ON SYMMETRIC EQUILIBRIUM OF AN ISOTHERMAL GAS WITH A FREE BOUNDARY AND A BODY FORCE

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The equation of symmetric equilibrium of an isothermal gas with an unknown boundary in the field of a body force is considered. Conditions for solvability and insolvability of the problem as well as for uniqueness and nonuniqueness of solutions are presented. Examples of finite, countable, or continual sets of solutions are constructed including equipotential ones. Static stability of solutions is analyzed too.

1. Introduction

The problem of symmetric equilibrium of an isothermal gas with a free boundary consists in seeking of a pair $\{\rho,R\}$ of the density $\rho \in W^{1,1}(r_0,R)$, $\rho \geqslant 0$ and the radius of the free boundary R, $r_0 < R < \infty$, which satisfy the equilibrium equation

$$\frac{d\rho}{dr} = \rho f \quad \text{on } (r_0,R), \tag{1.1}$$

where $f \in L^1(r_0,a)$, for any $a > r_0$, is a given body force, together with the boundary condition

$$\rho(R) = \rho_\Gamma > 0 \tag{1.2}$$

and the constraint of a given total mass

$$\int_{r_0}^{R} \rho(r)\varkappa(r)\,dr = M > 0, \tag{1.3}$$

where $\varkappa(r) = r^k$ with $k = 0,1$, or 2, respectively, in the cases of the planar, cylindrical, and spherical symmetry. Here $r_0 \geqslant 0$ is given; physically, $r_0 > 0$ is the radius of a hard core which the gas is surrounding whereas the value $r_0 = 0$ covers the case without core.

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In the sequel we will call the problem (1.1)–(1.3) by *the problem* $\mathcal{S}$ and the pair $\{\rho, R\}$ by *the (equilibrium) solution*. Instead of the above explicit formula for $\varkappa$, we will only exploit the properties $\varkappa \in L^1(r_0, a)$, for any $a > 0$, $\varkappa > 0$ a.e. (i.e., almost everywhere) on (r_0, ∞) and $\int_{r_0}^{\infty} \varkappa\, dr = \infty$; the last assumption can be omitted in some of our results.

In the fixed boundary case where R is given and the boundary condition (1.2) is absent, one can easily check that there exists a unique solution ρ, moreover ρ is strictly positive. For more general barotropic situation where $dp(\rho)/dr$ replaces $d\rho/dr$, under general conditions on p, there always exists a solution, and the nonuniqueness can take place only in the case where ρ degenerates, for sufficiently large f [1, 2, 4–6, 11–13].

In contrast with these results, we prove that though ρ is always strictly positive for the problem $\mathcal{S}$, this can be unsolvable or can easily have nonunique solutions; moreover, there can exist finite, countable, or continual sets of solutions. We present corresponding necessary conditions and sufficient conditions on f together with particular examples.

Clearly the properties of the problem $\mathcal{S}$ are important in connection with the large-time behavior of the associated nonstationary one [3, 7–10, 14–16].

2. Properties of solutions

We introduce the quantity $f_0 := \rho_\Gamma / M$ and the primitive functions, for $a \geqslant r_0$

$$\begin{aligned} F_a(r) &:= \int_a^r f(q)\, dq, \\ K_a(r) &:= \int_a^r \varkappa(q)\, dq, \\ H_a(r) := (F_a - f_0 K_a)(r) &= \int_a^r (f - f_0\varkappa)(q)\, dq, \end{aligned} \tag{2.1}$$

and set $F := F_{r_0}$, $K := K_{r_0}$, and $H := F - f_0 K = H_{r_0}$; they will play an important role in what follows. Let us restate the problem $\mathcal{S}$.

Proposition 2.1. *$\{\rho, R\}$ is an equilibrium solution if and only if R serves as a solution to the nonlinear algebraic equation*

$$\Psi(r) \equiv \Psi[f](r) := \int_{r_0}^r e^{F(q) - F(r)} \varkappa(q)\, dq = \frac{1}{f_0}, \quad r > r_0, \tag{2.2}$$

which can also be rewritten in the equivalent form

$$\int_{r_0}^r e^{F(q)} \Big(f_0 - \frac{f}{\varkappa}(q) \Big) \varkappa(q)\, dq = 1, \quad r > r_0. \tag{2.3}$$

Given R, the explicit formula for ρ holds

$$\rho(r) = \rho_\Gamma e^{F(r) - F(R)} \quad \text{on } [r_0, R]. \tag{2.4}$$

Proof. By (1.2), $\rho(r) > 0$ on $(r_1, R]$, for some $r_1 \in [r_0, R)$, and thus by solving (1.1), formula (2.4) is valid on $(r_1, R]$. Since $\rho \in C[r_0, R]$, in fact $r_1 = r_0$ and (2.4) is also valid for the whole $[r_0, R]$. Therefore (1.3) is equivalent to (2.2).

