 1$ and $z_1 = z_2$.

The implication $(2) \Rightarrow (3)$ is obvious. To prove $(3) \Rightarrow (1)$ suppose that X is not strictly convex. Then there exist $x_0, x_1 \in S_X$, $x_0 \neq x_1$, such that $\|x_0 + x_1\| = 2$. The convexity of the function $\varphi(t) = \|x_0 + t(x_1 - x_0)\|$, $t \in [0; 1]$, implies $\|0 - \varphi(t)\| = 1 = d(0, [x_0; x_1])$ for all $t \in [0; 1]$. □

In uniformly convex spaces, stronger results on the existence and uniqueness of best approximation hold.

PROPOSITION 2.2. *If Z is a nonempty complete convex subset of a uniformly convex normed space X, then Z is a Chebyshev set in X.*

For the proof, we need a lemma.

LEMMA 2.3. *Let X be a uniformly convex normed space. If (x_n) is a sequence in X such that, for some $d > 0$, $\|x_n\| \to d$ for $n \to \infty$, and $\|x_n + x_m\| \to 2d$, for $n, m \to \infty$, then the sequence (x_n) is fundamental.*

Proof. For $\epsilon > 0$, let $\delta > 0$ be such that (1.5) holds. Take $\alpha \in \mathbb{R}$ satisfying the conditions:

$$0 < \alpha < 1, \qquad \alpha < d, \qquad \frac{2\alpha}{d + \alpha} < \delta. \tag{2.2}$$

By hypothesis, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \quad \|x_n\| \leq d + \alpha, \qquad \forall n, m \geq n_0, \quad \left\| \frac{x_n + x_m}{2} \right\| \geq d - \alpha. \tag{2.3}$$

Then $x'_n = (d + \alpha)^{-1} x_n \in B_X$ and

$$\left\| \frac{x'_n + x'_m}{2} \right\| = \frac{1}{d + \alpha} \left\| \frac{x_n + x_m}{2} \right\| \geq \frac{d - \alpha}{d + \alpha} = 1 - \frac{2\alpha}{d + \alpha} > 1 - \delta, \tag{2.4}$$

for all $n, m \geq n_0$. It follows that $\|x'_n - x'_m\| < \epsilon$, so that $\|x_n - x_m\| < \epsilon(d + \alpha) < \epsilon(d + 1)$ for all $n, m \geq n_0$. The lemma is proved. □

Proof of Proposition 2.2. For $x \in X \setminus Z$, let $d = d(x, Z) > 0$ and let (z_n) be a minimizing sequence in Z, that is, $\|x - z_n\| \to d$. Reasoning as in the proof of Proposition 2.1, it follows

$$\left\| \frac{x - z_n + x - z_m}{2} \right\| = \left\| x - \frac{z_n + z_m}{2} \right\| \longrightarrow d. \tag{2.5}$$

By the above lemma, the sequence $(x - z_n)$ is fundamental. But then (z_n) is also a fundamental sequence and, by the completeness of the set Z, it has a limit $z \in Z$. It follows $\|x - z\| = \lim_n \|x - z_n\| = d(x \cdot Z)$. Since every uniformly convex space is strictly convex, the uniqueness of z follows from Proposition 2.1.

Proposition 2.2 is proved. □

Stečkin's results on best approximation by elements of arbitrary sets. In the seminal paper [76], Stečkin proved some fundamental generic uniqueness and existence results for nearest points in arbitrary subsets of normed spaces, and posed several open questions. This drew the attention of mathematicians interested in abstract approximation theory or in Banach space geometry, to this challenging area of research, leading to a series of papers dealing with generic existence and uniqueness for the nearest-point and farthest-point problems in normed spaces.

Now, we will present Stečkin's results along with the problems raised by him in [76]. As Stečkin's paper is available only in Russian, I will present the proofs, taking into account some subsequent developments, mainly by De Blasi, Myjak, and Papini [17, 19]. (An English translation of Stečkin's paper can be obtained upon request from the author of the present paper.) A subset Z of a normed space X is called *relatively boundedly compact* if its intersection with any closed ball in X is relatively compact or, equivalently, if any bounded sequence in Z contains a convergent subsequence. If the intersection of Z with any closed ball is compact, then we call it *boundedly compact*.

We first present the results concerning uniqueness.

THEOREM 2.4. *Let X be strictly convex normed space. Then*
  (1) *the set $U(Z)$ is dense in X for any $Z \subset X$,*
  (2) *if X is Banach and Z is relatively boundedly compact, then $X \setminus U(Z)$ is of the first Baire category,*
  (3) *if X is Banach and Z is closed and relatively boundedly compact, then $X \setminus U(Z)$ is of the first Baire category and a G_δ set.*

The proof is based on the following simple remark.

LEMMA 2.5. *Let X be a strictly convex normed space, Z a nonempty subset of X, and $x_0 \in X \setminus Z$.*
  If $z_0 \in P_Z(x_0)$, then $P_Z(x) = \{z_0\}$ for every $x \in [z_0; x_0)$.

Proof. The assertion is obviously true for $x = z_0$. Let $x \in (z_0; x_0)$ and suppose that there exists $z \in Z \setminus \{z_0\}$ such that $\|x - z\| \le \|x - z_0\|$. If $z \in (z_0; x)$, then $\|x - z\| < \|x - z_0\|$, a contradiction. If z does not lie on the line $D(z_0, x_0)$, then, by the strict convexity of X, we obtain again a contradiction:

$$\|x_0 - z\| < \|x_0 - x\| + \|x - z\| \le \|x_0 - x\| + \|x - z_0\| = \|x - z_0\|. \qquad (2.6)$$
$\square$

Proof of Theorem 2.4.

Proof of assertion (1). If $x \notin U(Z)$, then $P_Z(x)$ contains at least two distinct points z_1, z_2. By Lemma 2.5, $[z_1; x) \subset U(Z)$, implying the density of $U(Z)$ in X.

Because $d(x, Z) = d(x, \overline{Z})$, we have $X \setminus U(Z) \subset X \setminus U(\overline{Z})$, so that assertion (2) is a consequence of assertion (3).

Proof of assertion (3). Let Z be a nonempty boundedly compact subset of the strictly convex space X and let

$$D(x) = \operatorname{diam} P_Z(x) = \sup\{\|z' - z''\| : z', z'' \in P_Z(x)\}. \qquad (2.7)$$

Put $Y = X \setminus U(Z)$ and let

$$Y_n = \left\{ x \in X : D(x) \geq \frac{1}{n} \right\}, \quad n \in \mathbb{N}. \tag{2.8}$$

Since $Y = \bigcup_{n=1}^{\infty} Y_n$, it suffices to prove that each set Y_n is closed and nowhere dense.

Y_n is closed. Let (x_k) be a sequence in Y_n converging to an element $x \in X$. Because Z is boundedly compact, the set $P_Z(x_k) = Z \cap B(x_k, d(x_k, Z))$ is compact for every $k \in \mathbb{N}$, so that there exist $z_k', z_k'' \in P_Z(x_k)$ such that

$$\|z_k' - z_k''\| = \operatorname{diam} P_Z(x_k) \geq \frac{1}{n}. \tag{2.9}$$

The inequalities $\|z_k' - x\| \leq \|z_k' - x_k\| + \|x_k - x\|$, $k \in \mathbb{N}$, and similarly for z_k'', show that the sequences (z_k') and (z_k'') are bounded. The set Z being boundedly compact, they will contain convergent subsequences. Therefore, without restricting the generality, we can suppose that there exist $z', z'' \in Z$ such that $z_k' \to z'$ and $z_k'' \to z''$ for $k \to \infty$, and $\|z_k' - z_k''\| \geq 1/n$, $k \in \mathbb{N}$. But then

$$\|z' - x\| = \lim_{k \to \infty} \|z_k' - x_k\| = \lim_{k \to \infty} d(x_k, Z) = d(x, Z), \tag{2.10}$$

and similarly for z_k'' and z''. It follows $z', z'' \in P_Z(x)$ and $\operatorname{diam} P_Z(x) \geq \|z' - z''\| \geq 1/n$, showing that $x \in Y_n$, that is, the set Y_n is closed.

