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Research Article

The Atkinson Theorem in Hilbert C^*-Modules over C^*-Algebras of Compact Operators

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The concept of unbounded Fredholm operators on Hilbert C^*-modules over an arbitrary C^*-algebra is discussed and the Atkinson theorem is generalized for bounded and unbounded Feredholm operators on Hilbert C^*-modules over C^*-algebras of compact operators. In the framework of Hilbert C^*-modules over C^*-algebras of compact operators, the index of an unbounded Fredholm operator and the index of its bounded transform are the same.

1. Introduction

Hilbert C^*-modules are an often used tool in operator theory and in operator algebras. The theory of Hilbert C^*-modules is very interesting on its own. Interacting with the theory of operator algebras and including ideas from noncommutative geometry, it progresses and produces results and new problems attracting attentions (see [1]).

A bounded operator T on a Hilbert space $\mathscr{H}$ is called Fredholm if there is a bounded operator S on $\mathscr{H}$ such that $1-ST$ and $1-TS$ are compact operators on $\mathscr{H}$. Atkinson theorem declares that a bounded operator is Fredholm if and only if its kernel and cokernel are finite dimensional and its range is closed (see [2, Theorem 14.1.1]). The Atkinson theorem has been extended to bounded operators on standard Hilbert module $\ell^2(\mathscr{A})$ in [2, Section 17.1]. In the present note, we give another generalization of the Atkinson theorem to Hilbert C^*-modules over C^*-algebras of compact operators.

A (left) pre-Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ (not necessarily unital) is a left $\mathscr{A}$-module E equipped with an $\mathscr{A}$-valued inner product $\langle \cdot,\cdot \rangle : E\times E\to\mathscr{A}$, which is $\mathscr{A}$-linear in the first variable and has the properties

$$\langle x,y\rangle = \langle y,x\rangle^*, \quad \langle x,x\rangle \geq 0 \text{ with equality if and only if } x = 0. \tag{1.1}$$

A pre-Hilbert $\mathcal{A}$-module E is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{1/2}$. The basic theory of Hilbert C^*-modules can be found in [3, 2].

We denote, by $B(E)$, the C^*-algebra of all adjointable operators on E (i.e., of all maps $T : E\to E$ such that there exists $T^* : E\to E$ with the property $\langle Tx,y\rangle = \langle x,T^*y\rangle$, for all $x,y \in E$). It is well known that each adjointable operator is necessarily bounded $\mathcal{A}$-linear in the sense that $T(ax) = aT(x)$, for all $a \in \mathcal{A}$, $x \in E$. In general, bounded $\mathcal{A}$-linear operator may fail to possess an adjoint (cf., [3]). However, if E is a Hilbert C^*-module over the C^*-algebra $\mathcal{K}$ of all compact operators on a Hilbert space, then it is known that each bounded $\mathcal{K}$-linear operator on E is necessarily adjointable (see, e.g., [4, Remark 5]).

Given elements $x,y \in E$, we define $\Theta_{x,y} : E\to E$ by $\Theta_{x,y}(z) = \langle z,x\rangle y$ for each $z \in E$, then $\Theta_{x,y} \in B(E)$, with $(\Theta_{x,y})^* = \Theta_{y,x}$. The closure of the span of $\{\Theta_{x,y} : x,y \in E\}$ in $B(E)$ is denoted by $K(E)$, and elements from this set will be called $\mathcal{A}$-compact operators.

In various contexts where the Hilbert C^*-modules arise, one also needs to study "unbounded adjointable operators" or what are now known as regular operators. These were first introduced by Baaj and Julg in [5], where they gave a nice construction of Kasparov bimodules in KK theory using regular operators, Lance gave a brief indication in his book [3] about Hilbert modules and regular operators on them. Let us quickly recall the definition of a regular operator. An operator T from a Hilbert $\mathcal{A}$-module E to E is said to be regular if

(i) T is closed and densely defined,

(ii) its adjoint T^* is also densely defined, and

(iii) range of $1 + T^*T$ is dense in E.

Note that as we set $\mathcal{A} = \mathbb{C}$, that is, if we take E to be a Hilbert space, then this is exactly the definition of a closed operator, except that in that case, both the second and third conditions follow from the first one. Before starting the other parts, let us fix the rest of our notation.

Throughout the paper, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ will be the C^*-algebra of all compact operators on a Hilbert space $\mathcal{H}$, and $\mathcal{A}$ will be an arbitrary C^*-algebra (not necessarily unital). In this paper, we will deal with bounded and unbounded operators at the same time. To simplify the notation, we will, as a general rule, denote bounded operators by capital letters and unbounded operators by small letters, also we will use $\mathrm{Dom}(\cdot)$ for the domain of unbounded operators. We use Ker and Ran for the kernel and the range of operators, respectively.

