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Research Article

Asymptotics of Time Harmonic Solutions to a Thin Ferroelectric Model

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We introduce new model equations to describe the dynamics of the electric polarization in a ferroelectric material. We consider a thin cylinder representing the material with thickness ε and discuss the asymptotic behavior of the time harmonic solutions to the model when ε tends to 0. We obtain a reduced model settled in the cross-section of the cylinder describing the dynamics of the plane components of the polarization and electric fields.

1. Introduction

In this work, we are interested in the model equations of ferroelectric materials introduced in [1] and discussed in [1–3] for example. We consider time harmonic solutions to the model as studied in [4]. We first rewrite the equations of the model given in [1] to precise the boundary conditions we will use. Let (E,H) be the electromagnetic field acting on the ferroelectric material Ω, which is a bounded and a regular domain of $\mathbb{R}^3$. Let P be the electric polarization induced in Ω. The electric displacement is then given by $D = \epsilon(E+P)$ where $\epsilon > 0$ is the electric permittitivity of the vacuum. The Maxwell equations satisfied by the electromagnetic field are

$$\begin{gathered} \mu\partial_t H - \operatorname{curl} E = 0, \\ \epsilon\partial_t(E+P) + \operatorname{curl} H + \sigma E = 0, \end{gathered} \tag{1.1}$$

where $\mu > 0$ is the magnetic permeability of the vacuum and $\sigma > 0$ is the conductivity constant of the ferroelectric material. The behavior of the electric polarization P is driven by the nonlinear Maxwell equation

$$\partial_t^2 P + \lambda^2 \operatorname{curl}^2 P + a\partial_t P = -b\left(E_{\mathrm{eq}}(P) - E\right), \tag{1.2}$$

where $\operatorname{curl}^2 P = \operatorname{curl}(\operatorname{curl} P)$, $E_{\mathrm{eq}}(P)$ is the nonlinear equilibrium electric field which will be given later, and $\lambda^2 = 1/\epsilon\mu$. The parameters a and b are some physical positive constants. This model is obtained as follow (see [1]). Denoting by $\mathbf{m}$ the internal magnetization and by $\mathbf{j}$ the current density which is driven by the difference between the equilibrium field $E_{\mathrm{eq}}(P)$ and the electric field E, then with the internal polarization field P they satisfy the set of equations

$$\begin{aligned} \epsilon\left(\partial_t P + \delta^{-1}\mathbf{j}\right) &= \operatorname{curl}\mathbf{m}, \\ \mu\left(\partial_t \mathbf{m} + \alpha\delta^{-1}\mathbf{m}\right) &= -\operatorname{curl} P, \\ \left(\partial_t \mathbf{j} + \alpha\delta^{-1}\mathbf{j}\right) &= \beta\delta^{-1}\left(E_{\mathrm{eq}}(P) - E\right) \end{aligned} \tag{1.3}$$

which reduces to the nonlinear Maxwell equation (1.2) satisfied by P. The internal magnetization $\mathbf{m}$ satisfies the boundary condition $\mathbf{m} \times n = 0$, then the second equation of (1.3) implies that P satisfies

$$\operatorname{curl} P \times n = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

In this work, we consider, on $\partial\Omega$, Leontovitch-type boundary conditions for E extending the one used in [1], that is,

$$H \times n + \beta n \times (E \times n) = 0, \qquad \operatorname{curl} P \times n = 0, \tag{1.5}$$

where β is some nonnegative function defined on $\partial\Omega$ and n is the unit outward normal to $\partial\Omega$.

The equilibrium field is assumed to be the gradient of a potential function $\phi(|P|^2)$. We have $E_{\mathrm{eq}}(P) = 2P\phi'(|P|^2)$ where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a C^2 convex function satisfying the hypotheses given in [1], more precisely, we assume that there exist $0 < r_1 < r_0$ and $C_2 > 0$ such that

$$\begin{gathered} \phi(0) = \phi(r_0) = 0, \qquad \phi'(r_1) = 0, \qquad \phi'(0) < 0, \\ \left(s\phi'(s^2)\right)' \le C_2 \quad \forall s \ge 0. \end{gathered} \tag{1.6}$$

Hence, for all $s \ge R > r_1$, there exists $C_R > 0$ such that $\phi'(s) \ge C_R$ and for all $s \ge 0$

$$\left|\phi'(s)\right| \le C_*, \qquad 0 \le s^2\phi^{(2)}(s^2) \le C_*, \tag{1.7}$$

where $C_* = \max(|\phi'(0)|, C_2)$. Examples of such potentials defined on $\mathbb{R}^+$ satisfying the hypotheses are the following: $\phi_1(s) = b(1+s^2)^{1/2} - as - 1$ with $0 < a < b$, $0 < b < 1$, $\phi_2(s) = s/2 - \log(1+s)$, $\phi_3(s) = as + 1 - (1+s)^\alpha$ with $a > 0$ and $a < \alpha < 1$.

With these hypotheses, the vector-valued function $E_{\mathrm{eq}}(P) = 2P\phi'(|P|^2)$ satisfies the estimate

$$\left|E_{\mathrm{eq}}(P) - E_{\mathrm{eq}}(Q)\right| \le C_* |P - Q| \tag{1.8}$$

for all $P, Q \in \mathbb{R}^3$.

Let us mention other interesting models for ferroelectric materials, see [5–7] for example. In the first two papers, the authors consider deformable ferroelectric materials and give the evolution equation for the spontaneous polarization. The model obtained is different from the one given in [1], since it includes the deformation of the bodies. In the second one, a theoretical model is proposed explaining the lamellar morphology of domains of opposite polarization observed in ferroelectric crystals in their polar phases. The jump conditions for the electric field and polarization vector across domain walls play an important role in the characterization of the free energy. Many interesting mathematical problems, as the dimension reduction of domains, are contained in both papers.

In this paper, we are dealing with time harmonic solutions to the model (1.1)-(1.2). We write $H(t,x) = e^{\imath t}H(x)$, $E(t,x) = e^{\imath\omega t}E(x)$, $P(t,x) = e^{\imath\omega t}P(x)$, and $F(t,x) = e^{\imath\omega t}F(x)$ with $\omega > 0$ fixed. The new complex field (E,P) satisfies the set of equations

$$\begin{aligned}(\zeta_1(\omega) + \lambda^2 \operatorname{curl}^2)E &= \omega^2 P + \imath\omega F\\ (\zeta_2(\omega) + \lambda^2 \operatorname{curl}^2)P &= -b\left(2P\phi'(|P|^2) - E\right),\end{aligned} \tag{1.9}$$

where $\zeta_1(\omega) = -\omega^2 + \imath a_1$, $\zeta_2(\omega) = -\omega^2 + \imath\omega a_2$ with $a_1 = \sigma/\epsilon$ and $a_2 = a$. The magnetic field H is recovered from the electric field E by the formula $H = \operatorname{curl} E/\imath\omega\mu$. The boundary conditions on $\partial\Omega$ write

$$\operatorname{curl} E \times n + \imath\omega\mu\beta n \times (E \times n) = 0, \qquad \operatorname{curl} P \times n = 0. \tag{1.10}$$

The main difficulty in this problem is related to the lack of regularity of the polarization field P to prove the stability of the nonlinear equilibrium field $E_{\mathrm{eq}}(P)$ with respect to the weak convergence of a sequence P_m. It is easy to prove that a sequence of solutions (E_m, P_m) of (1.9) is such that P_m, $\operatorname{curl} P_m$ are bounded in $L^2(\Omega)$. Even if we prove that $\operatorname{div} P_m$ is also bounded in $L^2(\Omega)$, the boundary condition $\operatorname{curl} P_m \times n = 0$ satisfied by P_m does not allow to deduce compactness in $L^2(\Omega)$ of the sequence P_m. Note that, in [8], the compactness of the sequence (P_m) is obtained in the case of the boundary condition $\operatorname{curl} P \times n + \beta n \times (P \times n) = 0$. To avoid this difficulty, we derive new model equations as follows.