The set Y_n is nowhere dense. As the set Y_n is closed, this is equivalent to $\operatorname{int} Y_n = \varnothing$, where by $\operatorname{int} A$ we denote the interior of a set A. To prove this, we will use again Lemma 2.5. If $x \in Y_n$ then there are $z', z'' \in P_Z(x)$ with $\|z' - z''\| \geq 1/n$. But then, $[z'; x) \subset U(Z) \subset X \setminus Y_n$, so that Y_n does not contain any ball with center x.

Theorem 2.4 is completely proved. $\qquad \square$

The following theorem shows that the density result in the above theorem characterizes in fact the strict convexity of X.

THEOREM 2.6. *If X is Banach and $\overline{U(Z)} = X$ for every $Z \subset X$, then X is strictly convex.*

Proof. If the space X is strictly convex, then, by Proposition 2.1, the set $U(Z)$ is dense in X for any nonempty subset Z of X.

Suppose now that X is not strictly convex, and let $x_1, x_2 \in S_X$, $x_1 \neq x_2$, such that $\|x_1 + x_2\| = 2$. Put $x_0 = (x_1 + x_2)/2$ and let $x^* \in X^*$ be such that $\|x^*\| = 1$ and $x^*(x_0) = \|x_0\| = 1$. It follows $x^*(x_1) = x^*(x_2) = 1$. The hyperplane $H = \{x \in X : x^*(x) = 1\}$ is proximinal, because x^* attains its norm on S_X (at x_0). By the well known formula for the distance to a hyperplane, for every $x \notin H$, $d(x, H) = |x^*(x) - 1|$ and $h_k = x - (x^*(x) - 1)x_k \in P_H(x)$, $k = 1, 2$. It follows that $U(H) = H$ is not dense in X. $\qquad \square$

In the presence of local uniform convexity and of uniform convexity, one obtains better topological properties of the sets $U(Z)$ and $EU(Z)$.

THEOREM 2.7. *If X is locally uniformly convex Banach space, then $X \setminus U(Z)$ is of the first Baire category for any $Z \subset X$.*

In order to present the result in uniformly convex spaces, we recall the notion of well posedness. For $Z \subset X$ and $x \in X$, denote by $\min(x,Z)$ the problem of best approximation of x by elements in Z: find $z_0 \in Z$ such that $\|x - z_0\| = d(x,Z)$. A sequence (z_n) in Z is called a *minimizing sequence* if $\lim_n \|x - z_n\| = d(x,Z)$. It is clear that minimizing sequences always exist. We say that the problem $\min(x,Z)$ is *well posed* if it has a unique solution $z_0 \in Z$ and every minimizing sequence converges to z_0. A thorough presentation of well posedness in various problems of optimization and optimal control is given in the monograph by Dontchev and Zolezzi [23].

In uniformly convex Banach spaces, the following remarkable result holds.

THEOREM 2.8. *If X is a uniformly convex Banach space, then for every nonempty closed subset Z of X, the complement of the set of all $x \in X$ for which the problem $\min(x,Z)$ is well posed is of the first Baire category.*

For the proofs of these theorems, we need some results on some special subsets of locally uniformly convex and of uniformly convex normed spaces.

Let $x_0 \in X$, $r > 0$, $0 < \alpha \le 1/2$, $\beta > 0$, and $z \in S(x_0,r)$. Put $x_1 = x_0 + \alpha(z - x_0)$ and let

$$M_\beta(x_0,z;\alpha) = B(x_1,(1-\alpha)r+\beta) \setminus B'(x_0,r). \tag{2.11}$$

Following De Blasi et al. [19], we call a set of this kind a *lens*, a term suggested by its geometric shape in the Euclidean space $\mathbb{R}^3$.

The inequality from the following proposition will be the key tool in the proofs of uniqueness and existence results in locally uniformly and uniformly convex space.

PROPOSITION 2.9. *Let $x_1 = x_0 + \alpha(x_0 - z_0)$ and, for $y \in M_\beta(x_0,z_0;\alpha)$, let $z = x_1 + (1-\alpha)r(y - x_1)/\|y - x_1\|$. Then the following inequality holds:*

$$\|y - x_0\| \le r + \beta - 2\alpha r\delta, \tag{2.12}$$

where

$$\delta = \delta_X\left(\frac{\|z - z_0\|}{(1-\alpha)r}, \frac{z_0 - x_1}{(1-\alpha)r}\right) \quad or \quad \delta = \delta_X\left(\frac{\|z - z_0\|}{(1-\alpha)r}\right), \tag{2.13}$$

according to the fact that X is locally uniformly convex or uniformly convex.

Proof. Suppose that X is locally uniformly convex. Since $\|z_0 - x_1\| = \|z - x_1\| = (1-\alpha)r$, we have

$$\|z + z_0 - 2x_1\| \le 2(1-\alpha)r(1-\delta), \tag{2.14}$$

where

$$\delta = \delta_X\left(\frac{\|z - z_0\|}{(1-\alpha)r}, \frac{z_0 - x_1}{(1-\alpha)r}\right). \tag{2.15}$$

Writing

$$y - x_0 = y - z + z - x_1 + x_1 - x_0 = (y - z) + (z - x_1) + \frac{\alpha}{1 - \alpha}(z_0 - x_1)$$
$$= (y - z) + \left(1 - \frac{\alpha}{1 - \alpha}\right)(z - x_1) + \frac{\alpha}{1 - \alpha}(z + z_0 - 2x_1), \tag{2.16}$$

we obtain

$$\|y - x_0\| \leq \beta + \left(1 - \frac{\alpha}{1 - \alpha}\right)(1 - \alpha)r + \frac{2\alpha}{1 - \alpha}(1 - \alpha)r(1 - \delta) = r + \beta - 2\alpha r\delta. \tag{2.17}$$

The proof in the uniformly convex case proceeds in the same way, working with $\delta = \delta_X(\|z - z_0\|/(1 - \alpha)r)$. $\qquad\square$

PROPOSITION 2.10. (1) *If X is locally uniformly convex, then*

$$\lim_{\beta \searrow 0} \operatorname{diam} M_\beta(x_0, z; \alpha) = 0. \tag{2.18}$$

(2) *If X is uniformly convex, then for every $\epsilon > 0$, there exists $\gamma = \gamma(\epsilon, \alpha)$ such that*

$$\operatorname{diam} M_\beta(x_0, z; \alpha) < \epsilon \tag{2.19}$$

for all β, $0 < \beta \leq \gamma$, and all $z \in S_X$, that is, $\operatorname{diam} M_\beta(x_0, z; \alpha)$ tends to zero for $\beta \searrow 0$, uniformly with respect to $z \in S_X$.