One can easily transform equivalently (2.3) into (2.2). $\square$

Corollary 2.2. *If $f/\varkappa \geqslant f_0$ a.e. on (r_0,a), for some $a > r_0$, then $R > a$ for any equilibrium solution $\{\rho,R\}$.*

In particular, if $f/\varkappa \geqslant f_0$ a.e. on (r_0,∞), then the problem $\mathcal{S}$ has no solution.

Proof. The result is straightforward according to (2.3). □

By (2.4), $\min_{[r_0,R]}\rho > 0$. Notice also that the problem $\mathcal{S}$ is unsolvable for $\rho_\Gamma = 0$.

Clearly $\Psi(r_0) = 0$ and $\Psi(r) > 0$ for $r > r_0$. We set $\Psi_{\sup} := \sup_{r>r_0}\Psi(r)$. If $\Psi_{\sup} = \infty$, then (2.2) has a solution for any $f_0 > 0$. If $\Psi_{\sup} < \infty$, then the solution exists for any $f_0 > 1/\Psi_{\sup}$ only (the value $f_0 = 1/\Psi_{\sup}$ is also admissible provided that $\Psi_{\sup} = \Psi(r_*)$ for some $r_0 < r_* < \infty$).

For example, if $f(r) \leqslant 0$ a.e. on (a,∞) for some $a \geqslant r_0$, then $\Psi(r) \geqslant K(r) - K(a)$ for $r \geqslant a$ and therefore $\Psi_{\sup} = \infty$ (since $K(\infty) = \int_{r_0}^{\infty}\varkappa(r)\,dr = \infty$). If $f/\varkappa \geqslant \alpha = \text{const} > 0$, then $\Psi(r) \leqslant \Psi_\alpha(r) := \alpha^{-1}(1 - e^{-\alpha K(r)})$; thus $\Psi_{\sup} \leqslant \Psi_\alpha(\infty) = \alpha^{-1}$ (clearly $\Psi(r) < \alpha^{-1}$ for $r \geqslant r_0$); in particular, for $f/\varkappa \equiv \alpha$, obviously $\Psi = \Psi_\alpha$ and $\Psi_{\sup} = \alpha^{-1}$.

Corollary 2.3 (the comparison theorem). *If the problem $\mathcal{S}$ has a solution $\{\rho^{(1)},R^{(1)}\}$ for $f = f_1$, then $\mathcal{S}$ has also a solution $\{\rho^{(2)},R^{(2)}\}$ with $R^{(2)} \leqslant R^{(1)}$, for any $f = f_2 \leqslant f_1$ on $(r_0,R^{(1)})$; moreover $\rho^{(2)}(r) \geqslant \rho^{(1)}(r)$ on $[r_0,R^{(2)}]$ provided that $f_1 \geqslant 0$ on $(R^{(2)},R^{(1)})$.*

If the problem $\mathcal{S}$ has no solution for $f = f_1$, then $\mathcal{S}$ has no solution for any $f = f_2 \geqslant f_1$.

Proof. The result follows from the property $\Psi[f_2] \geqslant \Psi[f_1]$ on $[r_0,r_1]$ in the case of $f_2 \leqslant f_1$ on (r_0,r_1) (and also from formula (2.4)). □

Now we present a necessary condition for the insolvability of the problem $\mathcal{S}$ and consequently a sufficient condition for its solvability.

Proposition 2.4. *If the problem $\mathcal{S}$ has no solution, then F satisfies the limiting property*

$$\liminf_{r\to\infty} H(r) > -\infty; \tag{2.5}$$

consequently $\liminf_{r\to\infty}(F/K)(r) \geqslant f_0$.

Proof. We introduce the primitive function $\Phi(r) := \int_{r_0}^{r} e^F\varkappa\,dq$ for $r \geqslant r_0$ and rewrite (2.2) in the form

$$\frac{1}{\varkappa}\frac{d\Phi}{dr}(r) = f_0\Phi(r) \quad \text{with some } r > r_0, \tag{2.6}$$

considering that the equality $(1/\varkappa)(d\Phi/dr)(r) = e^{F(r)}$ holds for all $r \geqslant r_0$. Clearly $\Phi(r_0) = 0$ and $\Phi(r) > 0$ for all $r > r_0$.

