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Research Article

Nonlinear Periodic Systems with the p-Laplacian: Existence and Multiplicity Results

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We study second-order nonlinear periodic systems driven by the vector p-Laplacian with a nonsmooth, locally Lipschitz potential function. Under minimal and natural hypotheses on the potential and using variational methods based on the nonsmooth critical point theory, we prove existence theorems and a multiplicity result. We conclude the paper with an existence theorem for the scalar problem, in which the energy functional is indefinite (unbounded from both above and below).

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1. Introduction

In this paper, we study the following nonlinear nonautonomous second-order periodic system driven by the one-dimensional p-Laplacian:

$$\begin{gathered}-\left(\|x'(t)\|^{p-2}x'(t)\right)' \in \partial j(t,x(t)) + h(t), \quad \text{a.e. on } T=[0,b],\\ x(0)=x(b), \quad x'(0)=x'(b), \quad 1<p<+\infty.\end{gathered} \tag{I}$$

Here $h \in L^1(T,\mathbb{R}^N)$, the potential function $j(t,x)$ is in general nondifferentiable, locally Lipschitz and by $\partial j(t,x)$ we denote the generalized subdifferential. So problem (I) is a periodic hemivariational inequality. Hemivariational inequalities are a new type of variational expressions which arise naturally in mechanics and engineering when one wants to consider more realistic nonmonotone and multivalued laws. For several concrete applications we refer to the book of Naniewicz and Panagiotopoulos [1].

In the past, the works on nonautonomous periodic systems in which the existence of solutions is obtained as a critical point of the energy functional, focused on the semilinear case (i.e., $p = 2$) with a smooth potential (i.e., $j(t,\cdot) \in C^1(\mathbb{R}^N)$). We can mention the works of Berger and Schechter [2] who employ a coercivity condition, Mawhin [3], where the potential function is convex, Mawhin and Willem [4], where the right-hand side is L^1-bounded, Tang [5, 6] where the potential exhibits a strictly subquadratic growth, and Tang and Wu [7] who use a nonuniform coercivity condition which generalizes the condition in use in the aforementioned work of M. S. Berger and M. Schechter. In this paper, one can find also multiplicity results.

On the other hand, recently there has been increasing interest for nonlinear nonautonomous periodic problems driven by the ordinary p-Laplacian. Most works deal with scalar equations. Systems driven by the vector p-Laplacian or p-Laplacian-like operators were studied by Manásevich and Mawhin [8], Mawhin [9, 10], Kyritsi et al. [11], E. H. Papageorgiou and N. S. Papageorgiou [12]. In these works, the approach is different based on degree theory or nonlinear operators of monotone type and only the problem of existence of solutions is addressed. No multiplicity results are proved.

In the last years, using variational methods on the nonsmooth critical point theory (see [13]), the problem (I) has been studied in order to obtain existence and multiplicity results. We can mention the papers of E. H. Papageorgiou and N. S. Papageorgiou [14, 15], or the numerous results collected in the book of Gasiński and Papageorgiou (see [13, Section 3.4.1]).

In this paper, using the same critical point theory, under minimal and natural hypotheses we prove some existence results and a multiplicity theorem. In particular in Theorem 3.3, using a result obtained by Tang and Wu [7], we prove the existence of nontrivial solutions requiring, among the others, that the potential function $j(t,x)$ satisfies a locally, nonuniform anticoercivity condition (i.e., $j(t,x) \to -\infty$ as $\|x\| \to \infty$ for almost t in some positive-measure subset of T). Moreover we do not assume any polynomial growth of the subdifferential $\partial j(t,x)$. The result so obtained extends the analogous and just mentioned results of [13–15], in the sense that there exist potential functions satisfying our hypotheses but not those of the mentioned theorems (see Remark 3.5).

While in this first theorem we obtain the coercivity of the energy functional and so the result is obtained by an application of the least action principle, in Theorem 3.10 we consider the case in which the energy functional is bounded below but not coercive and in Theorem 3.14 this functional is indefinite (i.e., unbounded from both above and below). In particular in Theorem 3.14 we make an Ambrosetti-Rabinowitz-type assumption (see $H(j)_4$(iv)) which, together with the other hypotheses, implies a growth condition on $j(t,x)$ strictly less than p. Moreover, Theorem 4.1 gives us the existence of multiple solutions in the setting of local, nonuniform anticoercive potential function. All the last three theorems extend, in the sense explained above, analogous results of [13–15].