Proof. (1) Suppose, on the contrary, that there exists $\epsilon > 0$ and a strictly decreasing sequence of real numbers $\beta_n \searrow 0$ such that

$$\operatorname{diam} M_{\beta_n}(x_0, z; \alpha) > 2\epsilon \tag{2.20}$$

for all $n \in \mathbb{N}$. Put $M_n = M_{\beta_n}(x_0, z; \alpha)$ and choose $y_n \in M_n$ such that $\|y_n - z_n\| > \epsilon$. If $z_n' = x_1 + (1 - \alpha)r\|y - x_1\|^{-1}(y - x_1)$, then

$$\|z - z_n'\| \geq \|z - y_n\| - \|y_n - z_n'\| \geq \epsilon - \beta_n > \frac{\epsilon}{2}, \tag{2.21}$$

whenever $\beta_n < \epsilon/2$, which holds for sufficiently large n. By Proposition 2.9, we have

$$\|y_n - x_0\| \leq r + \beta_n - 2\alpha r\delta, \tag{2.22}$$

where

$$\delta = \delta_X\left(\frac{\epsilon}{2(1 - \alpha)r}, \frac{z - x_1}{(1 - \alpha)r}\right). \tag{2.23}$$

Since $\beta_n \searrow 0$ and δ do not depend on β_n, it follows that $\|y_n - x_0\| < r$ for sufficiently large n, in contradiction to $y_n \in M_n \subset X \setminus B'(x_0, r)$ (see the definition (2.11) of $M_n = M_{\beta_n}(x_0, z; \alpha)$).

(2) Proceeding again by contradiction, suppose that there exist $\epsilon > 0$ and the sequences (z_n) in $S(x_0, r)$ and (β_n) in $\mathbb{R}_+$ with $\beta_n \searrow 0$, such that

$$\operatorname{diam} M_{\beta_n}(x_o, z_n; \alpha) > 2\epsilon, \tag{2.24}$$

for all $n \in \mathbb{N}$. We can suppose $\beta_n < \epsilon/2$. Taking

$$\delta = \delta_X\left(\frac{\epsilon}{2(1-\alpha)r}\right), \tag{2.25}$$

and reasoning as above, we obtain

$$\|y_n - x_0\| \le r + \beta_n - 2\alpha r \delta. \tag{2.26}$$

It follows $\|y_n - x_0\| < r$ for sufficiently large n, in contradiction to $y_n \notin B'(x_0, r)$. □

De Blasi et al. [19, Lemma 2.1] gave a quantitative version of Proposition 2.10 and called it *Stečkin's lens lemma*.

Let X be a uniformly convex space with modulus of uniform convexity δ_X. For $\sigma \in (0; 1]$, define $\delta^*(\sigma)$ by

$$\delta^*(\sigma) = \sup\{\epsilon : 0 < \epsilon \le 2,\ \delta_X(\epsilon) \le \sigma\}. \tag{2.27}$$

It is immediate that $0 < \delta^*(\sigma) \le \delta^*(\sigma')$, for $0 < \sigma \le \sigma' \le 1$, and $\delta^*(\sigma) \to 0$ for $\sigma \searrow 0$.

PROPOSITION 2.11. *If X is a uniformly convex space, $x_0 \in X$, $r > 0$, $0 < \alpha \le 1/2$, $\beta > 0$, and $z \in S(x_0, r)$, then*

$$\operatorname{diam} M_\beta(x_0, z; \alpha) \le 2\beta + 2(1-\alpha)r\delta^*\left(\frac{\beta}{2\alpha r}\right), \tag{2.28}$$

where $M_\beta(x_0, z; \alpha)$ is the lens defined by (2.11).

Proof. Let $y \in M_\beta(x_0, z; \alpha)$ be such that $\|y - z\| > \beta$. If $z' = x_1 + (1-\alpha)r(y - x_1)/\|y - x_1\|$, then

$$\|z - z'\| \ge \|z - y\| - \|y - z'\| > \beta - \beta = 0. \tag{2.29}$$

By Proposition 2.9,

$$r \le \|y - x_0\| \le r + \beta - 2r\alpha\delta_X\left(\frac{\|z - z'\|}{2\alpha r}\right), \tag{2.30}$$

implying

$$\delta_X\left(\frac{\|z - z'\|}{2\alpha r}\right) \le \frac{\beta}{2\alpha r}. \tag{2.31}$$

By the definition of δ^*, this inequality yields

$$\frac{\|z - z'\|}{2\alpha r} \le \delta^*\left(\frac{\beta}{2\alpha r}\right), \tag{2.32}$$

so that

$$\|z - z'\| \leq (1 - \alpha)r\delta^* \left(\frac{\beta}{2\alpha r} \right),$$

$$\|y - z\| \leq \|y - z'\| + \|z' - z\| \leq \beta + (1 - \alpha)r\delta^* \left(\frac{\beta}{2\alpha r} \right).$$

(2.33)

Since this inequality trivially holds for $\|y - z\| \leq \beta$, it follows

$$\|y_1 - y_2\| \leq \|y_1 - z\| + \|z - y_2\| \leq 2\beta + 2(1 - \alpha)r\delta^* \left(\frac{\beta}{2\alpha r} \right)$$

(2.34)

for all $y_1, y_2 \in M_\beta(x_0, z; \alpha)$. $\qquad\square$

For $x \in X$, $Z \subset X$, and $\epsilon > 0$, let

$$d(x) = d(x, Z), \qquad D(x) = \operatorname{diam} P_Z(x);$$

$$P_Z^\epsilon(x) = \{z \in Z : \|x - z\| \leq \epsilon + d(x)\} = Z \cap B(x, \epsilon + d(x));$$

$$D_\epsilon(x) = \operatorname{diam} P_Z^\epsilon(x), \qquad D_0(x) = \lim_{\epsilon \searrow 0} D_\epsilon(x).$$

(2.35)

The quantities $D_\epsilon(x)$ and $D_0(x)$ are closely related to the well posedness of the problem of best approximation.

The following proposition is taken from De Blasi and Myjak [17].

PROPOSITION 2.12. *If Z is a nonempty complete subset of a normed space X, $x \in X$, and $D_0(x) = 0$, then the problem $\min(x, Z)$ is well posed.*

Conversely, if Z is a nonempty subset of X such that the problem $\min(x, Z)$ is well posed, then $D_0(x) = 0$.

Proof. Suppose that the set Z is complete and $\lim_{\epsilon \searrow 0} D_\epsilon(x) = 0$. If (z_n) is a minimizing sequence, then for every $\epsilon > 0$, there exists n_ϵ such that $z_n \in P_Z^\epsilon(x)$ for all $n \geq n_\epsilon$, implying $\|z_n - z_m\| \leq \operatorname{diam} P_Z^\epsilon(x) = D_\epsilon(x)$ for all $n, m \geq n_\epsilon$. It follows that the sequence (z_n) is fundamental, so that, by the completeness of Z, it converges to an element $z_0 \in Z$ which satisfies $\|x - z_0\| = d(x, Z)$. If x has two distinct elements of best approximation z_0, z_0' in Z, then $z_0, z_0', z_0 \cdot z_0', \dots$ is a minimizing sequence which is not fundamental, in contradiction with the fact we have just proved.

Suppose now that the problem $\min(x, Z)$ is well posed. If $z_n, z_n' \in P_Z^{1/n}(x)$, then the sequences (z_n) and (z_n') are both minimizing and converge to z_0—the solution of the problem, so that $\|z_n - z_n'\| \to 0$. It follows $\operatorname{diam} P_Z^{1/n}(x) \to 0$ for $n \to \infty$, and $\lim_{\epsilon \searrow 0} \operatorname{diam} P_Z^\epsilon(x) = 0$. $\qquad\square$

Consider the sets

$$Y_n = \left\{ x \in X : D(x) \geq \frac{1}{n} \right\}, \qquad F_n = \left\{ x \in X : D_0(x) \geq \frac{1}{n} \right\}.$$

(2.36)

The sets Y_n were used in the proof of Theorem 2.4. Because $P_Z(x) \subset P_Z^\epsilon(x)$, we have $D(x) \leq D_\epsilon(x)$, so that $D(x) \leq D_0(x)$. It follows $Y_n \subset F_n$, and $Y_n = F_n$ if Z is boundedly compact.