Supposing that (2.6) is valid nowhere, by continuity of e^F and Φ we find

$$\frac{1}{\varkappa}\frac{d\Phi}{dr}(r) > f_0\Phi(r) \quad \text{for } r \geqslant r_0. \tag{2.7}$$

Therefore, for any $a \geqslant r_0$,

$$\Phi(r) > \Phi(a)e^{f_0K_a(r)} \quad \text{for } r > a. \tag{2.8}$$

By virtue of two last inequalities

$$e^{F(r)} > f_0\Phi(r) > f_0\Phi(a)e^{f_0 K_a(r)} \quad \text{for } r > a. \tag{2.9}$$

Thus $F(r) > f_0(K(r) - K(a)) + \log(f_0\Phi(a))$ for $r > a > r_0$ which yields (2.5). □

Corollary 2.5. *If F satisfies the property* $\liminf_{r\to\infty} H(r) = -\infty$, *or a little bit more restrictive one*

$$\liminf_{r\to\infty} \frac{F}{K}(r) < f_0, \tag{2.10}$$

then the problem $\mathcal{S}$ has a solution.

Note that clearly the condition $f/\varkappa \geqslant f_0$ from Corollary 2.2 implies (2.5).

The more restrictive than (2.10) condition with lim sup replacing lim inf is known from [15] (see also [3, 10, 16]) in order to prove the uniform-in-time energy bound in the case of the nonstationary problem and to prove the existence for the barotropic equilibrium problem [3, 16, 17]. The condition $f/\varkappa \leqslant f_0 - \varepsilon_0$ a.e. on (a,∞), for some $\varepsilon_0 > 0$ and $a \geqslant r_0$ (see [14, 15]), implies this more restrictive condition.

Now we turn to the uniqueness of equilibrium solutions. First we present a necessary and sufficient condition for the existence of at least two solutions and derive a sufficient condition for the uniqueness.

Proposition 2.6. *Assume that the problem $\mathcal{S}$ has the solution for $R = R_1$. Then $\mathcal{S}$ has also the solution for $R = R_2 > R_1$ if and only if any of two equivalent conditions holds*

$$\int_{R_1}^{R_2} e^F \varkappa\, dr = \frac{1}{f_0}\left(e^{F(R_2)} - e^{F(R_1)}\right), \tag{2.11}$$

$$\int_{R_1}^{R_2} e^F \left(\frac{f}{\varkappa} - f_0\right) \varkappa\, dr = 0. \tag{2.12}$$

Proof. The result is straightforward after (2.2) and (2.3). □

It is essential that the values of f on (r_0, R_1) can be easily removed from conditions (2.11) and (2.12) by multiplying them by $e^{-F(R_1)}$ which leads to replacing e^F by $e^{F_{R_1}}$.

Corollary 2.7. *Assume that the problem $\mathcal{S}$ has solutions for $R = R_1, R_2$ with $R_2 > R_1$ and* $\operatorname{meas}\{r \in (R_1,R_2);\, (f/\varkappa)(r) \neq f_0\} > 0$. *Then f has the properties* $\operatorname{meas}\{r \in (R_1,R_2);\, (f/\varkappa)(r) < f_0\} > 0$ *and* $\operatorname{meas}\{r \in (R_1,R_2);\, (f/\varkappa)(r) > f_0\} > 0$.

Corollary 2.8. *If, for some $r_0 \leqslant a < b \leqslant \infty$, either $f/\varkappa < f_0$ a.e. on (a,b) or $f/\varkappa > f_0$ a.e. on (a,b), then the problem $\mathcal{S}$ has at most one solution for $R \in [a,b]$; hereafter $[a,b] = [a,\infty)$ for $b = \infty$.*

Corollary 2.9. *If, for some $a \geqslant r_0$, $f/\varkappa \geqslant f_0$ a.e. on (r_0,a) and $f/\varkappa < f_0$ a.e. on (a,∞), then the problem $\mathcal{S}$ has at most one solution.*

Proof. Condition (2.12) implies Corollary 2.7. Corollary 2.7 implies Corollary 2.8 by contradiction. Corollary 2.9 follows from Corollaries 2.2 and 2.8. □

Notice that the uniqueness condition in Corollary 2.9 is essentially broader than the known conditions that $f/\varkappa$ is nonincreasing [14, 15] or $f \leqslant 0$, for $k = 0$ [9]. Moreover, the above mentioned simple one-sided condition $f/\varkappa \leqslant f_0 - \varepsilon_0$ a.e. on (r_0, ∞), for some $\varepsilon_0 > 0$, ensures both the existence and uniqueness of the equilibrium solution.

There holds a partial proportionality between any two equilibrium solutions.