Finally in the last section, we consider a scalar problem, for which we prove (see Theorem 5.1) the existence of a solution by permitting, asymptotically at ∞, a partial interaction with λ_0 and λ_1, being λ_0 and λ_1 the first two eigenvalues of the negative scalar p-Laplacian with periodic boundary conditions. This theorem generalizes the results of [16] and [13, Theorem 3.4.9].

Many examples are given for showing the various comparisons.

2. Mathematical preliminaries

As we have said in the introduction, our approach is variational, based on the nonsmooth critical point theory. For the convenience of the reader, in this section we recall the main items of the mathematical background needed to follow this paper. Our main references are the books of Hu and Papageorgiou [17, 18] and Gasiński and Papageorgiou [13, 19].

Let X be a reflexive Banach space and X^* its topological dual. A function $\Phi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every bounded open set $U \subset X$, there exists a constant $k_U > 0$ such that $|\Phi(z) - \Phi(y)| \leq k_U \|z - y\|$ for all $z, y \in U$. For this kind of functions, define the generalized directional derivative $\Phi^o(x;h)$ at $x \in X$ in the direction $h \in X$ in this way:

$$\Phi^o(x;h) = \limsup_{x' \to x \ \lambda \downarrow 0} \frac{\Phi(x' + \lambda h) - \Phi(x')}{\lambda}. \tag{2.1}$$

It is known that the function $h \to \Phi^o(x;h)$ is sublinear, continuous and it is the support function of the nonempty, convex and w^*-compact set

$$\partial\Phi(x) = \{x^* \in X^* : (x^*, h) \leq \Phi^o(x;h) \ \forall h \in X\}. \tag{2.2}$$

The set $\partial\Phi(x)$ is called the generalized or Clarke subdifferential of Φ at x. If $\Phi, \Psi : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\Phi + \Psi)(x) \subseteq \partial\Phi(x) + \partial\Psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\Phi)(x) = \lambda\partial\Phi(x)$. Moreover, if $\Phi : X \to \mathbb{R}$ is also convex, then this subdifferential coincides with the subdifferential in the sense of convex analysis. If $\Phi : X \to \mathbb{R}$ is strictly differentiable, then $\partial\Phi(x) = \{\Phi'(x)\}$. A point $x \in X$ is a critical point of Φ if $0 \in \partial\Phi(x)$ while a critical value is the value assumed by Φ in a critical point. It is easy to check that if $x \in X$ is a local extremum (i.e., a local minimum or maximum), then x is a critical point.

The compactness conditions for locally Lipschitz functionals $\Phi : X \to \mathbb{R}$ that we consider are the following.

Φ satisfies the "Palais-Smale condition" ((PS)-condition in short) if any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\{\Phi(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) \to 0$, as $n \to \infty$, has a convergent subsequence (where $m(x_n) = \min_{x^* \in \partial\Phi(x_n)} \|x^*\|_{X^*}$; the existence of such an element follows from the fact that $\partial\Phi(x_n)$ is weakly compact and the norm functional on X^* is weakly lower semicontinuous).

A weaker compactness condition is given by the following.

Φ satisfies the "Cerami Palais-Smale condition" (C-(PS)-condition) if any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\{\Phi(x_n)\}_{n\geq 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$, as $n \to \infty$, has a convergent subsequence.

We say that $\Phi : X \to \mathbb{R}$ is coercive if $\Phi(x) \to +\infty$ as $\|x\| \to \infty$, while Φ is anticoercive if $\Phi(x) \to -\infty$ when $\|x\| \to \infty$.

Finally, let $A : X \to X^*$ be an operator. We recall the following definitions:

A is said to be monotone if $\langle Ax_1 - Ax_2, x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in X$;

A is said to be pseudomonotone if for any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$ weakly in X and $\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle \leq 0$ it follows that $\langle Ax, x - w\rangle \leq \liminf_{n\to\infty} \langle Ax_n, x_n - w\rangle$, for all $w \in X$;

A is said to be demicontinuous if for any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$ in X it follows that $Ax_n \to Ax$ weakly in X^*.