The sets F_n have a better topological behavior than the sets Y_n.

PROPOSITION 2.13. *If X is a Banach space, then the set F_n is closed for every $n \in \mathbb{N}$.*

Proof. Let (x_k) be a sequence in F_n converging to an element $x_0 \in X$. For a given $\epsilon > 0$, let $k_0 \in \mathbb{N}$ be such that $\|x_k - x_0\| \le \epsilon/3$ for all $k \ge k_0$.

Then for all $k \ge k_0$,

$$d(x_k) \le \|x_k - x_0\| + d(x_0) \le \frac{\epsilon}{3} + d(x_0), \tag{2.37}$$

implying

$$B\left(x_k, \frac{\epsilon}{3} + d(x_k)\right) \subset B\left(x_k, \frac{2\epsilon}{3} + d(x_0)\right) \subset B(x_0, \epsilon + d(x_0)),$$
$$P_Z^{\epsilon/3}(x_k) \subset P_Z^{\epsilon}(x_0). \tag{2.38}$$

It follows

$$\frac{1}{n} \le D_{\epsilon/3}(x_k) \le D_{\epsilon}(x_0). \tag{2.39}$$

Letting $\epsilon \searrow 0$, we get $D_0(x_0) \ge 1/n$, that is, $x_0 \in F_n$. $\qquad\square$

Now we are able to prove Theorem 2.7.

Proof of Theorem 2.7. Let Z be a nonempty subset of the locally uniformly convex space X. We have $X \setminus U(Z) = \bigcup_{n=1}^{\infty} Y_n$, so it suffices to show that each Y_n is nowhere dense in X, that is, that $\mathrm{int}\, \overline{Y_n} = \varnothing$ for every $n \in \mathbb{N}$. This is equivalent to the fact that $\overline{Y_n}$ is not a neighborhood of any point $x \in X$. In its turn, this is equivalent to

$$\forall x \in X, \ \forall r > 0, \quad \exists y \in B(x,r), \ \exists r' > 0 \text{ such that } B(y, r') \cap Y_n = \varnothing. \tag{2.40}$$

Let $x \in X$ and $r > 0$. If $B(x,r) \cap Y_n = \varnothing$, then there is nothing to prove. Suppose that there is an element $x_0 \in B(x,r) \cap Y_n$. It follows $x_0 \notin Z$.

If $z_0 \in P_Z(x_0)$ and $y \in (x_0; z_0)$, then, by Lemma 2.5, $P_Z(y) = \{z_0\}$ and $d(x, Z) = \|x_0 - y\| + d(y, Z)$. For $\epsilon > 0$,

$$P_Z^{\epsilon}(y) \subset B(y, \epsilon + d(y, Z)) \setminus B'(x_0, d(x_0, Z)) = M_{\epsilon}(x_0, z_0; \alpha), \tag{2.41}$$

where $\alpha = \|y - x_0\|/\|z_0 - x_0\|$.

We have $D_{\epsilon}(y) = \mathrm{diam}\, P_Z^{\epsilon}(y) \le \mathrm{diam}\, M_{\epsilon}(x_0, z_0; \alpha)$, so that, by Proposition 2.10, $D_{\epsilon}(y) \to 0$ for $\epsilon \searrow 0$. It follows $D_0(y) = 0 < 1/n$, showing that $y \notin F_n$. But, by Proposition 2.13, the set $X \setminus F_n$ is open, so that there exists $r' > 0$ such that $B(y, r') \subset X \setminus F_n \subset X \setminus Y_n$. Theorem 2.7 is proved. $\qquad\square$

Theorem 2.8 is an immediate consequence of the following result from [76] and of Proposition 2.12.

THEOREM 2.14. *If X is a uniformly convex Banach space, then for every nonempty subset Z of X, the set $\{x \in X : D_0(x) = 0\}$ is G_δ and dense in X.*

Proof. If $F_0 = \{x \in X : D_0(x) = 0\}$, then $F_0 = \bigcap_{n=1}^{\infty}(X \setminus F_n)$, so that

$$X \setminus F_0 = \bigcup_{n=1}^{\infty} F_n. \tag{2.42}$$

Since, by Proposition 2.13, every set F_n is closed, we have only to show that the set F_n is nowhere dense for every $n \in \mathbb{N}$, meaning that $B(x,r) \cap (X \setminus F_n) \neq \varnothing$, for every $x \in F_n$ and every $r > 0$. For $x \in F_n$, let $d = d(x,Z) > 0$ and consider a ball $B(x, \alpha d)$, with $0 < \alpha \leq 1/2$. By Proposition 2.10, there exists $\beta_0 > 0$ such that

$$\operatorname{diam} M_\beta(x, y; \alpha) < \frac{1}{n} \tag{2.43}$$

for all β, $0 < \beta \leq 2\beta_0$, and all $y \in S(x, \alpha d)$. Let $z \in P_Z^{\beta_0}(x)$, that is, $z \in Z$ and $d \leq \|x - z\| \leq d + \beta_0$, and put $y = x + \alpha dz/\|z\|$. We have

$$d(y,Z) \leq \|y - z\| = \|x - z\| - \|x - y\| \leq (1 - \alpha)d + \beta_0, \tag{2.44}$$

implying

$$P_Z^{\beta_0}(y) \subset M_{2\beta_0}(x, y; \alpha). \tag{2.45}$$

It follows $\operatorname{diam} P_Z^{\beta_0}(y) \leq \operatorname{diam} M_{2\beta_0}(x, y; \alpha) < 1/n$, that is, $y \in B(x, \alpha d) \cap (X \setminus F_n)$. $\square$

The following problems were raised in Stečkin's paper [76]. A normed space X is called *strongly convex* if

$$\lim_{t \to d} \operatorname{diam}(Z \cap tB_X) = 0 \tag{2.46}$$

for any convex subset Z of X, where $d = d(0,Z)$ and B_X denotes the closed unit ball of X.

Stečkin's problems. (1) Does assertion (2) of Theorem 2.4 remain true for an arbitrary subset Z of X? Also, if X is Banach and $\overline{U(Z)} = X$ (or $X \setminus U(Z)$ is of the first Baire category) for every compact subset Z, then must X be strictly convex?

(2) Is Theorem 2.7 true for strongly convex Banach spaces?

(3) Is the conclusion of Theorem 2.8 true for locally uniformly convex Banach spaces?

We mention also some related existence results obtained by Borwein and Fitzpatrick [9].

THEOREM 2.15 (see [9]). *Let X be a reflexive Banach space. Then the following assertions hold:*

(1) *the complement of every open convex proper subset of X is almost proximinal,*

(2) *if $\{U_i : i \in I\}$ is a collection of mutually disjoint open convex subsets of X, then the set $X \setminus \bigcup\{U_i : i \in I\}$ is almost proximinal in X, provided it is nonempty.*

Approximative compactness and Efimov-Stečkin spaces. A reflexive Banach space having the Kadec-Klee property is called an *Efimov-Stečkin (ES) space.*

A point $x \in X$ is called a point of *approximative compactness* for $Z \subset X$ if every minimizing sequence for $d(x,Z)$ contains a convergent subsequence. The set of points of approximative compactness for the set Z is denoted by $AC(Z)$. The set Z is called *approximatively compact* if $AC(Z) = X$.