For a given $P \in L^2(\Omega)$, the Hodge decomposition of P gives the orthogonal decomposition in $L^2(\Omega)$, $P = \nabla\varphi + U$ where $\varphi \in H^1(\Omega)$ and $U \in L^2(\Omega)$ satisfying $\operatorname{div} U = 0$, $U \cdot n = 0$ on $\partial\Omega$. The scalar potential φ is unique up to additive constants (See [9, Corollary 5, page 258]). Hence, $\operatorname{curl} P \times n = \operatorname{curl} U \times n = 0$. The field U may be decomposed on $\Gamma = \partial\Omega$ as follows: (see [10, page 75]) $U = (U \cdot n)n + U_\Gamma$, $U_\Gamma = n \times (U \times n)$. Thus, U satisfies the equivalent boundary condition on Γ (see [10, page 77]),

$$\frac{\partial U_\Gamma}{\partial n} + \mathcal{R}(U_\Gamma) = 0, \quad U \cdot n = 0, \tag{1.11}$$

where $\mathcal{R}$ is the symmetric curvature operator acting in the tangent plane.

In what follows, we are interested only in the regular part U of the polarization field P and assume that the potential φ is constant in $\overline{\Omega}$. Hence, we have $P = U$, then

$$\operatorname{div} P = 0 \quad \text{in } \Omega, \qquad P \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.12}$$

Next, we assume that the source term F satisfies in Ω the condition

$$\operatorname{div} F = 0. \tag{1.13}$$

By considering the equation satisfied by E in (1.9), we deduce the compatibility condition $\zeta_1(\omega) \operatorname{div} E - \omega^2 \operatorname{div} P = 0$ which implies $\operatorname{div} E = 0$. Hence, under the divergence free condition for P, (1.9) shows that (E, P) satisfies in Ω the new problem

$$\begin{gathered}
(\zeta_1(\omega) + \lambda^2 \operatorname{curl}^2) E - \omega^2 P = \imath\omega F, \\
(\zeta_2(\omega) - \lambda^2 \Delta) P + \nabla \pi = -b(E_{\mathrm{eq}}(P) - E), \\
\operatorname{div} P = 0, \\
\operatorname{curl} E \times n + \imath\omega\mu\beta n \times (E \times n) = 0, \\
\operatorname{curl} P \times n = 0, \quad P \cdot n = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.14}$$

where π is the Lagrange multiplier associated with the constraint $\operatorname{div} P = 0$ and where we used the relation $-\Delta = \operatorname{curl}^2 + \nabla \operatorname{div}$. Combining the condition $\operatorname{div} P = 0$ with the compatibility condition $\operatorname{div} E = 0$ and using the second equation of (1.14), we see that the equilibrium eelectric field should satisfy the condition $\operatorname{div} E_{\mathrm{eq}}(P) = \phi^{(2)}(|P|^2) P \cdot \nabla(|P|^2)$.

In the remainder of the paper, we assume that the ferroelectric domain is the cylinder $\Omega^\varepsilon = \Omega_T \times (0, \varepsilon)$ with thickness $\varepsilon > 0$ and the cross-section Ω_T which is an open, bounded, and regular set of $\mathbb{R}^2$. The generic point of Ω^ε is denoted by $x = (x_T, x_3)$ where $x_T = (x_1, x_2) \in \Omega_T$ and $0 \le x_3 \le \varepsilon$. We also assume that the function β appearing in the boundary condition satisfied by the electric field E depends on ε and is given by

$$\beta^\varepsilon(x_3) = \beta \quad \text{on } \partial\Omega_T \times (0, \varepsilon), \qquad \beta^\varepsilon(0) = \varepsilon\beta_0, \qquad \beta^\varepsilon(\varepsilon) = \varepsilon\beta_1 \quad \text{in } \Omega_T, \tag{1.15}$$

where β, β_k are positive constants. The boundary $\partial\Omega^\varepsilon$ writes as $(\Omega_T \times \{x_3 = 0\}) \cup (\Omega_T \times \{x_3 = \varepsilon\}) \cup (\partial\Omega_T \times (0, \varepsilon))$. We denote by $(E_\varepsilon, P_\varepsilon)$ the solutions satisfying (1.14) in Ω^ε.

Let us set some notations. We define the norm of the complex Lebesgue space $L^2(\Omega^\varepsilon)$ by setting $|F|_\varepsilon^2 = (1/\varepsilon) \int_0^\varepsilon \int_{\Omega_T} |F(x_T, x_3)|^2 dx_T dx_3$ and its scalar product by $(F; G)_\varepsilon = (1/\varepsilon) \int_0^\varepsilon \int_{\Omega_T} F \cdot G^* dx_T dx_3$ where G^* stands for the complex conjugate of G. We use the same notations for the Lebesgue space $L^2(\partial\Omega_T \times (0, \varepsilon))$. If $\Omega = \Omega_T \times (0, 1)$, we write $|\cdot|$ for the norm of $L^2(\Omega)$ and $(\cdot;\cdot)$ for its scalar product. We denote by $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ the canonical basis of $\mathbb{R}^3$.

The paper is organized as follows. In Section 2, we prove uniform bounds for the solution of the model equations (1.14). In Section 3, we then introduce a change of variable with respect to the vertical variable to transform the thin domain Ω^ε to the cylinder Ω with thickness 1. We deduce the uniform bounds for the scaled solutions satisfying the model equations (3.5). In Section 4, we pass to the limit in the weak formulation of (3.5) and deduce the reduced model. The last section is devoted to some remarks.

2. Uniform bounds

Since E_{eq} is a Lipschitz perturbation of order 0 of the operator $(\text{curl}^2, \text{curl}^2)$, then existence and uniqueness of the solution $(E_\varepsilon, P_\varepsilon)$ of problem (1.14) can be obtained by introducing, in the Hilbert space $\mathcal{H} = L^2(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon)$, the unbounded operator $\mathcal{A}$ with domain $D(\mathcal{A}) = \{(E,P) \in \mathcal{H},\ (\text{curl}^2 E, \text{curl}^2 P) \in \mathcal{H},\ \text{div}\, P = 0 \text{ in } \Omega^\varepsilon,\ \text{curl}\, E \times n + \imath\omega\mu\beta^\varepsilon n \times (E \times n) = 0,\ \text{curl}\, P \times n = 0,\ P \cdot n = 0 \text{ on } \partial\Omega^\varepsilon\}$ with $\mathcal{A}(E,P) = (\text{curl}^2 E, \text{curl}^2 P)$ for $(E,P) \in D(\mathcal{A})$. Problem (1.14) writes as $(\mathcal{B}(\omega) + \lambda^2\mathcal{A})(E,P) = \imath\omega\mathcal{F} + b\mathcal{G}(E,P)$ where $\mathcal{B}(\omega)$ is the diagonal block matrix with diagonal $\zeta_1(\omega)I$ and $\zeta_2(\omega)I$, $\mathcal{G}(E,P) = (0, E_{\text{eq}}(P) - E)$, and $\mathcal{F} = (F, 0)$. We use classical results (e.g., see [11–13]) to prove existence and uniqueness of the solution for $\omega > 0$ fixed.