2. Preliminaries

In this section, we would like to recall some definitions and present a few simple facts about unbounded regular operators and their bounded transforms. We give a definition of regular Fredholm operators on Hilbert C^*-modules and then state that a regular operator is Fredholm if and only if its bounded transform is a Fredholm operator.

Let E be a Hilbert $\mathcal{A}$-module, consider a closed $\mathcal{A}$-linear operator $t : \mathrm{Dom}(t) \subseteq E \to E$, where $\mathrm{Dom}(t)$ is a dense submodule of E. We define

$$\mathrm{Dom}(t^*) = \{y \in E : \exists\, y' \in E \text{ s.t. } \langle tx, y\rangle = \langle x, y'\rangle \ \forall x \in \mathrm{Dom}(t)\}. \tag{2.1}$$

This is the domain of a closed $\mathcal{A}$-linear operator $t^* : \mathrm{Dom}(t^*) \subseteq E \to E$ uniquely determined by $\langle tx, y\rangle = \langle x, t^* y\rangle$ for all $x \in \mathrm{Dom}(t)$, $y \in \mathrm{Dom}(t^*)$.

Definition 2.1. An operator t as above is called regular if t^* is densely defined and $1 + t^*t$ has dense range. The set of all regular operators on E is denoted by $R(E)$.

If t is a regular operator, so are t^* and t^*t, moreover, $t^{**} = t$, and t^*t is self adjoint (cf., [3, Corollaries 9.4, 9.6, and Proposition 9.9]), and also we can define

$$\begin{aligned} F_t &= t\left(1 + t^*t\right)^{-1/2}, \\ Q_t &= \left(1 + t^*t\right)^{-1/2}, \end{aligned} \tag{2.2}$$

then F_t, Q_t, and tQ_t^2 are in $B(E)$ and $\mathrm{Ran}\, Q_t = \mathrm{Dom}(t)$ (cf., [3, Chapter 9]).

The map $t \to F_t$ defines a bijection (cf., [3, Theorem 10.4])

$$R(E) \longrightarrow \{F \in B(E) : \|F\| \le 1 \text{ and } \mathrm{Ran}\left(1 - F^*F\right) \text{ is dense in } E\}. \tag{2.3}$$

This map is adjoint preserving, that is, $F_t^* = F_{t^*}$ and $F_t = tQ_t = t(1 + t^*t)^{-1/2}$ is called the bounded transform of a regular operator t. Moreover, we have $Q_t = (1 - F_t^*F_t)^{1/2}$ and $t = F_t(1 - F_t^*F_t)^{-1/2}$, $\|F_t\| \le 1$.

Recall that the composition of two regular operators $t, s \in R(E)$ is the unbounded operator ts with $\mathrm{Dom}(ts) = \{x \in \mathrm{Dom}(s) : sx \in \mathrm{Dom}(t)\}$ given by $(ts)x = t(sx)$ for all $x \in \mathrm{Dom}(ts)$. Note that, in general, the composition of regular operators will not be a regular operator.

Recall that a bounded operator $T \in B(E)$ is said to be Fredholm (or $\mathcal{A}$-Fredholm) if it has a pseudoleft as well as pseudoright inverse, that is, there are $S_1, S_2 \in B(E)$ such that $S_1T = 1 \bmod K(E)$ and $TS_2 = 1 \bmod K(E)$. This is equivalent to say that there exist $S \in B(E)$ such that $ST = TS = 1 \bmod K(E)$. For more details about bounded Fredholm operators on Hilbert C^*-modules and their applications, one can see [1, 2]. The theory of unbounded Fredholm operators on Hilbert spaces and on standard Hilbert $\mathcal{A}$-module $\ell^2(\mathcal{A})$ are discussed in [6, 7], respectively. These motivate us to study such operators on general Hilbert C^*-modules as follow.

Definition 2.2. Let t be a regular operator on a Hilbert $\mathcal{A}$-module E. An adjointable bounded operator $G \in B(E)$ is called a pseudoleft inverse of t if Gt is closable and its closure $\overline{Gt}$ satisfies $\overline{Gt} \in B(E)$ and $\overline{Gt} = 1 \bmod K(E)$. Analogously, G is called a pseudoright inverse if tG is closable and its closure $\overline{tG}$ satisfies $\overline{tG} \in B(E)$ and $\overline{tG} = 1 \bmod K(E)$. The regular operator t is called Fredholm (or $\mathcal{A}$-Fredholm) if it has a pseudoleft as well as a pseudoright inverse.