Proposition 2.10. *If $\{\rho_1, R_1\}$ and $\{\rho_2, R_2\}$ are two equilibrium solutions with $R_2 > R_1$, then*

$$\frac{\rho_2(r)}{\rho_1(r)} \equiv \delta \quad \text{on } [r_0, R_1] \quad \text{with } \delta := e^{F(R_1)-F(R_2)} \in (0,1). \tag{2.13}$$

Proof. The result is straightforward after formula (2.4). The property $\delta < 1$ follows from the mass constraint (1.3). □

Notice that in this proposition $\rho_2(R_1) < \rho_2(R_2)$ and $F(R_1) < F(R_2)$.

Next we complete Corollary 2.7 and present a situation where there exists a continuum of equilibrium solutions.

Proposition 2.11. *If $\{\rho_1, R_1\}$ is an equilibrium solution and*

$$\frac{f}{\varkappa} = f_0 \quad \text{a.e. on } (R_1, R_2), \tag{2.14}$$

then, for any $R \in (R_1, R_2]$, the pair $\{\rho, R\}$ with

$$\rho(r) = \begin{cases} e^{-f_0(K(R)-K(R_1))}\rho_1(r) & \text{for } r \in [r_0, R_1], \\ \rho_\Gamma e^{-f_0(K(R)-K(r))} & \text{for } r \in (R_1, R] \end{cases} \tag{2.15}$$

is an equilibrium solution as well.

Conversely, if the problem $\mathcal{S}$ has a solution for any $R \in (R_1, R_2)$, then property (2.14) holds.

Proof. The direct assertion follows from condition (2.12) and formula (2.4) since $F(r) - F(R_1) = f_0(K(r) - K(R_1))$ on $[R_1, R_2]$ after (2.14). The converse one is derived by the differentiation of (2.3) on (R_1, R_2). □

Next, having the equilibrium solution for $r = R_1$, we get a necessary condition for the absence of solution for $R \in (R_1, b]$ and consequently a sufficient condition for its existence.

Proposition 2.12. *Assume that the problem $\mathcal{S}$ has the solution for $R = R_1$. If $\mathcal{S}$ has no solution for $R \in (R_1, b]$ with some $b > R_1$, then the function $H_{R_1}(r) = \int_{R_1}^{r} (f - f_0\varkappa)dq$ has no zero on $(R_1, b]$.*

Clearly the result remains valid for (R_1, ∞) replacing $(R_1, b]$.

Proof. We can argue similarly to the proof of Proposition 2.4. Let us introduce the primitive function $\Phi_1(r) := \int_{R_1}^{r} e^{F_{R_1}(q)}\varkappa(q)dq$ for $r \geqslant R_1$ and rewrite condition (2.11) with

$R_2 = r$ in the form

$$\frac{1}{\varkappa}\frac{d\Phi_1}{dr}(r) = f_0\Phi_1(r) + 1 \tag{2.16}$$

considering that $(1/\varkappa)(d\Phi_1/dr)(r) = e^{F_{R_1}(r)}$ for all $r \geqslant R_1$. Assuming that equality (2.16) is valid nowhere on $(R_1,b]$, we find that the difference $(1/\varkappa)(d\Phi_1/dr) - f_0\Phi_1 - 1$ does not change its sign there. Let, for definiteness,

$$\frac{1}{\varkappa}\frac{d\Phi_1}{dr} - f_0\Phi_1 - 1 > 0 \quad \text{on } (R_1,b]. \tag{2.17}$$

By solving this linear differential inequality and using the equality $\Phi(R_1) = 0$, we get

$$\Phi_1(r) > \frac{1}{f_0}\left(e^{f_0K_{R_1}(r)} - 1\right) \quad \text{on } (R_1,b]. \tag{2.18}$$

Two last inequalities together yield

$$e^{F_{R_1}(r)} = \frac{1}{\varkappa}\frac{d\Phi_1}{dr}(r) > e^{f_0K_{R_1}(r)} \quad \text{on } (R_1,b]. \tag{2.19}$$

Finally $H_{R_1}(r) = F_{R_1}(r) - f_0K_{R_1}(r) > 0$ on $(R_1,b]$. □

Corollary 2.13. *If the problem $\mathcal{S}$ has the solution for $R = R_1$ and $H_{R_1}(b) = 0$ for some $b > R_1$, then $\mathcal{S}$ has the solution for some $R = R_2 \in (R_1,b]$ too.*

One can check that the converse assertions to Propositions 2.4 and 2.12 are not valid.