In order to obtain uniform estimates, we multiply the first equation of (1.14) by E_ε^* and the second one by P_ε^* and use the Green formula

$$-(\Delta P_\varepsilon; P_\varepsilon)_\varepsilon = \left|\nabla P_\varepsilon\right|_\varepsilon^2 + \int_{\Gamma^\varepsilon} \mathcal{R}(P_{\Gamma,\varepsilon}) \cdot P_{\Gamma,\varepsilon}^\star d\sigma. \tag{2.1}$$

We get (notice that $(\nabla\pi_\varepsilon, P_\varepsilon)_\varepsilon = 0$)

$$\begin{gathered}
\zeta_1(\omega)\left|E_\varepsilon\right|_\varepsilon^2 + \lambda^2\left|\text{curl}\, E_\varepsilon\right|_\varepsilon^2 + \imath\omega\lambda^2\mu\int_{\partial\Omega^\varepsilon} \beta^\varepsilon\left|E_\varepsilon \times n\right|^2 d\sigma = \omega^2(P_\varepsilon; E_\varepsilon)_\varepsilon + \imath\omega(F; E_\varepsilon)_\varepsilon, \\
\zeta_2(\omega)\left|P_\varepsilon\right|_\varepsilon^2 + \lambda^2\left|\nabla P_\varepsilon\right|_\varepsilon^2 + \lambda^2\int_{\partial\Omega^\varepsilon} \mathcal{R}(P_{\Gamma,\varepsilon}) \cdot P_{\Gamma,\varepsilon}^* d\sigma + b\int_{\Omega^\varepsilon}\left|P_\varepsilon\right|^2\phi'\left(\left|P_\varepsilon\right|^2\right)dx = b(E_\varepsilon; P_\varepsilon)_\varepsilon.
\end{gathered} \tag{2.2}$$

The real parts of each equation write as

$$\begin{gathered}
-\omega^2\left|E_\varepsilon\right|_\varepsilon^2 + \lambda^2\left|\text{curl}\, E_\varepsilon\right|_\varepsilon^2 = \omega^2\Re(P_\varepsilon; E_\varepsilon)_\varepsilon + \Re(\imath\omega(F; E_\varepsilon)_\varepsilon), \\
-\omega^2\left|P_\varepsilon\right|_\varepsilon^2 + \lambda^2\left|\nabla P_\varepsilon\right|_\varepsilon^2 + \int_{\partial\Omega^\varepsilon} \mathcal{R}(P_{\Gamma,\varepsilon}) \cdot P_{\Gamma,\varepsilon}^* d\sigma + b\int_{\Omega^\varepsilon}\left|P_\varepsilon\right|^2\phi'\left(\left|P_\varepsilon\right|^2\right)dx = b\Re(E_\varepsilon; P_\varepsilon)_\varepsilon
\end{gathered} \tag{2.3}$$

and the imaginary parts give

$$\begin{gathered}
a_1\left|E_\varepsilon\right|_\varepsilon^2 + \omega\mu\lambda^2\int_{\partial\Omega^\varepsilon} \beta^\varepsilon\left|E_\varepsilon \times n\right|^2 d\sigma = \omega^2\Im(P_\varepsilon; E_\varepsilon)_\varepsilon + \Im(\imath\omega(F; E_\varepsilon)_\varepsilon), \\
\omega a_2\left|P_\varepsilon\right|_\varepsilon^2 = b\Im(E_\varepsilon; P_\varepsilon)_\varepsilon.
\end{gathered} \tag{2.4}$$

Adding the last equalities and using the property $\Im(P_\varepsilon;E_\varepsilon)_\varepsilon+\Im(E_\varepsilon;P_\varepsilon)_\varepsilon=0$, we get

$$a_1 b\,|E_\varepsilon|_\varepsilon^2+b\omega\mu\lambda^2\int_{\partial\Omega^\varepsilon}\beta^\varepsilon\,|E_\varepsilon\times n|^2 d\sigma+a_2\omega^3\,|P_\varepsilon|_\varepsilon^2=\Im\big(\imath\omega b(F;E_\varepsilon)_\varepsilon\big). \tag{2.5}$$

Using the fact that $\mathcal{R}$ is independent of ε, then there exists $c>0$ which is independent of ε such that $|\int_{\Gamma^\varepsilon}\mathcal{R}(P_{\Gamma,\varepsilon})\cdot P^\star_{\Gamma,\varepsilon}d\sigma|\le c|P_{\Gamma,\varepsilon}|_\varepsilon^2\le c(\eta|\nabla P_\varepsilon|_\varepsilon^2+C_\eta|P_\varepsilon|_\varepsilon^2)$ for all $\eta>0$. We obtain, for η small enough, the following result.

Lemma 2.1. *There exists $C>0$ which is independent of ε (depending on ω and F) such that*

$$\begin{gathered}|E_\varepsilon|_\varepsilon+|\operatorname{curl}E_\varepsilon|_\varepsilon+\left|\sqrt{\beta^\varepsilon}E_\varepsilon\times n\right|_\varepsilon\le C,\\ |P_\varepsilon|_\varepsilon+|\nabla P_\varepsilon|_\varepsilon+|\pi^\varepsilon|_\varepsilon\le C.\end{gathered} \tag{2.6}$$

Moreover,

$$|\Delta P_\varepsilon|_\varepsilon+|\operatorname{curl}^2 E_\varepsilon|_\varepsilon\le C. \tag{2.7}$$

3. The scaled problem and convergences

We introduce the change of variable $z=x_3/\varepsilon$ for $x_T\in\Omega_T$ fixed. We define the cylinder $\Omega=\Omega_T\times(0,1)$ with generic point (x_T,z). For a given vector-valued function $G(x_T,x_3)$ defined on Ω^ε we set $\mathbf{G}^\varepsilon(x_T,z)=G(x_T,\varepsilon z)$ which is defined in Ω. We write $\mathbf{G}^\varepsilon=(G_T^\varepsilon,g^\varepsilon)$ where $\mathrm{G}^\varepsilon=(\mathbf{G}_1^\varepsilon,\mathbf{G}_2^\varepsilon)$ and $g^\varepsilon=\mathbf{G}_3^\varepsilon$. Denoting ∇_T the gradient with respect to the variable x_T we have $\nabla_T G_T=\nabla_T\mathbf{G}^\varepsilon$ and $\partial_{x_3}G=(1/\varepsilon)\partial_z\mathbf{G}^\varepsilon$. Let g be a scalar function and let $G_T=(\mathrm{G}_1,\mathrm{G}_2)$ be a vector-valued function both defined in Ω. We set

$$\begin{aligned}\operatorname{curl}_T g&=(\partial_2 g,-\partial_1 g), & \Delta_T g&=\partial_1^2 g+\partial_2^2 g,\\ \operatorname{Curl}\mathrm{G}_T&=\partial_1\mathrm{G}_2-\partial_2\mathrm{G}_1, & \operatorname{div}_T\mathrm{G}_T&=\partial_1\mathrm{G}_1+\partial_2\mathrm{G}_2.\end{aligned} \tag{3.1}$$