In what follows we employ on $\mathbb{R}^N$ where the Euclidean norm is denoted by $\|\cdot\|$ and the usual inner product is denoted by $(\cdot,\cdot)$. Also by $\|\cdot\|_p$ $(1 \leq p \leq +\infty)$ we denote the L^p norm. For the Sobolev space $W^{1,p}(T,\mathbb{R}^N)$ the norm will be denoted by $\|\cdot\|$ (there will not be confusion with the norm in $\mathbb{R}^N$ because it will be clear from the context which norm is used). Finally, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}(T,\mathbb{R}^N), W^{1,p}(T,\mathbb{R}^N)^*)$ or for $(W_0^{1,p}(T,\mathbb{R}^N), W^{-1,q}(T,\mathbb{R}^N))$ $(1/p+1/q=1)$ and by $(\cdot,\cdot)_{pq}$ the duality brackets for the pair $(L^p(T,\mathbb{R}^N), L^q(T,\mathbb{R}^N))$ $(1/p+1/q=1)$.

3. Existence results

In this section, we will prove some existence theorems under different conditions on the potential function $j(t,x)$ in order to cover a large class of problems for which we obtain the existence of nontrivial solutions.

For the first existence result our hypotheses on the nonsmooth potential function $j(t,x)$ are the following:

$H(j)_1$: $j: T\times\mathbb{R}^N \to \mathbb{R}$ is a function such that

(i) for all $x \in R^N$, $t \mapsto j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$ there exists $a_r \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \leq r$ and all $u \in \partial j(t,x)$, we have $\|u\| \leq a_r(t)$;

(iv) there exists $C \subset T$, $|C| > 0$, such that $j(t,x) \to -\infty$ as $\|x\| \to \infty$ for almost all $t \in C$;

(v) there exists $\beta \in L^1(T)_+$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have $j(t,x) \leq \beta(t)$;

(v) $j(\cdot,0) \in L^1(T)$ and $\exists x_0 \in \mathbb{R}^N\setminus\{0\}$ such that $\int_0^b j(t,x_0)dt > 0$ and $\int_0^b j(t,0)dt \leq 0$.

Remark 3.1. From the assumption that the function $j(\cdot,0)$ is in $L^1(T)$, from the mean value theorem (see [19, page 552]), and from $H(j)_1$(iii) we obtain that, for all $x \in R^N$, $j(\cdot,x) \in L^1(T)$.

Moreover, we observe that condition $H(j)_1$(iii) is general enough since we do not assume any polynomial growth on the subdifferential $\partial j(t,x)$. The following example puts in evidence this fact.

Example 3.2. Let $j: T\times\mathbb{R}^N \to \mathbb{R}$ be defined as

$$j(t,x) = k(t)\big(-e^{\|x\|^p} + \alpha\|x\| + 1\big), \tag{3.1}$$

where $\alpha > 1$ and

$$k(t) = \begin{cases} t^{-\beta}, & t \in (0,b], \\ 1, & t = b, \end{cases} \tag{3.2}$$

for some $\beta \in (0,1)$. The function $j(t,x)$ satisfies hypotheses $H(j)_1$ since

$$\partial j(t,x) = \begin{cases} \alpha k(t)\overline{B}_1, & \|x\| = 0, \\ k(t)\left(-e^{\|x\|^p} p\|x\|^{p-2}x + \alpha \dfrac{x}{\|x\|}\right), & \|x\| \neq 0, \end{cases} \tag{3.3}$$

where $\overline{B}_1$ is the closed unit ball in $\mathbb{R}^N$.

Theorem 3.3. *If hypotheses $H(j)_1$ hold and $h \in L^1(T,\mathbb{R}^N)$ is such that $\int_0^b h(t)dt = 0$, then problem (I) has a nontrivial solution $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$.*

Proof. We start by observing that, because of hypotheses $H(j)_1$(iii), using again the mean value theorem, we have that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \le r$

$$|j(t,x)| \le b_r(t)\widetilde{a}(\|x\|), \tag{3.4}$$

where $b_r(t) = |j(t,0)| + a_r(t)$ and

$$\widetilde{a}(s) = \begin{cases} 1, & 0 \le s \le 1, \\ s, & s > 1. \end{cases} \tag{3.5}$$

Hence, taking into account $H(j)_1$(iv), by virtue of [7, Lemma 2] applied to the function $-j(t,x)$, for all $\delta > 0$, there exists $C_\delta \subset C$ such that $|C - C_\delta| < \delta$ and $j(t,x) \to -\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in C_\delta$.