To complete the section, we consider particular families of equilibrium solutions. Let hereafter $\{r_n\}_{n=1}^N$, $2 \leqslant N \leqslant \infty$, be any increasing finite or countable sequence with $r_1 > r_0$, and $r_n \to \infty$ as $n \to \infty$ in the case $N = \infty$. We set $I_n := (r_n, r_{n+1})$.

(1) Let first there exist solutions for $R = r_1, r_2$ (thus condition (2.12), for $(R_1,R_2) = (r_1,r_2)$, be valid) and $N > 2$. We set $\Delta r = r_2 - r_1$ and consider $r_n := r_1 + (n-1)\Delta r$, for $2 \leqslant n \leqslant N$. In the case where f and $\varkappa$ satisfy the condition

$$f(r+\Delta r) = f(r), \quad \varkappa(r+\Delta r) = \varkappa(r) \quad \text{for any } r \in (r_1, r_{N-1}) \tag{2.20}$$

(i.e., both f and $\varkappa$ are "Δr-periodic" on (r_1,r_N)), clearly condition (2.12), for the interval (r_n,r_{n+1}) replacing (R_1,R_2), is also valid for all $2 \leqslant n < N$. This allows us to obtain finite or countable sequences of equilibrium solutions existing for $R = r_n$, with $1 \leqslant n \leqslant N$ and $n < \infty$.

If $N < \infty$, by choosing any f such that either $f(r) > f_0\varkappa$ for $r > r_N$ or $f(r) < f_0\varkappa$ for $r > r_N$, we may get the absence of solutions for $R > r_N$ according to Corollary 2.8.

Concerning $\varkappa$, "Δr-periodicity" condition is rather restrictive, so we consider other examples as well.

(2) By combining Corollaries 2.8 and 2.13, we can easily construct a broad family of finite or countable sequences of equilibrium solutions. Assume that there exists a solution for $R = r_1$ and $N > 2$. For any $1 \leqslant n < N$, let a function $\varphi_n \in L^1(I_n)$, $\varphi_n > 0$ a.e. on I_n, be arbitrary. We set $f := f_0\varkappa + j_0(-1)^n c_n\varphi_n$ on I_n, with $j_0 = 1$ or -1 independently of n and

$c_n > 0$ being parameters. Since $\operatorname{sign}(f - f_0\varkappa) = j_0(-1)^n$ a.e. on I_n, by Corollary 2.8 there exists at most one solution for $R \in \bar{I}_n$. Let $c_1 > 0$ and points $\bar{r}_n \in I_n$, for $2 \leqslant n < N$, be arbitrary. We choose c_{n+1} recurrently such that

$$\int_{R_n}^{\bar{r}_{n+1}} (f - f_0\varkappa)\,dr \equiv j_0(-1)^{n+1}\left(c_{n+1}\int_{r_{n+1}}^{\bar{r}_{n+1}} \varphi_{n+1}\,dr - c_n\int_{R_n}^{r_{n+1}} \varphi_n\,dr\right) = 0 \tag{2.21}$$

assuming that, for induction, $R_n = r_1$, for $n = 1$, or there exists an equilibrium solution for $R = R_n \in I_n$, for $2 \leqslant n < N - 1$, and $c_n > 0$. Clearly $c_{n+1} > 0$. By Corollary 2.13, there exists a solution for some $R = R_{n+1} \in (R_n, \bar{r}_{n+1}]$. Since for $R \in \bar{I}_n$ a solution exists for $R = R_n$ only, in fact $R_{n+1} \in (r_{n+1}, \bar{r}_{n+1}] \subset I_{n+1}$.

Finally, for $R \in \bar{I}_n$, an equilibrium solution exists for a unique $R = r_1$, for $n = 1$, or a unique $R = R_n$ (in fact $R_n \in (r_n, \bar{r}_n)$), for $2 \leqslant n < N$. For $N = \infty$, the condition $r_n \to \infty$ as $n \to \infty$ ensures the property $f \in L^1(r_0, a)$ for any $a > r_0$.

In this example, for the smooth functions φ_n vanishing together with their derivatives at r_n and r_{n+1} for any n, the function $f - f_0\varkappa$ is smooth on $[r_1, r_N]$ as well.