With the change of variable, we have $(1/\varepsilon)\int_0^\varepsilon|G(x_3)|^2dx_3=\int_0^1|\mathbf{G}^\varepsilon(z)|^2dz$ and the differential operators become $\operatorname{curl}G=\operatorname{curl}_\varepsilon\mathbf{G}^\varepsilon$, $\operatorname{div}G=\operatorname{div}_\varepsilon\mathbf{G}^\varepsilon$, $\Delta G=\Delta_\varepsilon\mathbf{G}^\varepsilon$ with

$$\begin{gathered}\operatorname{curl}_\varepsilon\mathbf{G}^\varepsilon=-\frac{1}{\varepsilon}\partial_z(\mathbf{G}^\varepsilon\times\mathbf{u}_3)+\operatorname{curl}_T g^\varepsilon+(\operatorname{Curl}_T\mathrm{G}_T^\varepsilon)\mathbf{u}_3,\\ \operatorname{div}_\varepsilon\mathbf{G}^\varepsilon=\operatorname{div}_T\mathrm{G}_T^\varepsilon+\frac{1}{\varepsilon}\partial_z g^\varepsilon,\\ \Delta_\varepsilon\mathbf{G}^\varepsilon=\Delta_T\mathrm{G}_T^\varepsilon+\frac{1}{\varepsilon^2}\partial_z^2\mathbf{G}^\varepsilon,\qquad \nabla_\varepsilon \mathrm{g}^\varepsilon=\left(\nabla_{x_T}\mathrm{g}^\varepsilon,\frac{1}{\varepsilon}\partial_z\mathrm{g}^\varepsilon\right).\end{gathered} \tag{3.2}$$

We rewrite $\operatorname{curl}_\varepsilon\mathbf{G}^\varepsilon$ as follows:

$$\operatorname{curl}_\varepsilon\mathbf{G}^\varepsilon=(\theta^\varepsilon,\operatorname{Curl}_T\mathrm{G}_T^\varepsilon),\quad \theta^\varepsilon=\left(\partial_2 g^\varepsilon-\frac{1}{\varepsilon}\partial_z\mathrm{G}_2^\varepsilon,\frac{1}{\varepsilon}\partial_z\mathrm{G}_1^\varepsilon-\partial_1 g^\varepsilon\right). \tag{3.3}$$

Notice that $\theta^\varepsilon\cdot\mathbf{u}_3=0$ a.e. Here we have identified the 2D vectors θ^ε and $\operatorname{curl}_T g^\varepsilon$ with the vectors $(\theta^\varepsilon,0)$ of $\mathbb{R}^3$ and $(\partial_2 g^\varepsilon,-\partial_1 g^\varepsilon,0)$, respectively. This identification will be used throughout this manuscript.

Let $(E_\varepsilon, P_\varepsilon)$ be a solution to problem (1.14) associated with the source term F satisfying the hypothesis

$$F = (\mathrm{F}_T(x_T), 0), \qquad \operatorname{div}_T \mathrm{F}_T = 0. \tag{3.4}$$

Using the previous notations, let $\mathbf{E}^\varepsilon = (\mathrm{E}_T^\varepsilon, e^\varepsilon)$ and $\mathbf{P}^\varepsilon = (\mathrm{P}_T^\varepsilon, p^\varepsilon)$ be the scaled solution to (1.14) and let Π^ε be the scaled function associated with π^ε. Then $(\mathbf{E}^\varepsilon, \mathbf{P}^\varepsilon)$ satisfies in Ω the system of equations

$$\begin{gathered}
(\zeta_1(\omega) + \lambda^2 \operatorname{curl}_\varepsilon^2)\mathbf{E}^\varepsilon = \omega^2 \mathbf{P}^\varepsilon + \imath\omega \mathbf{F}(x_T), \\
(\zeta_2(\omega) - \lambda^2 \Delta_\varepsilon)\mathbf{P}^\varepsilon + \nabla_\varepsilon \Pi^\varepsilon = -b(E_{\mathrm{eq}}(\mathbf{P}^\varepsilon) - \mathbf{E}^\varepsilon), \\
\operatorname{div}_\varepsilon \mathbf{P}^\varepsilon = 0, \\
\operatorname{curl}_\varepsilon \mathbf{E}^\varepsilon \times n + \imath\omega\mu\beta^\varepsilon n \times (\mathbf{E}^\varepsilon \times n) = 0 \quad \text{on } \partial\Omega, \\
\operatorname{curl}_\varepsilon \mathbf{P}^\varepsilon \times n = 0, \quad \mathbf{P}^\varepsilon \cdot n = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.5}$$

where $n = (n_T, n_3)$ is the unit outward normal to Ω. We have $n = \mathbf{u}_3$ for $z = 1$, $n = -\mathbf{u}_3$ for $z = 0$, and $n = n_T = (n_1, n_2)$ on $\partial\Omega_T$ for $0 \le z \le 1$.

Let θ^ε be the 2D vector appearing in the definition of $\operatorname{curl}_\varepsilon \mathbf{E}^\varepsilon$. We have

$$\theta^\varepsilon = \left(\partial_2 e^\varepsilon - \frac{1}{\varepsilon}\partial_z \mathrm{E}_2^\varepsilon, \frac{1}{\varepsilon}\partial_z \mathrm{E}_1^\varepsilon - \partial_1 e^\varepsilon\right). \tag{3.6}$$

The boundary conditions satisfied by $(\mathbf{E}^\varepsilon, \mathbf{P}^\varepsilon)$ are rewritten as follows. On $z = 0$ and $z = 1$, we have

$$\begin{gathered}
(\theta^\varepsilon \times \mathbf{u}_3)(x_T, 1) = -\imath\omega\mu\beta_1\varepsilon \mathrm{E}^\varepsilon(x_T, 1), \qquad (\theta^\varepsilon \times \mathbf{u}_3)(x_T, 1) = \imath\omega\mu\beta_0\varepsilon \mathrm{E}^\varepsilon(x_T, 0), \\
p^\varepsilon(x_T, 1) = p^\varepsilon(x_T, 0) = 0, \\
\partial_z \mathrm{P}_T^\varepsilon(x_T, 1) + \varepsilon\mathcal{R}(\mathrm{P}_T^\varepsilon(x_T, 1)) = -\partial_z \mathrm{P}_T^\varepsilon(x_T, 0) + \varepsilon\mathcal{R}(\mathrm{P}_T^\varepsilon(x_T, 0)) = 0,
\end{gathered} \tag{3.7}$$

and on $\partial\Omega_T \times (0, 1)$, we have

$$\begin{gathered}
\operatorname{Curl}_T \mathrm{E}_T^\varepsilon = \imath\omega\beta \mathrm{E}^\varepsilon \times n_T, \qquad \operatorname{Curl}_T \mathrm{P}_T^\varepsilon = 0, \\
\frac{\partial \mathbf{P}_\Gamma^\varepsilon}{\partial n_T} + \mathcal{R}(\mathbf{P}_\Gamma^\varepsilon) = 0, \qquad \mathrm{P}_T^\varepsilon \cdot n_T = 0,
\end{gathered} \tag{3.8}$$

where $\mathrm{E}_T^\varepsilon \times n_T = \mathbf{E}_1^\varepsilon n_2 - \mathbf{E}_2^\varepsilon n_1$. Recall that $\mathbf{P}_\Gamma^\varepsilon = n_T \times (\mathbf{P}^\varepsilon \times n_T)$. Applying the uniform bounds of Lemma 2.1 to the scaled solution $(\mathbf{E}^\varepsilon, \mathbf{P}^\varepsilon)$ and using (3.7), we get the following.