The notion of approximative compactness was introduced by Efimov and Stečkin [31], in connection with the problem of convexity of Chebyshev sets. It was also used by Reich [68] to study continuity properties of the metric projection with applications to fixed point theorems of Ky Fan type.

Although, at the first glance, this notion looks a little tautological, it turned to be a very useful one in the study best approximation problems. It differs from the notion of compactness: Borodin [6] has shown that every separable infinite-dimensional Banach space contains a bounded approximatively compact set which is not compact. In [7], he showed that every reflexive Banach space contains a noncompact bounded convex approximatively compact set.

Efimov and Stečkin proved the following.

THEOREM 2.16. *Let X be a uniformly convex smooth Banach space and Z a Chebyshev subset of X.*

The set Z is convex if and only if it is approximatively compact.

The term Efimov-Stečkin space was proposed by Singer [74].

There are several conditions equivalent to the *ES* property.

THEOREM 2.17 (see Singer [74] and Konyagin and Tsar'kov [49]). *For a Banach space X, the following conditions are equivalent:*
   (1) *X is an ES space;*
   (2) *X is reflexive and Kadec-Klee;*
   (3) *any sequentially closed subset of X is compact;*
   (4) *every weakly closed subset of X is compact.*

There are also some characterizations of *ES* spaces in terms of approximative compactness.

THEOREM 2.18 (see Singer [74] and Konyagin and Tsar'kov [49]). *For a Banach space X, the following conditions are equivalent:*
   (1) *X is an ES space;*
   (2) *every weakly sequentially closed subset of X is approximatively compact;*
   (3) *every weakly closed subset of X is approximatively compact;*
   (4) *every closed convex set is approximatively compact;*
   (5) *every weakly sequentially closed set is an existence set with lower semicontinuous metric projection;*
   (6) *every weakly closed set is an existence set with lower semicontinuous metric projection;*
   (7) *every weakly sequentially closed set is an existence set with lower semicontinuous metric projection;*
   (8) *every closed convex set is an existence set with lower semicontinuous metric projection.*

Condition (2) from the above theorem was taken by Singer [74] as the definition of *ES* spaces.

ES spaces are also important in the study of generic existence for best approximation. Denote by $\mathcal{P}_{\mathrm{cl}}(X)$ the family of all nonempty closed subsets of a normed space X.

Theorem 2.19 (see Konyagin and Tsar'kov [49]). *For a Banach space X, the following conditions are equivalent:*

(1) *X is an ES space;*
(2) *for all $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, the set $AC(Z)$ is residual in X;*
(3) *for all $Z \in \mathcal{P}_{\mathrm{cl}}B(X)$, the set $AC(Z)$ is dense in X;*
(4) *for all $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, the set $E(Z)$ is residual in X;*
(5) *for all $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, the set $E(Z)$ is dense in X;*
(6) *for all $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, the set $E(Z)$ is weakly dense in X;*
(7) *for all $Z \in \mathcal{P}_{\mathrm{cl}}B(X)$, the set $AC(Z)$ is connex;*
(8) *for all $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, the set $E(Z)$ connex.*
(9) *for every $Z \in \mathcal{P}_{\mathrm{cl}}(X)$, there exists a residual set A such that $A \subset E(Z)$ and the set-valued map $P_Z|A$ is continuous;*
(10) *for every $Z \in \mathcal{P}_{\mathrm{cl}}B(X)$, there exists a residual set B such that the set-valued map P_Z has a continuous selection on B.*

Strongly convex spaces. This notion was introduced by Šmulian [75] and, for further properties, one can consult also the paper by Fan and Glicksberg [32]. As we have already mentioned, Stečkin used it in his paper [76] on approximation properties of Banach spaces. Recall that a Banach space X is called strongly convex if $\lim_{t \to d} \mathrm{diam}(Z \cap tB_X) = 0$, for any convex subset Z of X.

Sometimes the strong convexity (or strong rotundity) is denoted by (K), the fact that X is Banach and satisfies (K) is denoted by (D), and the Kadec-Klee property by (H). In fact, in [32], there is a list of geometric properties that a normed space could satisfy, labelled from (A) to (H), and the Kadec-Klee property is labelled as (H).

The following proposition puts in evidence some connections of this notion with other geometric properties of Banach spaces.

Theorem 2.20. *The following assertions hold:*

(1) *if X is uniformly convex, then X is strongly convex;*
(2) *if X is reflexive and locally uniformly convex, then X is strongly convex;*
(3) *if X is strongly convex, then X is strictly convex and Kadec-Klee,*
(4) *if X is Banach, then X is strongly convex if and only if it is reflexive strictly convex and Kadec-Klee.*

For a good presentation of these results and of other geometric properties of Banach spaces, one can consult, for instance, Megginson [58, Section 5.3].

The following theorem contains some equivalent characterizations of strongly convex spaces.

Theorem 2.21. *For a normed space X, the following conditions are equivalent:*

(1) *X is strongly convex;*
(2) *for every $x^* \in S_{X^*}$, $\lim_{\epsilon \to 0+} \mathrm{diam}\{x \in B_X : x^*(x) \geq 1 - \epsilon\} = 0$;*
(3) *if $(x_n) \subset S_X$ and $x^*(x_n) \to 1$, for some $x^* \in S_{X^*}$, then (x_n) is a Cauchy sequence;*

(4) *for every convex subset Z of X and $x \in X$, every minimizing sequence $(z_n) \subset Z$ is a Cauchy sequence.*

If X is Banach, then the above conditions are also equivalent to the following one:

(5) *every nonempty closed convex subset of X is an approximatively compact Chebyshev set.*

Some definitive results concerning the problems raised by Stečkin were obtained by Konyagin [42, 43, 44, 45, 46, 47] and Konyagin and Tsar'kov [49]. A good account is given in the survey paper [48]. In the following theorem, we collect some of these results.

THEOREM 2.22. *For a Banach X, the following conditions are equivalent:*
   (1) *for all $Z \subset X$, the set $E(Z)$ is dense in X;*
   (2) *for all $Z \subset X$, the set $E(Z)$ is residual in X;*
   (3) *for all $Z \subset X$, the set $AC(Z)$ is dense in X;*
   (4) *for all $Z \subset X$, the set $AC(Z)$ is residual in X;*
   (5) *X is an ES space.*

The implication (5)⇒(1) was proved by Lau [51] and the implication (1)⇒(5) by Konyagin [42]. The others are trivial.

Konyagin [42] (see also [9]) proved the following result.

THEOREM 2.23. *Let X be a Banach space that is neither reflexive nor Kadec-Klee. Then there exists a closed bounded set Z of X and an open nonempty subset U of $X \setminus Z$ such that*
   (i) *no $x \in U$ has a nearest point in Z;*
   (ii) *the distance function $d(\cdot, Z)$ is affine on U, so that, in particular, $d(\cdot, Z)$ is Fréchet differentiable on U.*

Konyagin [42] has also proved.