When we are dealing with Fredholm operators, a useful connection between unbounded operators and their bounded transforms (on $\ell^2(\mathcal{A})$) has been given in [7, Lemma 2.2]. Fortunately its proof can be repeated word by word to get the following result.

Theorem 2.3. *Let E be a Hilbert $\mathcal{A}$-module and t a regular operator on E. Then t is Fredholm if and only if F_t is.*

Proof. Refer to [7, Lemma 2.2]. □

3. Bounded Fredholm operators

In this section, we introduce a concept of an orthonormal basis for Hilbert C^*-modules and then we briefly discuss bounded Fredholm operators in Hilbert $\mathcal{K}$-modules. For this aim, we borrow some definitions from [4].

Let E be a Hilbert $\mathcal{A}$-module, a system (x_λ), $\lambda \in \Lambda$ in E is orthonormal if each x_λ is a basic vector (i.e., $e = \langle x_\lambda, x_\lambda\rangle$ is minimal projection in $\mathcal{A}$) and $\langle x_\lambda, x_\mu\rangle = 0$ for all $\lambda \neq \mu$. An orthonormal system (x_λ) in E is said to be an orthonormal basis for E if it generates a dense submodule of E. Immediately, a previous definition implies that if $x \in E$ satisfies $\langle x, x\rangle = e$ for some projection (not necessarily minimal) $e \in \mathcal{A}$, then $\langle ex - x, ex - x\rangle = 0$, so $ex = x$. In particular, the same is true for all basic vectors in E.

Theorem 3.1. *Let E be a Hilbert $\mathcal{K}$-module, then there exists an orthonormal basis for E.*

Proof. Refer to [4, Theorem 4]. □

Definition 3.2. Let E be a Hilbert $\mathcal{K}$-module. The orthonormal dimension of E (denoted by $\dim_{\mathcal{K}} E$) is defined as the cardinal number of any of its orthonormal bases.

Note that every two orthonormal bases of E have the same cardinal number (see [4]).

Now, let us fix a minimal projection $e_0 \in \mathcal{K}$ and denote by $E_{e_0} = e_0E = \{e_0x : x \in E\}$ then E_{e_0} is a Hilbert space with respect to the inner product $(\cdot,\cdot) = \operatorname{tr}(\langle\cdot,\cdot\rangle)$, which is introduced in [4, Remark 4], moreover, $\dim_{\mathcal{K}} E = \dim E_{e_0}$.

Suppose $B(E_{e_0})$ and $K(E_{e_0})$ are the C^*-algebras of all bounded linear operators and compact operators on Hilbert space E_{e_0}, respectively, then $B(E)$ and $K(E)$ can be described by the following theorem.

Theorem 3.3. *Let E be a Hilbert $\mathcal{K}$-module and let e_0 be an arbitrary minimal projection in $\mathcal{K}$. Then the map $\Psi : B(E) \to B(E_{e_0})$, $\Psi(T) = T|_{E_{e_0}}$ is a $*$-isomorphism of C^*-algebras. Moreover, T is a compact operator on E if and only if $\Psi(T) = T|_{E_{e_0}}$ is a compact operator on the Hilbert space E_{e_0}.*

Proof. Refer to [4, Theorems 5, 6]. □

Remark 3.4. Let E be a Hilbert $\mathcal{K}$-module and X a closed submodule in E. It is well known that X is orthogonally complemented in E, that is, $E = X \oplus X^\perp$ (cf., [8]).

Remark 3.5. Let e be an arbitrary minimal projection in $\mathcal{K}$ and suppose Ψ has the same meaning as in Theorem 3.3, then $\operatorname{Ran}\Psi(T) = e\operatorname{Ran}T$ and $\operatorname{Ker}\Psi(T) = e\operatorname{Ker}T$.

Remark 3.6. Let E be a Hilbert $\mathcal{K}$-module and $T \in B(E)$, like in the general theory of Banach spaces, one can easily see that every bounded below operator T on $(\operatorname{Ker}T)^{\perp}$ has a closed range (cf., [9, page 21]). It is easily checked that $T : E \to E$ is not bounded below on $(\operatorname{Ker}T)^{\perp}$ if and only if there is a sequence of unit elements x_n in $(\operatorname{Ker}T)^{\perp}$ such that $\lim_{n\to\infty} Tx_n = 0$. This fact will be used in the following theorem.