Lemma 3.1. *There exists $C > 0$ which is independent of ε such that*

$$\begin{gathered}
|\mathbf{E}^\varepsilon| + |\mathbf{P}^\varepsilon| + \left|\Pi^\varepsilon\right| \le C, \\
|\theta^\varepsilon| + |\operatorname{Curl}_T E_T^\varepsilon| + |\nabla_T \mathbf{P}^\varepsilon| + \frac{1}{\varepsilon}|\partial_z \mathbf{P}^\varepsilon| \le C, \\
|\operatorname{curl}_\varepsilon^2 \mathbf{E}^\varepsilon| + |\Delta_\varepsilon \mathbf{P}^\varepsilon| \le C.
\end{gathered} \tag{3.9}$$

Moreover, the traces of the solution satisfy the estimates

$$\begin{gathered} \left| E^{\varepsilon}_{T|z=k} \right| \leq C, \quad \left| \theta^{\varepsilon}_{|z=k} \right| \leq C\varepsilon, \quad \text{for } k = 1,2, \\ \left| \mathbf{E}^{\varepsilon} \times n_{|\partial\Omega_T \times (0,1)} \right| \leq C. \end{gathered} \tag{3.10}$$

We will prove the following general result which is useful in the sequel.

Proposition 3.2. *Let $\mathbf{U}^{\varepsilon} = (U^{\varepsilon}_T, u^{\varepsilon})$ be a bounded sequence of $L^2(\Omega)$ such that $\operatorname{curl}_{\varepsilon} \mathbf{U}^{\varepsilon} = (\theta^{\varepsilon}, \operatorname{Curl}_T U^{\varepsilon}_T)$ is bounded in $L^2(\Omega)$ and assume that the tangential trace $\mathbf{U}^{\varepsilon} \times n$ is uniformly bounded in $L^2(\partial\Omega)$. Then, there exists a subsequence, still denoted, $\mathbf{U}^{\varepsilon}$ such that $\mathbf{U}^{\varepsilon} = (U^{\varepsilon}_T, u^{\varepsilon}) \rightharpoonup \mathbf{U} = (U_T, u)$ weakly in $L^2(\Omega)$, $\operatorname{Curl}_T U^{\varepsilon}_T \rightharpoonup \operatorname{Curl}_T U_T$ weakly in $L^2(\Omega)$. Moreover, U_T is independent of z and*

$$\begin{gathered} \left(U^{\varepsilon}_T \times \mathbf{u}_3\right)_{|z=k} \rightharpoonup U_T \times \mathbf{u}_3 \quad \text{weakly in } L^2(\Omega_T) \text{ for } k = 0,1, \\ \int_0^1 U^{\varepsilon}_T \times n_T dz \rightharpoonup U_T \times n_T \quad \text{weakly in } L^2(\partial\Omega_T). \end{gathered} \tag{3.11}$$

Proof. Let $\mathbf{U} = (\mathrm{U}_T, u)$ be the weak limit in $L^2(\Omega)$ of a subsequence of $\mathbf{U}^{\varepsilon}$. Let $\varphi \in \mathcal{D}(\overline{\Omega})$ be a test function. The Green formula gives

$$\begin{aligned} \int_{\Omega} \operatorname{curl}_{\varepsilon} \mathbf{U}^{\varepsilon} \cdot \varphi dx = {} & \int_{\Omega} \mathbf{U}^{\varepsilon} \cdot \operatorname{curl}_{\varepsilon} \varphi dx - \int_{\partial\Omega_T \times (0,1)} \mathbf{U}^{\varepsilon} \times n \cdot \varphi d\sigma \\ & - \frac{1}{\varepsilon} \int_{\Omega_T} \left(\mathbf{U}^{\varepsilon} \times \mathbf{u}_3\right)_{|z=1} \cdot \varphi_{|z=1} dx_T + \frac{1}{\varepsilon} \int_{\Omega_T} \left(\mathbf{U}^{\varepsilon} \times \mathbf{u}_3\right)_{|z=0} \cdot \varphi_{|z=0} dx_T. \end{aligned} \tag{3.12}$$

Firstly, we choose in the Green formula $\varphi = \varepsilon\phi$ with $\phi = (\phi_1, \phi_2, 0) = (\phi_T, 0) \in \mathcal{D}(\Omega)$. Since $\operatorname{curl}_{\varepsilon} \varphi = -\partial_z(\phi \times \mathbf{u}_3) + \varepsilon(\operatorname{Curl}_T \phi_T)\mathbf{u}_3$, then passing to the limit in (3.12), we get

$$\int_{\Omega} -\mathrm{U}_1 \partial_z \phi_2 + \mathrm{U}_2 \partial_z \phi_1 dx = 0 \tag{3.13}$$

which implies that $\partial_z \mathrm{U}_T = 0$ in the sense of distributions so, U_T is independent of the variable z. Next, let A_j be the weak limit in $L^2(\Omega_T)$ of a subsequence of the traces $(\mathbf{U}^{\varepsilon} \times \mathbf{u}_3)_{|z=j}$ for $j = 0, 1$. To identify A_1, we choose in the Green formula $\varphi = \varepsilon z \phi$ with $\phi = (\phi_1(x_T), \phi_2(x_T), 0) \in (\mathcal{D}(\Omega_T))^3$. Passing to the limit in (3.12), we get

$$\int_{\Omega_T} -\mathrm{U}_1 \phi_2 + \mathrm{U}_2 \phi_1 dx - \int_{\Omega_T} A_1 \cdot \phi dx_T = 0 \tag{3.14}$$

which shows that $A_1 = \mathrm{U}_T \times \mathbf{u}_3$. Secondly, we use the test function $\varphi = \varepsilon(1-z)\phi$ with $\phi = (\phi_1(x_T), \phi_2(x_T), 0) \in (\mathcal{D}(\Omega_T))^2$ in the Green formula (3.12) and pass to the limit, we get

$$\int_{\Omega_T} \mathrm{U}_1\phi_2 - \mathrm{U}_2\phi_1 dx + \int_{\Omega_T} A_0 \cdot \phi dx_T = 0. \tag{3.15}$$

Thus, we get $A_0 = \mathrm{U}_T \times \mathbf{u}_3$ and $A_0 = A_1$. Finally, let g be the weak limit in $L^2(\partial\Omega_T \times (0,1))$ of a subsequence of the traces $\mathrm{U}^\varepsilon \times n_{T|\partial\Omega_T \times (0,1)}$. To characterize g, we consider the test function $\varphi = (0,0,\phi_3(x_T))$ with $\phi_3 \in \mathcal{D}(\overline{\Omega_T})$. Observing that $\mathrm{curl}_\varepsilon\, \varphi = \mathrm{curl}_T\, \phi_3 = (\partial_2\phi_3, -\partial_1\phi_3, 0)$ and passing to the limit in (3.12), since U is independent of the variable z, we deduce that

$$\int_{\Omega_T} \phi_3 \,\mathrm{Curl}_T\, \mathrm{U}_T dx_T = \int_{\Omega_T} \mathrm{U}_T \cdot \mathrm{curl}_T\, \phi_3 dx_T - \int_{\partial\Omega_T} \left(\int_0^1 g dz \right) \phi_3 dx_T. \tag{3.16}$$

Now, since $\mathrm{Curl}_T\, \mathrm{U}_T \in L^2(\Omega_T)$, then $\mathrm{U}_T \times n_T$ is well defined, we finally deduce that $\int_0^1 g dz = \mathrm{U}_T \times n_T$. Hence, Proposition 3.2 is proved. $\square$

Applying Proposition 3.2 to the fields $\mathbf{E}^\varepsilon = (\mathrm{E}_T^\varepsilon, e^\varepsilon)$ and $\mathrm{curl}_\varepsilon\, \mathbf{E}^\varepsilon = (\theta^\varepsilon, \mathrm{Curl}_T\, \mathrm{E}_T^\varepsilon)$, we get the following.