Apply [7, Lemma 3] to the function $-j(t,x)$ in C_δ to obtain the existence of a function $G \in C(\mathbb{R}^N,\mathbb{R})$ and a function $\gamma \in L^1(C_\delta)$, such that, for almost all $t \in C_\delta$ and all $x \in \mathbb{R}^N$, we have

$$j(t,x) \le -G(x) + \gamma(t), \tag{3.6}$$

where G satisfies the following properties:

(a) $G(x+y) \le G(x) + G(y)$ for all $x, y \in \mathbb{R}^N$ (i.e., G is subadditive),

(aa) $G(x) \to +\infty$ as $\|x\| \to \infty$ (i.e., G is coercive),

(aaa) $G(x) \le \|x\| + 4$, for all $x \in \mathbb{R}^N$.

Let $W^{1,p}_{\text{per}}(T,\mathbb{R}^N) = \{x \in W^{1,p}(T,\mathbb{R}^N) : x(0) = x(b)\}$ and let $\varphi : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$ be the energy functional defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))dt - \int_0^b (h(t),x(t))dt, \tag{3.7}$$

for all $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$. From Remark 3.1, φ is well defined and it is locally Lipschitz (see [20, page 617]). Moreover, we know that $W^{1,p}_{\text{per}}(T,\mathbb{R}^N) = V \oplus \mathbb{R}^N$ with $V = \{x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N) : \int_0^b x(t)dt = 0\}$. So let $x = \hat{x} + \overline{x}$, with $\hat{x} \in V$ and $\overline{x} \in \mathbb{R}^N$. The properties (a) and (aaa) imply that

$$G(\overline{x}) \le G(x(t)) + \|\hat{x}(t)\| + 4, \quad \forall t \in T, \tag{3.8}$$

therefore, from (3.6), we obtain

$$-j(t,x(t)) \geq G(\overline{x}) - ||\hat{x}(t)|| - 4 - \gamma(t), \quad \text{a.e. on } C_\delta. \tag{3.9}$$

Using the previous inequality and $H(j)_1$(v), we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt - \int_0^b (h(t),x(t))\,dt \\
&\geq \frac{1}{p}\|\hat{x}'\|_p^p + G(\overline{x})\,|\,C_\delta\,| - \int_{T\setminus C_\delta} j(t,x(t))\,dt - \|\hat{x}\|_1 - \|h\|_1\|x\|_\infty - c_1 \\
&\geq \frac{1}{p}\|\hat{x}'\|_p^p + G(\overline{x})\,|\,C_\delta\,| - c_2\|\hat{x}'\|_p - c_1 - \|\beta\|_1,
\end{aligned} \tag{3.10}$$

for some $c_1, c_2 > 0$. Here we have used the Poincaré-Wirtinger inequality (see [18, page 866]). From (3.10), (aa), and since $\|x\| \leq k\|\hat{x}'\|_p + \|\overline{x}\|$, for a constant $k > 0$, we infer that φ is coercive.

By virtue of the compact embedding of $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ into $C(T,\mathbb{R}^N)$ and the weakly lower semicontinuity of the norm functional in a Banach space, we have that φ is weakly lower semicontinuous. So invoking Weierstrass theorem (see [13, page 711]), we can find $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ such that

$$-\infty < m = \inf_{W^{1,p}_{\text{per}}(T,\mathbb{R}^N)} \varphi = \varphi(x). \tag{3.11}$$

From $H(j)_1$(vi), we deduce that $\varphi(x) \leq \varphi(x_0) < 0 \leq \varphi(0)$, so

$$x \neq 0, \qquad 0 \in \partial\varphi(x). \tag{3.12}$$

Now let $A : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to W^{1,p}_{\text{per}}(T,\mathbb{R}^N)^*$ and $J : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$ be, respectively, the nonlinear operator and the integral functional defined by

$$\begin{aligned}
\langle A(x),y\rangle &= \int_0^b ||x'(t)||^{p-2}(x'(t),y'(t))\,dt, \quad \forall x,y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N), \\
J(x) &= \int_0^b j(t,x(t))\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N).
\end{aligned} \tag{3.13}$$