Theorem 3.7. *Let E be a Hilbert $\mathcal{K}$-module and $T \in B(E)$. Then T is Fredholm if and only if the range of T is a closed submodule and both* $\dim_{\mathcal{K}}\operatorname{Ker}T$ *and* $\dim_{\mathcal{K}}\operatorname{Ker}T^*$ *are finite.*

Proof. Let e be a minimal projection in $\mathcal{K}$ and let $\Psi : B(E) \to B(E_e)$ be the isomorphism from Theorem 3.3. Suppose T is a Fredholm operator on E, then there exist an operator $S \in B(E)$ and two compact operators $K_1, K_2 \in K(E)$ such that $TS - 1 = K_1$ and $ST - 1 = K_2$. Since $\Psi(K(E)) = K(E_e)$, the operator $\Psi(T)$ is Fredholm on the Hilbert space E_e. In particular, $\Psi(T)$ has a closed range and $\operatorname{Ker}\Psi(T)$ and $\operatorname{Ker}\Psi(T^*)$ are finite dimensional, so by applying Remark 3.5 to the Hilbert modules $\operatorname{Ker}T$ and $\operatorname{Ker}T^*$, respectively, we get $\dim_{\mathcal{K}}\operatorname{Ker}T$, $\dim_{\mathcal{K}}\operatorname{Ker}T^* < \infty$.

To prove that the range of T is closed, it is enough to show that $T|_{(\operatorname{Ker}T)^{\perp}}$ is bounded from below.

Suppose, on the contrary, $T|_{(\operatorname{Ker}T)^{\perp}}$ is not bounded below. Then there exists a sequence of unit elements $(x_n) \in (\operatorname{Ker}T)^{\perp}$ such that $\lim_{n\to\infty} Tx_n = 0$. This implies $\lim_{n\to\infty} T(ex_n) = e\lim_{n\to\infty} Tx_n = 0$, and since $(ex_n) \in (\operatorname{Ker}\Psi(T))^{\perp}$ and the range of $\Psi(T)$ is closed, we obtain a contradiction.

Conversely, suppose $T \in B(E)$ has a closed range and both $\dim_{\mathcal{K}}\operatorname{Ker}T$ and $\dim_{\mathcal{K}}\operatorname{Ker}T^*$ are finite, then Remark 3.5 implies that the range of $\Psi(T)$ is closed and both $\operatorname{Ker}\Psi(T)$ and $\operatorname{Ker}\Psi(T^*)$ are finite dimensional. Therefore, $\Psi(T)$ is a Fredholm operator on the Hilbert space E_e, that is, there exists an operator $S \in B(E_e)$ such that $\Psi(T)S - 1$, $S\Psi(T) - 1 \in K(E_e)$. Utilizing the mapping Ψ^{-1}, we have $T\Psi^{-1}(S) - 1$, $\Psi^{-1}(S)T - 1 \in K(E)$, and T is therefore a Fredholm operator on E. □

Definition 3.8. Let E be a Hilbert $\mathcal{K}$-module and $T \in B(E)$ a Fredholm operator. The Fredholm index of T is an integer defined by

$$\operatorname{ind}T = \dim_{\mathcal{K}}\operatorname{Ker}T - \dim_{\mathcal{K}}\operatorname{Ker}T^*. \tag{3.1}$$

The preceding discussion shows how the theory of Fredholm operators on Hilbert $\mathcal{K}$-modules is reduced to the classical theory of Fredholm operators on Hilbert spaces. In fact, if E is a Hilbert $\mathcal{K}$-module, all properties of an operator $T \in B(E)$ can be deduced by a simple procedure: first, consider a Hilbert space operator $\Psi(T)$ (by using Theorem 3.3) and then lift the relevant information back to $B(E)$. This enables us to conclude some results, for example, if $T \in B(E)$, then T is Fredholm operator with $\operatorname{ind}T = 0$ if and only if T is a compact perturbation of an invertible operator.

4. Unbounded Fredholm operators

Let E be a Hilbert $\mathcal{K}$-module. We recall that a densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to E$ is said to be regular if t^* is densely defined and $1 + t^*t$ has dense range, moreover,

$$\begin{aligned} F_t = tQ_t = t(1+t^*t)^{-1/2} &\in B(E), \qquad F_t^* = F_{t^*}, \\ Q_t = (1+t^*t)^{-1/2} &\in B(E), \qquad \mathrm{Ran}\,Q_t = \mathrm{Dom}(t). \end{aligned} \tag{4.1}$$

In this section, we are going to prove the Atkinson theorem for unbounded regular Fredholm operators.