(3) To demonstrate exploiting of (2.2) and (2.12), we construct an example where the equilibrium solutions exist if and only if $R = r_n$, for any $1 \leqslant n \leqslant N$ and $n < \infty$, with a rather simple f such that $f/\varkappa$ is a piecewise constant function. Let $\tilde{r}_n$ be any point in I_n, for $1 \leqslant n < N$, and $\tilde{r}_N = \infty$ for $N < \infty$. We set $f = \alpha_0\varkappa$ on $(r_0, \tilde{r}_1)$ and $f = \alpha_n\varkappa$ on $(\tilde{r}_n, \tilde{r}_{n+1})$ for any $1 \leqslant n < N$ and recurrently choose the parameters α_0 and α_n. Equation (2.2), for $r = r_1$, and (2.12), for $(R_1, R_2) = I_n$, take the forms

$$\begin{aligned}
G^{(0)}(\alpha_0) &:= \int_{r_0}^{r_1} e^{\alpha_0(K(r)-K(r_1))}\varkappa(r)\,dr = \frac{1}{f_0},\\
G_n(\alpha_n) &:= (\alpha_n - f_0)\int_{\tilde{r}_n}^{r_{n+1}} e^{\alpha_n(K(r)-K(\tilde{r}_n))}\varkappa(r)\,dr = d_{n-1}\\
&:= -(\alpha_{n-1} - f_0)\int_{r_n}^{\tilde{r}_n} e^{\alpha_{n-1}(K(r)-K(\tilde{r}_n))}\varkappa(r)\,dr.
\end{aligned} \tag{2.22}$$

All these integrals can be calculated explicitly, for example, $G^{(0)}(\alpha_0) = \alpha_0^{-1}(1 - e^{-\alpha_0 K(r_1)})$ for $\alpha_0 \neq 0$. Thus one can easily check that the functions $G^{(0)}$ and G_n are smooth on $\mathbb{R}$ and satisfy $G^{(0)}(-\infty) = +\infty$, $G^{(0)}(+\infty) = 0$ and $G_n(-\infty) = -1$, $G_n(+\infty) = +\infty$, moreover $d_{n-1} > -1$. This means that the equations have solutions α_0 and α_n; in addition $\alpha_0 \neq f_0$, and $\alpha_n \neq f_0$ by induction too; moreover, $\operatorname{sign}(\alpha_n - f_0) = (-1)^{n+1}$. Consequently there exist equilibrium solutions for $R = r_n$, $1 \leqslant n \leqslant N$ and $n < \infty$, and they do not exist for other $R > r_0$ according to Corollary 2.8.

3. Equipotential solutions and stable solutions

3.1. Equipotential solutions. In the isothermal case, for any pair $\{\rho, R\}$ such that $\rho \in C[r_0, R]$, $\rho > 0$, and $\rho(R) = \rho_\Gamma$ as well as $R > r_0$, the potential energy is given by the formula

$$\mathcal{F}\{\rho, R\} := \int_{r_0}^{R} (\rho\log\rho + \rho_\Gamma - \rho F)\varkappa\,dr; \tag{3.1}$$

for example see [3, 16, 17].

Proposition 3.1. *The potential energy of any equilibrium solution* $\{\rho, R\}$ *is given by the simple formula*

$$\mathcal{F}\{\rho, R\} = -MH(R) + M\log\rho_\Gamma. \tag{3.2}$$

Proof. The required formula follows from the formula $\log\rho = F - F(R) + \log\rho_\Gamma$, see (2.4), and the mass constraint (1.3). □

According to this proposition, the property that two equilibrium solutions $\{\rho_1, R_1\}$ and $\{\rho_2, R_2\}$ with $R_2 > R_1$ are *equipotential*, that is, have the same potential energy, means that

$$H_{R_1}(R_2) \equiv \int_{R_1}^{R_2} (f - f_0\varkappa)\,dr = 0. \tag{3.3}$$

Equipotential equilibrium solutions are especially interesting when describing global behavior for the nonstationary problem [3, 16].

Notice that the solutions from Proposition 2.11 are equipotential; the solutions from example (1) in the previous section are equipotential too provided that condition (3.3) is valid.

Proposition 3.2. *Conditions (2.12) and (3.3) together are equivalent to the following ones*

$$\int_{R_1}^{R_2} g e^{f_0 K_{R_1}} \varkappa\,dr = 0, \tag{3.4}$$

$$g(R_2) = 0 \tag{3.5}$$

on the function $g := e^{H_{R_1}} - 1$ *on* $[R_1, R_2]$ *which also satisfies*

$$g \in W^{1,1}(R_1, R_2), \qquad g(R_1) = 0, \qquad \min_{[R_1, R_2]} g > -1. \tag{3.6}$$

Moreover, for any function g satisfying (3.6) together with (3.4) and (3.5), the function

$$f := f_0\varkappa + \frac{1}{1+g}\frac{dg}{dr} \in L^1(R_1, R_2) \tag{3.7}$$

satisfies conditions (2.12) and (3.3).

Proof. Condition (2.12) can be rewritten in the form

$$\int_{R_1}^{R_2} e^{f_0 K_{R_1}}\, d(e^{H_{R_1}} - 1) = 0. \tag{3.8}$$

By integration by parts and exploiting the equivalence of (3.3) and (3.5), this is transformed into (3.4).