THEOREM 2.24. *For a Banach space X, the following conditions are equivalent:*
   (1) *for all $Z \subset X, EU(Z)$ is dense in X;*
   (2) *for all $Z \subset X, EU(Z)$ is residual in X;*
   (3) *X is strictly convex and ES.*

A point x of a closed bounded convex subset Z of a Banach space X is called an *exposed point* of Z if x is the only point of Z for which $x^*(x) = \sup x^*(Z)$, for some $x^* \in X^*$, $x^* \neq 0$. One says that the functional x^* *exposes* the point x. The point x is called a *strongly exposed point* of Z if it is an exposed point of Z with exposing functional x^* and, for every sequence (z_n) in Z such that $x^*(z_n) \to \sup x^*(Z)$, we have $z_n \to x$ in the norm topology of X. This is equivalent to diam $S(Z, x^*, \epsilon) \to 0$ for $\epsilon \to 0+$, where, for $\beta = \sup x^*(Z)$, $S(Z, x^*, \epsilon) = \{z \in Z : x^*(z) > \beta - \epsilon\}$. The notion of exposed point is due to Straszewicz [77].

The notion of exposed point is closely related to two other geometric properties of Banach spaces—the Radon-Nikodým property and Asplund property. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, that is, Ω is a set, $\mathcal{A}$ a σ-algebra of subsets of Ω, and $\mu : \mathcal{A} \to [0; \infty)$ a σ-additive finite positive measure on $\mathcal{A}$. One says that a Banach space X has the *Radon-Nikodým* property with respect to the measure space $(\Omega, \mathcal{A}, \mu)$ if for every σ-additive vector measure $G : \mathcal{A} \to X$, there exists $g \in L^1(\mu, X)$ such that $G(A) = \int_A g \, d\mu, A \in \mathcal{A}$. The

Banach space X has the *Radon-Nikodým property* if it has the Radon-Nikodým property with respect to every finite measure space $(\Omega, \mathcal{A}, \mu)$. The classic on Radon-Nikodým property is the book by Diestel and Uhl [22] (see also [11]). A Banach space X is said to have the *Asplund property* if every continuous convex function defined on a nonempty open subset Z of X is Fréchet differentiable on a dense G_δ subset of Z. The term was chosen to honor Asplund [2], who made important contributions to the subject. A good presentation of the results concerning differentiability properties of convex functions on Banach spaces can be found in the books [33, 67].

Denote by $\mathrm{near}(Z)$ the set of all $z \in Z$ such that $\|x - z\| = d(x, Z)$ for some $x \in X$. Let also $\mathrm{spt}(Z)$ denote the set of all support points of the set Z, that is, those points $z \in Z$ for which there exists a nonzero $x^* \in X^*$ such that $x^*(z) = \sup x^*(Z)$. It follows $\mathrm{near}(Z) \subset \mathrm{spt}(Z) \subset \mathrm{bd}(Z)$.

The problem of the existence of convex sets Z such that $\mathrm{near}(Z) \neq \mathrm{spt}(Z)$ was investigated by Godini [34], Borwein and O'Brien [10], and Borwein [8].

THEOREM 2.25. *If the Banach space X has the Radon-Nikodým property, then*
(1) $\overline{\mathrm{co}}(\mathrm{near}(Z)) = Z$ *for every closed convex subset Z of X,*
(2) *the same relation holds for any weakly compact convex subset Z of an arbitrary Banach space X.*

Borwein and O'Brien [10] constructed an example of a proximinal set Z in a nonreflexive Banach space X such that $\mathrm{bd}(Z) = \mathrm{spt}(Z) \neq \mathrm{near}(Z)$, where bd stands for the topological boundary of a set in a topological space.

A Banach space X is said to have the property (σ) if every closed bounded convex subset of X is the closed convex hull of its strictly exposed points. A remarkable result in the Banach space theory says that a Banach space X has the property (σ) if and only if it has the Radon-Nikodým property (see [67, Theorem 5.21]). A point x^* of a subset Z of a conjugate Banach space $Y = X^*$ is called strongly w^*-exposed if there exists $x \in X$ such that $x^*(x) = \sup Z(x)$ and if (z_n^*) in Z satisfies $z_n^*(x) \to x^*(x)$, then $z_n^* \to x^*$ in the norm topology. We mention also the following important result: a Banach space X is an Asplund space if and only if every nonempty w^*-compact convex subset of X^* is the w^*-closed convex hull of its strongly w^*-exposed points (see [67, Theorem 5.12]). Since a Banach space X is an Asplund space if and only if its dual space X^* has the Radon-Nikodým property (see [67, Theorem 5.7]), it follows that the dual of an Asplund space has the property (σ), a result contained in [25, Proposition 2.2].

Some interesting connections between the existence of nearest points and of strongly exposed points was established by Edelstein [25].

THEOREM 2.26. *If Z is a closed bounded convex subset of a Banach space having the property (σ), then the set $E(Z)$ of all $x \in X$ having a nearest point in Z is weakly dense in X.*
Moreover, $E(Z)$ contains infinitely many rays emanating from points of Z.

The actual status of Stečkin's problems.   Concerning problem (3), it was shown by Cobzaş [12, 13] (see also [26]) that the space c_0 equipped with Day's locally uniformly convex norm contains a closed bounded symmetric antiproximinal convex body Z, that is, such that no point outside Z has a nearest point in Z, showing that Theorem 2.8 does not hold, in general, for locally uniformly convex Banach spaces. A complete solution to this

problem and to the related one, problem (2), was given by Konyagin: strongly convex Banach spaces are exactly those for which Theorem 2.8 is true.

Problem (2) is still open. We will expose, following [48], some progress made towards the solution of "the most intriguing of the Stečkin's problems," as Konyagin calls it in [48], and on the closely related problem (1). There are some particular cases when the answer to problem (2) is yes as, for instance, if the strongly convex space X is separable (see [41, 79]), or if there exists a Fréchet differentiable bump function on X (see [42]). These results were obtained independently by Zhivkov [83, 84], who proved that the answer is yes for a class of strictly convex Banach spaces, containing the weakly compactly generated Banach spaces. Stečkin [76] proved that if Z is a boundedly compact subset of a strictly convex Banach space X, then $X \setminus U(Z)$ is of the first Baire category. Kenderov [38, 39, 40] extended this result to more general classes of subsets by proving it for approximatively compact subsets of a strictly convex Banach space. Konyagin [42] proved that in a strictly convex Banach space, the set $AC(Z) \setminus U(Z)$ is of the first Baire category for any subset Z, but this property does not characterize strictly convex Banach spaces, because, as it was shown by Kamuntavichyus [36], this property holds in the space $L^1(\Omega, \mathcal{A}, \mu)$ with a nonatomic measure μ, and these spaces are not strictly convex.

3. The problem of farthest points

Generic existence of farthest points for closed bounded sets was proved by Edelstein [24] for uniformly convex spaces, by Asplund [1] for reflexive locally uniformly convex spaces, and by Lau [50] for weakly compact subsets of arbitrary Banach spaces.

THEOREM 3.1 (see Lau [50]). *If Z is a weakly compact subset of a Banach space X, then the set $X \setminus e(Z)$ is of the first Baire category.*

Deville and Zizler [21] proved a kind of converse of this result.

THEOREM 3.2. *Let X be a Banach space and Z a closed convex subset of X. If the set Z is almost $\| \cdot \|_1$-remotal for any equivalent norm $\| \cdot \|_1$ on X, then the set Z is weakly compact.*

Edelstein and Lewis [27] have shown that there exists a closed bounded convex subset Z of ℓ_2 such that no point in the dense subspace ℓ_0, of all finitely supported sequences, has a farthest point in Z. Also no point in ℓ_0 has a farthest point in $Z \cap \ell_0$.

Interesting connections between the existence of farthest points and Mazur intersection property were established. One says that a Banach space X has the *Mazur intersection property* (MIP in short) if any bounded closed convex subset of X can be written as an intersection of closed balls. Mazur [57] proved that MIP holds in finite-dimensional Euclidean spaces. Phelps [66] gave a dual characterization of finite-dimensional Banach normed spaces which have the MIP and proved that Banach spaces with Fréchet differentiable norm have the MIP, resuscitating the interest to this important geometric property. The property turned out to have very interesting connections with other geometric properties of Banach spaces, from which we will emphasize only those related to the existence of farthest points.