It is simple to see that J is locally Lipschitz and $\partial J(x) \subset L^1(T,\mathbb{R}^N)$ (see [21, Theorem 2.2]). Moreover there exists $u \in \partial J(x)$ such that

$$0 = A(x) - u - h, \tag{3.14}$$

so (see [22, page 76]) $u(t) \in \partial j(t,x(t))$, a.e. on T, and for every test function $\psi \in C_0^1((0,b),\mathbb{R}^N)$ we have

$$\langle A(x),\psi\rangle = \int_0^b (u(t)+h(t),\psi(t))\,dt, \tag{3.15}$$

that means

$$\int_0^b ||x'(t)||^{p-2}(x'(t),\psi'(t))\,dt = \int_0^b (u(t)+h(t),\psi(t))\,dt. \tag{3.16}$$

Since $(\|x'\|^{p-2}x')' \in W^{-1,q}(T,\mathbb{R}^N)$ (where $1/p+1/q=1$) (see [20, page 362] or [13, Theorem 1.1.8]), if by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}(T,\mathbb{R}^N), W^{-1,q}(T,\mathbb{R}^N))$, we deduce that

$$-\langle(\|x'\|^{p-2}x')',\psi\rangle_0 = \langle u+h,\psi\rangle_0. \tag{3.17}$$

Note that $C_0^1((0,b),\mathbb{R}^N)$ is dense in $W_0^{1,p}(T,\mathbb{R}^N)$, therefore $(\|x'(t)\|^{p-2}x'(t))' = u(t)+h(t)$, a.e. on T, which implies that $\|x'\|^{p-2}x' \in W^{1,1}(T,\mathbb{R}^N) \hookrightarrow C(T,\mathbb{R}^N)$ and

$$-(||x'(t)||^{p-2}x'(t))' \in \partial j(t,x(t)) + h(t), \quad \text{a.e. on } T=[0,b] \tag{3.18}$$

with $x(0)=x(b)$. Now the function $\xi:\mathbb{R}^N\to\mathbb{R}^N$ defined as $\xi(v)=\|v\|^{p-2}v$ if $v\neq 0$, $\xi(0)=0$ is a homeomorphism, so, since $\xi^{-1}(\|x'(t)\|^{p-2}x'(t)) = x'(t)$, we obtain that $x\in C^1(T,\mathbb{R}^N)$.

Finally, for every $\vartheta\in C^1(T,\mathbb{R}^N)$ from (3.14) we have

$$\langle A(x),\vartheta\rangle = \int_0^b (u(t)+h(t),\vartheta(t))\,dt \tag{3.19}$$

and so, using Green's identity we obtain

$$\begin{aligned} &||x'(b)||^{p-2}(x'(b),\vartheta(b)) - ||x'(0)||^{p-2}(x'(0),\vartheta(0)) \\ &\quad - \int_0^b ((||x'(t)||^{p-2}x'(t))',\vartheta(t))\,dt = \int_0^b (u(t)+h(t),\vartheta(t))\,dt. \end{aligned} \tag{3.20}$$

This implies that

$$||x'(b)||^{p-2}(x'(b),\vartheta(b)) = ||x'(0)||^{p-2}(x'(0),\vartheta(0)), \quad \forall\,\vartheta\in C^1(T,\mathbb{R}^N), \tag{3.21}$$

therefore $x'(0)=x'(b)$. So $x\in C^1(T,\mathbb{R}^N)$ is a nontrivial solution of problem (I). □

Remark 3.4. If we drop hypothesis $H(j)_1$(vi), assuming only that $j(\cdot,0)\in L^1(T)$, we can still have a solution but we cannot guarantee that it is nontrivial.

Remark 3.5. We want to observe that our Theorem 3.3 extends the analogous existence results given in [14, 15] and the analogous theorems collected in [13, Section 3.4.1], as it is evident by considering again Example 3.2.