Lemma 3.3. *There exist subsequences, still denoted, $\mathbf{E}^\varepsilon$ and θ^ε such that the following weak convergences in $L^2(\Omega)$ hold:*

$$E_T^\varepsilon \rightharpoonup E_T, \qquad e^\varepsilon \rightharpoonup e, \qquad \mathrm{Curl}_T\, E_T^\varepsilon \rightharpoonup \mathrm{Curl}_T\, E_T, \qquad \theta^\varepsilon \rightharpoonup 0, \tag{3.17}$$

and E_T is independent of z. Moreover, the traces satisfy the convergences

$$\begin{gathered} E^\varepsilon_{T|z=k} \rightharpoonup E_T \text{ weak}, \qquad \theta^\varepsilon_{|z=k} \longrightarrow 0 \text{ strong}, \\ \int_0^1 E_T^\varepsilon \times n_T dz \rightharpoonup E_T \times n_T \end{gathered} \tag{3.18}$$

in $L^2(\Omega_T)$ for $k = 1,2$ and in $L^2(\partial\Omega_T)$, respectively.

Proof. Lemma 3.1 implies the strong convergence of $\theta^\varepsilon{}_{|z=j}$ to 0. Next, set $\mathbf{U}^\varepsilon = \mathrm{curl}_\varepsilon\, \mathbf{E}^\varepsilon$. As $\mathbf{U}^\varepsilon = (\theta^\varepsilon, \mathrm{Curl}_T\, \mathrm{E}_T^\varepsilon)$ satisfies the conditions of the previous proposition, then $\theta^\varepsilon \rightharpoonup \theta$ weakly in $L^2(\Omega)$ and θ is independent of the variable z. Using again Proposition 3.2, we get $(\theta^\varepsilon \times \mathbf{u}_3)_{|z=k} \rightharpoonup \theta \times \mathbf{u}_3$ weakly in $L^2(\Omega_T)$ for $k = 1,2$. Since $(\theta^\varepsilon \times \mathbf{u}_3)_{|z=k} \to 0$ strongly, then $\theta \equiv 0$ in Ω. $\square$

Now, we consider the convergences for $\mathbf{P}^\varepsilon$. We have the following.

Lemma 3.4. *There exists a subsequence, still denoted by* $\mathbf{P}^\varepsilon$ *such that*

$$\begin{gathered}\mathbf{P}^\varepsilon \longrightarrow \mathbf{P}=(P_T,0) \text{ strong}, \qquad \nabla_T\mathbf{P}^\varepsilon \rightharpoonup \nabla_T(P_T,0), \qquad \Pi^\varepsilon \rightharpoonup \Pi \text{ weak},\\ \partial_z\mathbf{P}^\varepsilon \longrightarrow 0, \qquad E_{\mathrm{eq}}(\mathbf{P}^\varepsilon) \longrightarrow E_{\mathrm{eq}}(P_T)=2P_T\phi'(|P_T|^2) \text{ strong}\end{gathered} \tag{3.19}$$

in $L^2(\Omega)$. *Moreover,* P_T *and* Π *are independent of the variable* z *and* $e=0$. *Finally,* P_T *satisfies on* $\partial\Omega_T$ *the boundary condition* $P_T\cdot n_T=0$.

Proof. Using the bounds in $L^2(\Omega)$ of $\nabla_\varepsilon\mathbf{P}^\varepsilon$ and $\mathbf{P}^\varepsilon$, we deduce that there exists a subsequence such that $\mathbf{P}^\varepsilon \rightharpoonup \mathbf{P}=(\mathrm{P}_T,p)$ weakly in $H^1(\Omega)$. Moreover, $\partial_z\mathbf{P}^\varepsilon\to 0$ strongly in $L^2(\Omega)$. It follows that $\mathbf{P}$ is independent of z. Furthermore, the pressure Π^ε converges weakly to Π in $L^2(\Omega)$. Next, the trace of $\mathbf{P}^\varepsilon$ on $\partial\Omega$ converges weakly in $H^{1/2}(\partial\Omega)$ to the trace of $\mathbf{P}$. Since we have $p^\varepsilon(x_T,1)=p^\varepsilon(x_T,0)=0$ and p is independent of z, then $p=0$. The trace $\mathbf{P}^\varepsilon\times n \rightharpoonup \mathbf{P}\times n$ weakly in $L^2(\partial\Omega_T\times(0,1))$. We may pass to the limit in the boundary condition to get $\mathbf{P}\cdot n=0$ on $\partial\Omega$ which gives $\mathrm{P}_T\cdot n_T=0$. We rewrite the Neumann boundary condition in its original form $\operatorname{curl}_\varepsilon\mathbf{P}^\varepsilon\times n=0$. We write $\operatorname{curl}_\varepsilon\mathbf{P}^\varepsilon=(\theta_1^\varepsilon,\operatorname{Curl}_T\mathrm{P}_T^\varepsilon)$, where θ_1^ε is defined as in Section 3, then the boundary condition becomes

$$\begin{gathered}\theta_1^\varepsilon(x_T,1)=\theta_1^\varepsilon(x_T,0)=0,\\ \theta_1^\varepsilon\times n_T=0, \quad \operatorname{Curl}_T\mathrm{P}_T^\varepsilon=0, \quad \text{on } \partial\Omega_T.\end{gathered} \tag{3.20}$$