As in the case of nearest points, denote by far(Z) the set of all points $z \in Z$ such that $z \in Q_Z(x)$ for some $x \in X$. Recall that $Q_Z(x)$ denotes the set of farthest points from x in Z.

THEOREM 3.3. *(1) If X has MIP, then* $\overline{\mathrm{co}}(\mathrm{far}(Z)) = Z$ *for any w-compact convex subset Z of X.*

(2) If X is reflexive, then X has MIP if and only if $\overline{\mathrm{co}}\,\mathrm{far}(Z) = Z$ *for any closed convex set* $Z \subset X$.

Remark 3.4. The first assertion of the above theorem was proved by Edelstein [24] and the second one by Lau [50]. Some extensions of these results as well as a discussion on the relations between w^*-MIP and the existence of farthest points of w^*-compact subsets of the dual space X^* of X were done by Bandyopadhyay [4].

Concerning the existence of farthest points of w^*-compact subsets of dual Banach spaces, we mention the following result of Deville and Zizler [21].

THEOREM 3.5. *If the Banach space X has the Radon-Nikodým property, then every* w^*- *compact subset of* X^* *is almost remotal.*

As in the case of nearest points, the strict convexity of the space X guarantees the density of the elements having a unique farthest point in a subset. More exactly, the following theorem holds.

THEOREM 3.6 [21]. *Suppose that X is a strictly convex Banach space and Z is a norm closed bounded subset of X, such that the set e(Z) is dense in X. Then the set eu(Z) is also dense in X.*

Another interesting question is that of the relations between farthest points and exposed points. In a locally uniformly convex Banach space, every farthest point of Z is a strongly exposed point of Z, and this was the method used by Straszewicz [77] to obtain exposed points of sets. Also, in a smooth Banach space X, we have $\exp(Z) \subset \mathrm{far}(Z)$, for any compact convex subset Z of X. The relations between farthest points and exposed point is complicated enough, as was shown by Bernau [5] and Edelstein and Lewis [27]. Bernau [5] has shown that every normed space of dimension greater than one contains a compact convex set Z for which $\exp(Z)$ is not a subset of $\mathrm{far}(Z)$. Edelstein and Lewis [27] constructed two-dimensional compact convex sets Z with $\mathrm{far}(Z) \cap \exp(Z) = \varnothing$.

Deville and Zizler [21] proved that the set

$$Z = \left\{ x \in \ell^1 : \sum_{n=1}^{\infty} \left(|x_n| + |x_n|^2 \right) \le 1 \right\} \tag{3.1}$$

is a w^*-compact convex subset of the Banach space ℓ^1 which has no farthest points in ℓ^1, that is, $Q_Z(x) = \varnothing$ for all $x \in \ell^1$. Since the space ℓ^1 has the Radon-Nikodým property, the set Z is the closed convex hull of its strongly exposed points, so it furnishes an example of a set having strongly exposed points but without farthest points. For results concerning strongly exposed points and Radon-Nikodým property, one can consult the book by Bourgin [11].

A bounded subset Z of a normed space X is called *uniquely remotal* if, for all $x \in X$, $\mathrm{card}\,Q_Z(x) = 1$, that is, any point in X has a unique farthest point in Z. An important problem in the study of farthest points is the following one: under what conditions on the space X is every uniquely remotal subset of X a singleton? This problem is closely

connected with the problem of convexity of Chebyshev sets. Both these problems are unsolved till now, and there are some opinions (see, e.g., [61]) that the solution of one of them will lead to the solution of the other one too. For instance, and this is the most challenging question, it is unknown whether every Chebyshev subset of a Hilbert space must be convex. A new approach was proposed by Ricceri in [73] (see also [37]). Good survey papers on the convexity of Chebyshev sets are those by Vlasov [78], Balaganskiĭ and Vlasov [3], and Narang [60], and, for the problem of farthest points, Narang [59, 61].

4. Other results

De Blasi, Myjak, and Papini [20] extended the problems of nearest and farthest points to subsets $A, Z \subset X$. Let

$$
\begin{aligned}
\lambda_{AZ} &= \inf \{\|x - z\| : x \in A, \ z \in Z\}, \\
\mu_{AZ} &= \sup \{\|x - z\| : x \in A, \ z \in Z\}.
\end{aligned}
\tag{4.1}
$$

The problem $\min(A, Z)$ (resp., $\max(A, Z)$) consists in finding $x_0 \in A$ and $z_0 \in Z$ such that $\|x_0 - z_0\| = \lambda_{AZ}$ (resp., $\|x_0 - z_0\| = \mu_{AZ}$).

Denote by $\mathcal{B}(X)$ the family of all nonempty closed bounded subsets of the normed space X, by $\mathcal{C}(X)$ the family of all nonempty closed bounded convex subsets of X, and by $\mathcal{K}(X)$ the family of all nonempty compact convex subsets of X. If X is a Banach space, then $\mathcal{B}(X)$, $\mathcal{C}(X)$, and $\mathcal{K}(X)$ are all complete with respect to the Pompeiu-Hausdorff metric H (see [35]). For $A \in \mathcal{B}(X)$, let

$$
\mathcal{B}_A(X) = \mathrm{cl}\,\{Z \in \mathcal{B}(X) : \lambda_{AZ} > 0\},
\tag{4.2}
$$

where the closure cl is taken with respect to the Pompeiu-Hausdorff metric H, and let $\mathcal{C}_A(X) = \mathcal{C}(X) \cap \mathcal{B}_A(X)$. In the following, all the topological notions concerning the spaces $\mathcal{B}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$ will be considered with respect to the Pompeiu-Hausdorff metric H.

One says that the problem $\min(A, Z)$ is *well posed* if it has a unique solution (y_0, z_0) and every minimizing sequence (i.e., a sequence $(y_n, z_n) \in A \times Z$ such that $\|y_n - z_n\| \to \lambda_{A,Z}$) converges to (y_0, z_0). The well posedness of the problem $\max(A, Z)$ is defined similarly.

The following results were proved in [20].

THEOREM 4.1. *Let X be a uniformly convex Banach space.*
(1) *For $A \in \mathcal{B}(X)$, the set of all $Z \in \mathcal{C}_A(X)$ for which the problem $\min(A, Z)$ is well posed is G_δ and dense in $\mathcal{C}_A(X)$.*
(2) *For $A \in \mathcal{B}(X)$, the set of all $Z \in \mathcal{C}(X)$, for which the problem $\max(A, Z)$ is well posed is G_δ and dense in $\mathcal{C}(X)$.*

Li [52] proved similar results for the family $\mathcal{K}(X)$ of all nonempty compact convex subsets of a strongly convex Banach spaces X. Recall that, by Theorem 2.20(4), a Banach space is strongly convex if and only if it is reflexive, strictly convex, and Kadec-Klee.

For $A \in \mathcal{B}(X)$, put $\mathcal{K}_A(X) = \mathcal{K}(X) \cap \mathcal{B}_A(X)$.

THEOREM 4.2 [52]. *Let X be a strongly convex Banach space.*

(1) *If A is a nonempty closed subset of X, then the set of all $Z \in \mathcal{K}_A(X)$ for which the problem $\min(A,Z)$ is well posed contains a dense G_δ subset of $\mathcal{K}_A(X)$.*

(2) *If A is a nonempty closed bounded subset of X, then the set of all $Z \in \mathcal{K}(X)$ for which the problem $\max(A,Z)$ is well posed contains a dense G_δ subset of $\mathcal{K}(X)$.*

Li [52] has also shown that the property (1) in the above theorem characterizes strongly convex spaces.