Consider now the following hypotheses on $j(t,x)$:

$H(j)_2$: $j: T\times\mathbb{R}^N\to\mathbb{R}$ is a function such that

(i) for all $x\in\mathbb{R}^N$, $t\mapsto j(t,x)$ is measurable;

(ii) for almost all $t\in T$, $x\mapsto j(t,x)$ is locally Lipschitz;

(iii) for every $r>0$ there exists $a_r \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \le r$ and all $u \in \partial j(t,x)$, we have $\|u\| \le a_r(t)$;
(iv) $j(t,x) \to -\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in T$;
(v) $j(\cdot,0) \in L^1(T)$ and $\exists\, x_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^b j(t,x_0)dt > 0$ and $\int_0^b j(t,0)dt \le 0$.

As a simple consequence of the previous theorem we can state the following.

Corollary 3.6. *If hypotheses $H(j)_2$ hold and $h \in L^1(T,\mathbb{R}^N)$ is such that $\int_0^b h(t)dt = 0$, then problem (I) has a nontrivial solution $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$.*

Proof. It is sufficient to show that under conditions $H(j)_2$, the potential function $j(t,x)$ satisfies the hypotheses of the previous theorem. In fact from $H(j)_2$(iv), fixed $L>0$, there exists $M>0$ such that $j(t,x) \le -L$, a.e. on T and for all $\|x\| > M$. So from the mean value theorem (see [19, page 552]) and $H(j)_2$(iii) we obtain that $j(t,x) \le j(t,0) + a_M(t)M$, a.e. on T, for all $\|x\| \le M$. Denoting, therefore, by $\beta(t)$ the function $\beta(t) = |j(t,0)| + a_M(t)M$, we have that $H(j)_1$(v) is satisfied. Applying the previous result we obtain the existence of a nontrivial solution for our problem (I). □

Remark 3.7. The function given in Example 3.2 satisfies also hypotheses $H(j)_2$ while in the next example we show a function satisfying $H(j)_1$ but not $H(j)_2$.

Example 3.8. Let $j : T \times \mathbb{R}^N \to \mathbb{R}$ be defined as

$$j(t,x) = -\frac{1}{p}\chi_C(t)\|x\|^p + \|x\|, \tag{3.22}$$

where χ_C is the characteristic function of a set C strictly contained in T and with positive measure.

In the previous existence results, the energy functional φ was coercive and so the solution was obtained by an application of the least action principle. In the next existence theorem, the energy functional φ is bounded below but not necessarily coercive. In this case, the hypotheses on the nonsmooth potential $j(t,x)$ are the following:

$H(j)_3$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that
(i) for all $x \in R^N$, $t \mapsto j(t,x)$ is measurable;
(ii) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz;
(iii) for every $r>0$ there exists $a_r \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \le r$ and all $u \in \partial j(t,x)$, we have $\|u\| \le a_r(t)$;
(iv) there exist $j_\infty \in L^1(T)$ and $M>0$ such that $\lim_{\|x\|\to\infty} j(t,x) = j_\infty(t)$ for almost all $t \in T$ and $j(t,x) \ge j_\infty(t)$ for almost all $t \in T$ and all $\|x\| \ge M$;
(v) there exists $\beta \in L^1(T)_+$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have $j(t,x) \le \beta(t)$;
(v) $j(\cdot,0) \in L^1(T)$ and $\exists\, x_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^b j(t,x_0)dt > 0$ and $\int_0^b j(t,0)dt \le 0$.

To deal with this case, first we consider the following auxiliary periodic problem:

$$\begin{aligned} -\left(\|x'(t)\|^{p-2}x'(t)\right)' &= h(t), \quad \text{a.e. on } T=[0,b] \\ x(0)=x(b), \quad x'(0)&=x'(b), \quad 1<p<+\infty, \end{aligned} \tag{II}$$

for which it is simple to prove the next proposition. For the convenience of the reader, we give also the relative proof.

Proposition 3.9. *If $h \in L^1(T,\mathbb{R}^N)$ is such that $\int_0^b h(t)dt = 0$, then problem (II) has a unique solution $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$ such that $\int_0^b x(t)dt = 0$.*

Proof. We start by considering the C^1-functional $\psi : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$, defined by

$$\psi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b (h(t),x(t))dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.23}$$

Since $W^{1,p}_{\text{per}}(T,\mathbb{R}^N) = V \oplus \mathbb{R}^N$, let $\widehat{\psi}$ be the restriction of ψ to V. Using again the Poincaré-Wirtinger inequality it is simple to see that $\widehat{\psi}$ is coercive on V and weakly lower semicontinuous. Therefore, we can find $v_0 \in V$ such that

$$\inf_V \widehat{\psi} = \widehat{\psi}(v_0), \tag{3.24}$$

and so, from the differentiability of $\widehat{\psi}$ we obtain that $\widehat{\psi}'(v_0) = 0$.