We apply Proposition 3.2 to $\mathbf{U}^\varepsilon=\operatorname{curl}_\varepsilon\mathbf{P}^\varepsilon$. Since we have $\Delta_\varepsilon\mathbf{P}^\varepsilon=\operatorname{curl}_\varepsilon(\operatorname{curl}_\varepsilon\mathbf{P}^\varepsilon)$ because we have $\operatorname{div}_\varepsilon\mathbf{P}^\varepsilon=0$, then by Lemma 3.1, it follows that $\operatorname{curl}_\varepsilon\mathbf{U}^\varepsilon$ is bounded in $L^2(\Omega)$. Applying Proposition 3.2, we deduce that θ_1, the weak limit of θ_1^ε, is independent of z where θ_1 is the weak limit of θ_1^ε. Finally, using the boundary condition satisfied by θ_1^ε at $z=0$ and $z=1$, we deduce that $\theta_1=0$. Next, we use the bound in $L^2(\Omega)$ of $\nabla_\varepsilon\mathrm{P}_T^\varepsilon$ to deduce that $\operatorname{Curl}_T\mathrm{P}_T^\varepsilon\rightharpoonup\operatorname{Curl}_T\mathrm{P}_T$ and $\mathrm{P}_T^\varepsilon\cdot n_T\rightharpoonup\mathrm{P}_T\cdot n_T$ weakly in $H^{1/2}(\partial\Omega)$. To end the proof of the lemma, we will prove that Π is independent of z and $e=0$. We set $\sigma^\varepsilon=\partial_z((1/\varepsilon)p^\varepsilon)$. The condition $\operatorname{div}_\varepsilon\mathbf{P}^\varepsilon=0$ is rewritten as $\operatorname{div}_T\mathrm{P}_T^\varepsilon+\sigma^\varepsilon=0$ and from the equation satisfied by $\mathbf{P}^\varepsilon$ we deduce that $-\lambda^2\partial_z\sigma^\varepsilon+\partial_z\Pi^\varepsilon=\varepsilon R^\varepsilon$ where the remainder term R^ε is bounded in $L^2(\Omega)$. Since σ^ε is bounded in $L^2(\Omega)$, then passing to the limit we get $\operatorname{div}\mathrm{P}_T+\sigma=0$ and $-\lambda^2\partial_z\sigma+\partial_z\Pi=0$. Since P_T is independent of z, then so is σ which implies that $\partial_z\Pi=0$. Let us consider the equation satisfied by $\mathbf{E}^\varepsilon$. We multiply the equation by the test function $\varphi=(0,0,\phi)$ with $\phi\in\mathcal{D}(\Omega)$. Since we have $\operatorname{curl}_\varepsilon\varphi=(\partial_2\phi,-\partial_1\phi,0)$, then $\operatorname{curl}_\varepsilon\mathbf{E}^\varepsilon\cdot\operatorname{curl}_\varepsilon\varphi=\theta^\varepsilon\cdot\operatorname{curl}_T\phi$, then we get after an integration by parts (recall that the third component of $\mathbf{F}$ is 0)

$$\int_\Omega\left(\zeta_1(\omega)e^\varepsilon-\omega^2p^\varepsilon\right)\phi\,dx+\lambda^2\int_\Omega\theta^\varepsilon\cdot\operatorname{curl}_T\phi\,dx=0. \tag{3.21}$$

Passing to the limit, we obtain $\zeta_1(\omega)e-\omega^2p=0$. Using that $p=0$, we get $e=0$. □

4. The reduced problem

Let us introduce the Hilbert space $H(\mathrm{Curl}_T,\Omega_T)=\{U\in L^2(\Omega_T)^2,\mathrm{Curl}_T\,U\in L^2(\Omega_T)\}$. We will prove the following main result describing the dimensional reduction of the thin ferroelectric cylinder.

Theorem 4.1. *Let $F\in(L^2(\Omega_T))^2$ be such that $\mathrm{div}_T\,F=0$ in Ω_T. Then for $\omega>0$ fixed, there exists a unique solution $(E,P)\in H(\mathrm{Curl}_T,\Omega_T)\times H(\mathrm{Curl}_T,\Omega_T)$ of the reduced problem*

$$\begin{gathered}
\zeta_1(\omega)E+\lambda^2\,\mathrm{curl}_T\left(\mathrm{Curl}_T\,E\right)+\imath\omega\mu(\beta_1+\beta_0)E-\omega^2P=\imath\omega F,\\
\mathrm{Curl}_T\,E+\imath\omega\mu\beta E\times n_T=0,\quad \text{on } \partial\Omega_T,\\
\zeta_2(\omega)P-\lambda^2\Delta P+\nabla_T\Pi=-b\big(E_{\mathrm{eq}}(P)-E\big),\\
\mathrm{div}_T\,P=0,\\
\mathrm{curl}_T\,P=0,\quad P\cdot n_T=0,\quad \text{on } \partial\Omega_T.
\end{gathered}\tag{4.1}$$

Furthermore, $\mathrm{Curl}_T\,E,\mathrm{Curl}_T\,P\in H^1(\Omega_T)$ and the solution is obtained as the limit of the sequence $(\mathbf{E}^\varepsilon,\mathbf{P}^\varepsilon)$ of the model problem (3.5).

Proof. To prove this theorem, we pass to the limit in the weak formulation of equation (3.5). Since the limit solution $(\mathbf{E},\mathbf{P})$ is independent of z and $e=p=0$, we choose test functions of the type $\varphi=(\varphi_1,\varphi_2,0)$, $\psi=(\psi_1,\psi_2,0)$ and a scalar function ϕ which are independent of the variable z. We suppose that $\phi,\varphi,\psi\in\mathcal{D}(\overline{\Omega_T})$ with $\mathrm{div}_T\,\psi=0$ and $\psi\cdot n_T=0$ on $\partial\Omega_T$. Multiplying the first equation by φ^*, the second by ψ^*, and the constraint $\mathrm{div}_\varepsilon\,\mathbf{P}^\varepsilon=0$ by ϕ^* then integrating by parts we get

$$\begin{gathered}
\int_\Omega\big(\zeta_1(\omega)\mathrm{E}_T^\varepsilon-\omega^2\mathrm{P}_T^\varepsilon\big)\cdot\varphi^*\,dx+\lambda^2\int_\Omega\mathrm{Curl}_T\,\mathrm{E}_T^\varepsilon\cdot\mathrm{Curl}_T\,\varphi^*\,dx\\
+\imath\mu\omega\beta\int_{\partial\Omega_T\times(0,1)}\big(\mathrm{E}_T^\varepsilon\times n_T\big)\cdot\big(\varphi^*\times n_T\big)d\sigma+\imath\mu\omega\beta_0\int_{\Omega_T}\big(\mathrm{E}_T^\varepsilon\times\mathbf{u}_3\big)\cdot\big(\varphi^*\times\mathbf{u}_3\big)(x_T,0)\,dx_T\\
+\imath\mu\omega\beta_1\int_{\Omega_T}\big(\mathrm{E}_T^\varepsilon\times\mathbf{u}_3\big)\cdot\big(\varphi^*\times\mathbf{u}_3\big)(x_T,1)\,dx_T=\imath\omega\int_\Omega\mathrm{F}\cdot\varphi^*\,dx,\\
\int_\Omega\big(\zeta_2(\omega)\mathrm{P}_T^\varepsilon+b\big(\mathrm{E}_{\mathrm{eq}}(\mathrm{P}^\varepsilon)-\mathrm{E}_T^\varepsilon\big)\big)\cdot\psi^*\,dx+\lambda^2\int_\Omega\mathrm{Curl}_T\,\mathrm{P}_T^\varepsilon\cdot\mathrm{Curl}_T\,\psi^*\,dx=0,\\
\int_\Omega\mathrm{P}_T^\varepsilon\cdot\nabla_T\phi^*\,dx=0,
\end{gathered}\tag{4.2}$$

where we set $E_{\mathrm{eq}}(\mathbf{P}^\varepsilon)=(\mathrm{E}_{\mathrm{eq}}(\mathbf{P}^\varepsilon),e_{\mathrm{eq}}(\mathbf{P}^\varepsilon))$. We used $-\Delta_\varepsilon\mathbf{P}^\varepsilon=\mathrm{curl}_\varepsilon^2\,\mathbf{P}^\varepsilon$, $\mathrm{div}_\varepsilon\,\mathbf{P}^\varepsilon=0$, $\mathrm{curl}_\varepsilon\,\mathbf{E}^\varepsilon=(\theta^\varepsilon,\mathrm{Curl}_T\,\mathrm{E}_T^\varepsilon)$, $\mathrm{curl}_\varepsilon\,\varphi=(0,0,\mathrm{Curl}_T\,\varphi)$, and the same properties for $\mathbf{P}^\varepsilon$.