THEOREM 4.3. *Let X be a Banach space. If for every nonempty closed set A the set of all $Z \in \mathcal{K}_A(X)$ for which the problem $\min(A,Z)$ is well posed contains a dense G_δ subset of $\mathcal{K}_A(X)$, then the space X is strongly convex.*

The same is true with "G_δ dense" replaced by "dense" in the above assertion.

Supposing the set A relatively boundedly weakly compact (i.e., its intersection with any ball is relatively weakly compact), Li and Xu [54] succeeded in dispensing with the reflexivity of the space X.

THEOREM 4.4. *Let X be a strictly convex Banach space satisfying the Kadec-Klee condition.*

(1) *If A is a relatively boundedly compact subset of X, then the family of all $Z \in \mathcal{B}_A(X)$ for which the problem $\min(A,Z)$ is well posed is a G_δ dense subset of $\mathcal{B}_A(X)$.*

The same is true for the family $\mathcal{K}_A(X)$.

(2) *If A is a nonempty closed relatively weakly compact bounded subset of X, then the family of all $Z \in \mathcal{B}_A(X)$ for which the problem $\max(A,Z)$ is well posed is a dense G_δ subset of $\mathcal{B}_A(X)$.*

The same is true for the family $\mathcal{K}(X)$.

Porosity results.   Many of the above generic existence results can be strengthened by using the notion of porosity. A subset Z of a metric space X is called *porous* if there exist α, $0 < \alpha \le 1$, and $r_0 > 0$ such that for every $x \in X$ and $0 < r \le r_0$, there is $y \in X$ such that $B'(y,\alpha r) \subset B'(x,r) \setminus Z$. Here $B'(x,r)$ denotes the open ball of center x and radius r. One obtains the same thing if we replace in the above definition "for every $x \in X$" by "for every $x \in Z$." A set Z is called σ-*porous* if it is a countable union of porous subsets of X. A porous set is nowhere dense in X, so that a σ-porous set is of the first Baire category. In general (e.g., if X is a Banach space), the family of sets of the first Baire category strictly contains the family of all σ-porous sets and, if $X = \mathbb{R}^n$, then a σ-porous set is also of Lebesgue measure zero. Since $\mathbb{R}^n$ can be written as the union of a set of the first Baire category and of a set of Lebesgue measure zero, it follows that there are sets of Lebesgue measure zero which are not σ-porous. A good account of various questions related to porosity is given in the survey papers by Zajíček [80, 81]. Remark that for what we call here "porous," Zajíček uses the term "uniformly very porous," in his definition of porosity, the number α may depend on x.

De Blasi et al. [19] proved porosity existence results for nearest- and farthest-point problems in uniformly convex Banach spaces, extending some known generic existence results.

As in the case of the problem of nearest points, one says that, for $x \in X$ and a bounded subset Z of X, the problem of farthest points $\max(x,Z)$ is *well posed* if it has a unique

solution $z(x)$ and every maximizing sequence $(z_n) \subset Z$ converges to $z(x)$. Denote by nwp(Z) and fwp(Z) the set of all points $x \in X$ for which the problem of nearest points (resp., of farthest points) is well posed.

A result similar to Proposition 2.12 holds for the problem of farthest points too. Suppose that $(e(x,Z) = \sup\{\|x - z\| : z \in Z\} > 0$. For $0 < \epsilon < e(x,Z)$, let

$$Q_Z^{\epsilon}(x) = Z \cap B(x, e(x,Z) - \epsilon). \tag{4.3}$$

PROPOSITION 4.5 [17]. *For a Banach space X, a bounded subset Z of X, and $x \in X$, the problem $\max(x,Z)$ is well posed if and only if $\lim_{\epsilon \searrow 0} \operatorname{diam} Q_Z^{\epsilon}(x) = 0$.*

De Blasi et al. [19] proved the following result.

THEOREM 4.6. *Let X be a uniformly convex Banach space.*
  (1) *If Z is a nonempty closed subset of X, then $X \setminus$ nwp(Z) is a σ-porous subset of X.*
  (2) *If Z is nonempty bounded closed subset of X, then $X \setminus$ fwp(Z) is a σ-porous subset of X.*

Lau [51] proved that for very closed subset Z of a reflexive Banach space X satisfying the Kadec-Klee property, the set $E(Z)$ is G_δ and dense in X. Concerning the problem of farthest points, he proved in [50] that for very weakly compact subset Z of an arbitrary Banach space X, the set of all points in X having a farthest point in Z is G_δ and dense in X.

In connection with these results, the following questions naturally arise.

Problems. (1) Does the first assertion of Theorem 4.6 remain true for a reflexive Banach space satisfying the Kadec-Klee property?

(2) The same question is for the second assertion and a weakly compact subset of an arbitrary Banach space X.

Li and Xu [54] proved some porosity results for the well posedness with respect to families of subsets of a Banach space equipped with the Pompeiu-Hausdorff metric.

THEOREM 4.7 (see De Blasi et al. [19]). *Let X be a uniformly convex Banach space.*
  (1) *If A is a nonempty closed subset of X and $\operatorname{minwp}(A)$ denotes the family of all sets $Z \in \mathcal{B}_A(X)$ for which the problem $\min(A,Z)$ is well posed, then the set $\mathcal{B}_A(X) \setminus \operatorname{minwp}(A)$ is σ-porous in $\mathcal{B}_A(X)$.*
  (2) *If A is a nonempty bounded closed subset of X and $\operatorname{maxwp}(A)$ denotes the family of all $Z \in \mathcal{B}(X)$ for which the problem $\max(A,Z)$ is well posed, then $\mathcal{B}(X) \setminus \operatorname{maxwp}(A)$ is σ-porous in $\mathcal{B}(X)$.*

The proof of assertion (1) uses in an essential way Stečkin's lens lemma and Proposition 2.11.

Final remarks. I tried to give a general idea about some questions and results on the existence of nearest- and farthest-points for arbitrary subsets of Banach spaces, with emphasis on genericity and porosity results. Of course, a lot of interesting related questions were not included. Beside the results mentioned here, there are many other ones by De Blasi and Myjak, De Blasi and Zhivkov, De Blasi and Kenderov who, for a long period

of time, systematically investigated various generic properties concerning nearest- and farthest-points problems. Reich and Zaslavski [69, 70, 71, 72] have recently also made contributions in this area. Penot [63, 64, 65] obtained results on generic existence in optimization in connection with convex and nonsmooth analysis. There are also negative results, meaning the existence of closed convex and bounded antiproximinal sets in Banach spaces. A survey on this topic is given in [14] (see also [16]).

Another direction of investigation, very active lately, is that of the study of the so-called generalized optimization problems. One considers an absorbing convex subset C of a normed space X and the Minkowski functional

$$p_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}, \quad x \in X. \tag{4.4}$$

The functional p_C is subadditive and positive, but not necessarily absolutely homogeneous, since the set C need not be symmetric. If C is also radially bounded, then one calls p_C an *asymmetric norm* on X. A study of analogs of the geometric properties of a Banach space in the case of a space with asymmetric norm was done by Zanco and Zucchi [82]. Contributions to existence results for optimization problems with respect to asymmetric norms have been made by De Blasi and Myjak [18], Li and Ni [53], Ni [62], and Maâden [55, 56].

Acknowledgment

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