Lemma 4.1. *Let E be a Hilbert $\mathcal{K}$-module and $t : \mathrm{Dom}(t) \subseteq E \to E$ a regular operator, then* $\mathrm{Ker}\,t = \{x \in \mathrm{Dom}(t) : tx = 0\}$ *and* $\mathrm{Ker}\,t^* = \{x \in \mathrm{Dom}(t^*) : t^*x = 0\}$ *are closed submodules of E.*

Proof. Let (x_n) be a sequence in $\mathrm{Ker}\,t$ and $x_n \to x$, then $t(x_n) = 0$, for all $n \in \mathbb{N}$. Therefore, $x_n \to x$ and $t(x_n) \to 0$. It follows, by the closedness of t, that $x \in \mathrm{Dom}(t)$ and $tx = 0$, that is, $x \in \mathrm{Ker}\,t$, and so $\mathrm{Ker}\,t$ is closed. Since t is regular, so is t^* and similarly $\mathrm{Ker}\,t^*$ is a closed submodule of E. □

Lemma 4.2. *Let t be a regular operator on a Hilbert $\mathcal{K}$-module E, then*

(i) $\mathrm{Ran}\,t = \mathrm{Ran}\,F_t$ *and* $\mathrm{Ran}\,t^* = \mathrm{Ran}\,F_{t^*}$;

(ii) $\mathrm{Ker}\,t^* = (\mathrm{Ran}\,t)^{\perp}$ *and* $\mathrm{Ker}\,t = (\mathrm{Ran}\,t^*)^{\perp}$;

(iii) $\mathrm{Ker}\,t = \mathrm{Ker}\,F_t$ *and* $\mathrm{Ker}\,t^* = \mathrm{Ker}\,F_{t^*}$.

Proof. (i) Recall that $F_t = tQ_t$ and $\mathrm{Ran}\,Q_t = \mathrm{Dom}(t)$, then $\mathrm{Ran}\,t = \mathrm{Ran}\,F_t$. Since t is regular and so is t^*, thus $\mathrm{Ran}\,t^* = \mathrm{Ran}\,F_{t^*}$.

(ii) We notice that $y \in \mathrm{Ker}\,t^*$ if and only if $\langle tx, y\rangle = \langle x, 0\rangle = 0$ for all $x \in \mathrm{Dom}(t)$, or if and only if $y \in (\mathrm{Ran}\,t)^{\perp}$. Thus we have $\mathrm{Ker}\,t^* = (\mathrm{Ran}\,t)^{\perp}$. The second equality follows from the first equality and [3, Corollary 9.4].

(iii) By [2, Theorem 15.3.5], we have $\mathrm{Ker}\,F_{t^*} = (\mathrm{Ran}\,F_t)^{\perp}$, and therefore,

$$\mathrm{Ker}\,F_{t^*} = (\mathrm{Ran}\,F_t)^{\perp} = (\mathrm{Ran}\,t)^{\perp} = \mathrm{Ker}\,t^*. \tag{4.2}$$

Similarly, we have $\mathrm{Ker}\,F_t = \mathrm{Ker}\,t$. □

Now, we are ready to prove the main theorem of this paper.

Theorem 4.3. *Let E be a Hilbert $\mathcal{K}$-module and t a regular operator on E. Then t is Fredholm if and only if the range of t is a closed submodule of E, and both* $\dim_{\mathcal{K}}\mathrm{Ker}\,t$ *and* $\dim_{\mathcal{K}}\mathrm{Ker}\,t^*$ *are finite.*

Proof. By Lemma 4.2, we have $\dim_{\mathcal{K}}\mathrm{Ker}\,t = \dim_{\mathcal{K}}\mathrm{Ker}\,F_t$, $\dim_{\mathcal{K}}\mathrm{Ker}\,t^* = \dim_{\mathcal{K}}\mathrm{Ker}\,F_{t^*}$, and since $\mathrm{Ran}\,t = \mathrm{Ran}\,F_t$, the statement is deduced from Theorems 2.3 and 3.7. □

Let t be a regular Fredholm operator on Hilbert $\mathcal{K}$-module E, then we can define an index of t formally, that is, we can define

$$\mathrm{ind}\,t = \dim_{\mathcal{K}}\mathrm{Ker}\,t - \dim_{\mathcal{K}}\mathrm{Ker}\,t^*, \tag{4.3}$$

and since $\dim_{\mathcal{K}}\mathrm{Ker}\,t = \dim_{\mathcal{K}}\mathrm{Ker}\,F_t$, $\dim_{\mathcal{K}}\mathrm{Ker}\,t^* = \dim_{\mathcal{K}}\mathrm{Ker}\,F_{t^*}$, we have $\mathrm{ind}\,t = \mathrm{ind}\,F_t$.

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