Applying our convergence results proved in Section 3 and passing to the limit in the weak formulation, we obtain

$$\begin{aligned}
&\int_{\Omega_T} (\zeta_1(\omega)\mathrm{E}-\omega^2\mathrm{P})\cdot\varphi^*\,dx_T+\lambda^2\int_{\Omega_T}\mathrm{Curl}_T\,\mathrm{E}\cdot\mathrm{Curl}_T\,\varphi^*\,dx_T\\
&\qquad +\imath\omega\mu(\beta_1+\beta_0)\int_{\Omega_T}(\mathrm{E}\times\mathbf{u}_3)\cdot(\varphi^*\times\mathbf{u}_3)\,dx_T+\imath\omega\beta\mu\int_{\partial\Omega_T}(\mathrm{E}\times n_T)\cdot(\varphi^*\times n_T)\,d\sigma\\
&\quad =\imath\omega\int_{\Omega_T}\mathrm{F}\cdot\varphi^*\,dx_T,\\
&\quad \int_{\Omega_T}\big(\zeta_2(\omega)\mathrm{P}+b\big(\mathrm{E}_{\mathrm{eq}}(\mathrm{P})-\mathrm{E}\big)\big)\cdot\psi^*\,dx_T+\lambda^2\int_{\Omega_T}\mathrm{Curl}_T\,\mathrm{P}\cdot\mathrm{Curl}_T\,\psi^*\,dx_T=0,\\
&\qquad\qquad \int_{\Omega_T}\mathrm{P}\cdot\nabla_T\phi^*\,dx_T=0.
\end{aligned}\tag{4.3}$$

Observe that the condition $\mathrm{div}_T\,\mathrm{P}=0$ shows that $\mathrm{curl}_T(\mathrm{Curl}_T\,\mathrm{P})=-\Delta\mathrm{P}$. Our main result is then proved. □

5. Concluding remarks

Let us conclude this work by the following remarks. If we impose that the regular part of the polarization P is 0, then $P=\nabla\varphi$ with $\varphi=c$ on $\partial\Omega$. The equation satisfied by the polarization field writes in Ω as

$$\big(\zeta_2(\omega)+2b\phi'\big(|\nabla\varphi|^2\big)\big)\nabla\varphi=bE\tag{5.1}$$

while the electric field E satisfies the Maxwell equation

$$\zeta_1(\omega)E+\lambda^2\,\mathrm{curl}^2\,E-\omega^2\nabla\varphi=\imath\omega F.\tag{5.2}$$

We set, for $X\in\mathbb{C}$, $a(|X|^2)=2b\phi'(|X|^2)-\omega^2+\imath\omega a_2$. We will show that the map

$$X\in\mathbb{C}\longmapsto H(X)=a(|X|^2)X\in\mathbb{C}\tag{5.3}$$

is onto. The equation $a(|X|^2)X=Y$ gives $|a(r)|^2r=t$ with $r=|X|^2$ and $t=|Y|^2$ or equivalently as $((2b\phi'(r)-\omega^2)^2+\omega^2a_2^2)r=t$. Let

$$\theta(r)=\left|a(r)\right|^2r=\big(2b\phi'(r)-\omega^2\big)^2r+\omega^2a_2^2r,\tag{5.4}$$

then we have $\theta'(r)=(2b\phi'(r)-\omega^2)^2+a_2^2\omega^2+4br\phi^{(2)}(r)(2b\phi'(r)-\omega^2)$. It follows that $\theta'(r)=(2b\phi'(r)-\omega^2+2br\phi^{(2)}(r))^2+a_2^2-4b^2r^2(\phi^{(2)}(r))^2$. Assuming that $0\le r\phi^{(2)}(r)\le a_2\omega/(2b)$ for all $r\ge0$, we get $\theta'(r)>0$ for all $r\ge0$. Consequently, θ is invertible and for all $t\ge0$ there exists a unique $r\ge0$ given by $r=\theta^{-1}(t)$. Hence, for all $Y\in\mathbb{C}^3$, the equation $\theta(|X|^2)=|Y|^2$ gives $|X|^2=\theta^{-1}(|Y|^2)$. Finally, for given $Y\in\mathbb{C}^3$, the equation $a(|X|^2)X=Y$ admits a unique solution $X\in\mathbb{C}^3$ given by $X=Y/a(\theta^{-1}(|Y|^2))$.

Now, let $E \in L^2(\Omega)$, then there exists a unique $U \in L^2(\Omega)$ solution to the equation $a(|U|^2)U = bE$ which is given by

$$U = \frac{bE}{a(\theta^{-1}(b^2|E|^2))}. \tag{5.5}$$

Finally, E should satisfy the nonlinear Maxwell equation

$$\begin{gathered} \zeta_1(\omega)E + \lambda^2 \operatorname{curl}^2 E = \omega^2 b\sigma(|E|^2)E + \imath\omega F, \\ \operatorname{curl}(\sigma(|E|^2)E) = 0, \\ E \times n = 0, \end{gathered} \tag{5.6}$$

with $\sigma(|E|^2) = 1/a(\theta^{-1}(b^2|E|^2))$. The condition $\operatorname{curl}(\sigma(|E|^2)E) = 0$ allows to show that $\sigma(|E|^2)E$ is a gradient. We will come back to this problem in a forthcoming work.

In [5, Section 3], and [6, Section 2.2], the authors introduce the following model equations to describe the dynamic of the time-dependant spontaneous polarization p in a ferroelectric domain Ω (here we use the Daví notations):

$$\begin{gathered} \rho_m \partial_t^2 \mathrm{p} + (\mathrm{D} + \mathrm{G})\partial_t \mathrm{p} - \sigma^2 \Delta \mathrm{p} = \frac{\partial \mathcal{W}}{\partial \mathrm{p}}(\mathrm{p}) + \frac{\partial \varphi}{\partial \mathrm{p}}(\mathbf{F}, \mathrm{e}, \mathrm{p}) - \rho \mathrm{e}, \quad \text{in } \Omega \times (0,T), \\ \sigma^2 \frac{\partial \mathrm{p}}{\partial n} = \mathrm{t}, \quad \text{on } \partial\Omega \times (0,T), \\ \mathrm{p}(0) = \mathrm{p}_0, \qquad \rho_m \partial_t \mathrm{p}(0) = \mathrm{p}_1. \end{gathered} \tag{5.7}$$

The electric field $\mathbf{e}$ with p satisfies in $\mathbb{R}^3 \times (0,T)$ the electrostatic equations

$$\operatorname{div}(\rho \mathrm{p} + \mathrm{e}) = 0, \qquad \operatorname{curl} \mathrm{e} = 0, \tag{5.8}$$

with the natural jump conditions across the boundary $\partial\Omega \times (0,T)$. The parameters appearing in the equation are defined in [5, 6]. It is important to notice that the system is coupled to some elasticity model describing the dynamic of the deformation $\mathbf{F}$ (e.g., when we assume that $\mathbf{F} = \mathbf{I} + \nabla u$ where u is the mechanical displacement, see [6, Section 3]). Next, the nonhomogeneous boundary condition satisfied by p takes into account the density of the electric dipoles. This model is more complete than the one introduced in [1]. If we consider rigid body, then both models are essentially the same. An interesting question is to study the full model satisfied by $(\mathrm{e}, \mathrm{p}, u)$.

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