Moreover, for any function g satisfying (3.6) the relation $g = e^{H_{R_1}} - 1$ on $[R_1, R_2]$ can be inverted which leads to $H_{R_1} = \log(1+g)$ and (3.7). □

Obviously, if in addition $g \in W^{1,q}(R_1, R_2)$ (for some $q \in (1, \infty]$), $C^1[R_1, R_2]$, or $C^2[R_1, R_2]$ and so forth, then respectively $f - f_0\varkappa \in L^q(R_1, R_2)$, $C[R_1, R_2]$, or $C^1[R_1, R_2]$ and so forth, in (3.7). In addition, for $g \in C^1[R_1,R_2]$, we have

$$(f - f_0\varkappa)(R_1+0) = \frac{dg}{dr}(R_1), \qquad (f - f_0\varkappa)(R_2-0) = \frac{dg}{dr}(R_2). \tag{3.9}$$

To present an example of a family of finite or countable sequences of equipotential solutions, we first consider a function g_0 satisfying (3.6) and (3.5) as g and changing its sign over (R_1,R_2). We define an operation of a partial scaling

$$S_{R_1,R_2}g_0 := \begin{cases} g_0(r) & \text{on } E_- := \{r \in [R_1,R_2];\ g_0(r) \leqslant 0\}, \\ \alpha g_0(r) & \text{on } E_+ := \{r \in [R_1,R_2];\ g_0(r) > 0\}, \end{cases} \tag{3.10}$$

where α is chosen such that condition (3.4) is valid for the function $g(r) := S_{R_1,R_2}g_0$, that is,

$$-\int_{E_-} |g_0|\, e^{f_0K_{R_1}}\varkappa\, dr + \alpha\int_{E_+} g_0 e^{f_0K_{R_1}}\varkappa\, dr = 0. \tag{3.11}$$

Since g_0 changes its sign over (R_1,R_2), both integrals are positive and thus α is uniquely defined and $\alpha > 0$. Clearly $g = \alpha\max\{g_0,0\} - \max\{-g_0,0\}$; consequently g satisfies conditions (3.4)–(3.6) and also changes its sign over (R_1,R_2).

In addition, if $g_0 \in C^1[R_1,R_2]$ and $(dg_0/dr)(r_*) = 0$ at any point $r_* \in (R_1,R_2)$ such that $g_0(r_*) = 0$, then $g \in C^1[R_1,R_2]$ too.

Now we are in a position to consider a broad family of finite or countable sequences of equipotential solutions.

(4) Given a sequence $\{r_n\}_{n=1}^N$ (see the previous section), assume that there exists an equilibrium solution for $R = r_1$, and, for any $1 \leqslant n < N$, take an arbitrary function g_{0n} such that

$$g_{0n} \in W^{1,1}(I_n), \qquad g_{0n}(r_n) = g_{0n}(r_{n+1}) = 0, \qquad \min_{\bar{I}_n} g_{0n} > -1 \tag{3.12}$$

as well as changing its sign over I_n. We set $g_n := S_{r_n,r_{n+1}}g_{0n}$ and then

$$f := f_0\varkappa + \frac{1}{1+g_n}\frac{dg_n}{dr} \in L^1(I_n). \tag{3.13}$$

According to Proposition 3.2, for any $2 \leqslant n \leqslant N$ and $n < \infty$, there exists an equilibrium solution for $R = r_n$, which is equipotential with the original solution for $R = r_1$.

Notice that since g_{0n} changes its sign over I_n, there exists a point $r_{*n} \in I_n$ such that $g_{0n}(r_{*n}) = g(r_{*n}) = 0$. By virtue of the formula $H_{r_n} = \log(1+g_n)$ on $\bar{I}_n$ and Corollary 2.13, there exists an additional solution for some $R = r_n' \in (r_n, r_{*n}]$.

Let $f - f_0\varkappa \in C[r_0,r_1]$ and $(f - f_0\varkappa)(r_1-0) \leqslant 0$. It is not difficult to ensure an additional property $f - f_0\varkappa \in C[r_0,r_N]$. To do that, let us impose additional restrictions $g_{0n} \in C^1(\bar{I}_n)$, $g_{0n} \leqslant 0$ in a right-hand neighborhood of r_n and $g_{0n} \geqslant 0$ in a left-hand neighborhood of r_{n+1} as well as $(dg_{0n}/dr)(r_*) = 0$ at any point $r_* \in I_n$ such that $g_{0n}(r_*) = 0$.

Consequently $g_{0n} \in C^1(\bar{I}_n)$ and

$$\frac{dg_n}{dr}(r_n) = \frac{dg_{0n}}{dr}(r_n) \leqslant 0, \qquad \frac{dg_n}{dr}(r_{n+1}) = \alpha_n \frac{dg_{0n}}{dr}(r_{n+1}) \leqslant 0, \tag{3.14}$$

for suitable $\alpha_n > 0$. Moreover, let sequentially

$$\frac{dg_{0n}}{dr}(r_n) = (f - f_0\varkappa)(r_1 - 0), \quad \text{for } n = 1, \qquad \frac{dg_{0n}}{dr}(r_n) = \frac{dg_{n-1}}{dr}(r_n), \quad \text{for } 2 \leqslant n < N. \tag{3.15}$$

Taking into account equalities (3.9), we obtain the desired continuity of $f - f_0\varkappa$ in (3.13). □

3.2. Stable solutions. According to [17], an equilibrium solution $\{\rho, R\}$ is called *(statically) stable* provided that $\lambda_{\min}\{\rho, R\} > 0$, where $\lambda_{\min}\{\rho, R\}$ is the minimal eigenvalue of the second order ODE eigenvalue problem

$$-\frac{d}{dr}\left(\frac{1}{\rho\varkappa}\frac{dw}{dr}\right) = \lambda a_0 w \quad \text{on } (r_0, R),\ w(r_0) = 0, \qquad \left\{\frac{1}{\rho\varkappa}\frac{dw}{dr} - \frac{f}{\rho\varkappa}w\right\}\bigg|_{r=R} = 0, \tag{3.16}$$

with a function $a_0 \in L^1(r_0, R)$, $\operatorname{ess\,inf}_{(r_0,R)} a_0 > 0$. Here we suppose that $\operatorname{ess\,inf}_{(r_0,a)} \varkappa > 0$ for any $a > r_0$ and the function $f/\varkappa$ is continuous at $r = R$. $\{\rho, R\}$ is called *neutrally stable* provided that $\lambda_{\min}\{\rho, R\} = 0$ or *(statically) unstable* provided that $\lambda_{\min}\{\rho, R\} < 0$. These definitions do not depend on the choice of a_0.

Proposition 3.3. *An equilibrium solution $\{\rho, R\}$ is stable, neutrally stable or unstable in dependence with*

$$\frac{f}{\varkappa}(R) < f_0, \qquad \frac{f}{\varkappa}(R) = f_0, \qquad \text{or} \qquad \frac{f}{\varkappa}(R) > f_0. \tag{3.17}$$

Proof. By passing to the Lagrangian mass coordinate $m(r) := \int_{r_0}^{r} \rho\varkappa\, dq$ and choosing $a_0 := \rho\varkappa$, one can easily transform the eigenvalue problem (3.16) to the simplest one with constant coefficients

$$-\frac{d^2y}{dm^2} = \lambda y \quad \text{on } (0, M),\ y(0) = 0, \qquad \frac{dy}{dm}(M) - \frac{1}{\rho_\Gamma}\frac{f}{\varkappa}(R)y(M) = 0. \tag{3.18}$$

The latter problem has the eigenvalue $\lambda = 0$ if and only if $(f/\varkappa)(R) = f_0$ (in this case there exists the eigenfunction $y(m) = m$) and a negative eigenvalue $\lambda = -(\alpha/M)^2 < 0$ if and only if $(f/\varkappa)(R) > f_0$ (in this case there exists the eigenfunction $y(m) = \sinh(\alpha m/M)$ with $\alpha > 0$ such that $\tanh(\alpha) = \alpha f_0/(f/\varkappa)(R)$). □

According to Proposition 3.3, in Proposition 2.11, the solutions, for $R \in (R_1, R_2)$, are neutrally stable (it is supposed that $f/\varkappa \equiv f_0$ on (R_1, R_2)).

In example (2), the solutions, for $R = R_n$, $2 \leqslant n < N$, are stable for odd n or unstable for even n in the case $j_0 = 1$ or vice versa in the case $j_0 = -1$ (it is supposed that in addition $\varphi_n/\varkappa \in C(I_n)$).

In example (3), the solutions, for $R = r_n$, $1 \leqslant n \leqslant N$, and $n < \infty$, are stable for odd n or unstable for even n.

In example (4), the solutions, for $R = r_{n+1}$, $1 \leqslant n < N$, are either stable or neutrally stable accordingly to whether the derivative $(dg_{0n}/dr)(r_{n+1})$ is chosen negative or zero in (3.14)(it is supposed that $\varkappa \in C[r_1, r_N]$).

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