But the subspace $V \subset W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ has finite codimension, so we can find a projection operator $p : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to V$ such that $\psi = \widehat{\psi} \circ p$ and, if $p^* : V^* \to W^{1,p}_{\text{per}}(T,\mathbb{R}^N)^*$ is the adjoint operator of p, from the Chain-Rule theorem (see [19, page 553]) we obtain

$$\psi'(x) = \widehat{\psi}'(p(x))(p(x)) = p^*(\widehat{\psi}'(p(x))), \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.25}$$

If $\langle\cdot,\cdot\rangle_V$ denotes the duality brackets for the pair (V,V^*), then for all $x,y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ we have

$$\langle \psi'(x), y\rangle = \langle p^*(\widehat{\psi}'(p(x))), y\rangle = \langle \widehat{\psi}'(p(x)), p^*(y)\rangle_V, \tag{3.26}$$

and so

$$\langle \psi'(v_0), y\rangle = \langle \widehat{\psi}'(p(v_0)), p^*(y)\rangle_V = \langle \widehat{\psi}'(v_0), p^*(y)\rangle_V = 0, \quad \forall y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.27}$$

This implies that $\psi'(v_0) = 0$ and so $A(v_0) = h$.

From this equality, as in the proof of Theorem 3.3, via Green's identity, we conclude that $v_0 \in C^1(T,\mathbb{R}^N)$ is a solution of problem (II). Moreover, from the strict monotonicity of A, we have that this solution is unique. □

Now we can prove the following.

Theorem 3.10. *If hypotheses $H(j)_3$ hold and $h \in L^1(T,\mathbb{R}^N)$ is such that $\int_0^b h(t)dt = 0$, then problem (I) has a nontrivial solution $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$.*

Proof. As in Theorem 3.3, we consider the usual energy functional $\varphi : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$,

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt - \int_0^b (h(t),x(t))\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.28}$$

Let $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$, then $x = \hat{x} + \overline{x}$, with $\hat{x} \in V$ and $\overline{x} \in \mathbb{R}^N$. From our hypotheses, using the Poincaré-Wirtinger inequality, we have

$$\varphi(x) \geq \frac{1}{p}\|\hat{x}'\|_p^p - c_3\|h\|_1\|\hat{x}'\|_p - \|\beta\|_1, \tag{3.29}$$

for some $c_3 > 0$. It follows that φ is bounded below and so

$$-\infty < m = \inf_{W^{1,p}_{\text{per}}(T,\mathbb{R}^N)} \varphi. \tag{3.30}$$

Let $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ be a minimizing sequence, that is, $\varphi(x_n) \downarrow m$, as $n \to \infty$. So from (3.29) we have

$$\varphi(x_n) \geq \frac{1}{p}||\hat{x}_n'||_p^p - c_3\|h\|_1||\hat{x}_n'||_p - \|\beta\|_1, \tag{3.31}$$

where $x_n = \hat{x}_n + \overline{x}_n$, $\hat{x}_n \in V$ and $\overline{x}_n \in \mathbb{R}^N$. We deduce that $\{\hat{x}_n\}_n$ is bounded in V, therefore, by passing to a suitable subsequence, if necessary, we may assume that $\hat{x}_n \to \hat{x} \in V$ weakly in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ and $\hat{x}_n \to \hat{x}$ in $C(T,\mathbb{R}^N)$. If also $\{x_n\}_n$ is bounded, then there exists a subsequence, denoted again with $\{x_n\}_n$ which converges weakly to $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$. Therefore from the weak-lower semicontinuity of φ in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$, we deduce that $\varphi(x) = m$. Now proceeding as in Theorem 3.3 it follows that $x \in C^1(T,\mathbb{R}^N)$ is a solution of problem (I). Suppose now that $\{x_n\}_n$ is unbounded. Since $x_n = \hat{x}_n + \overline{x}_n$, $\{\overline{x}_n\}_n$ must be unbounded in $\mathbb{R}^N$, so, at least for a subsequence, we must have $\|\overline{x}_n\| \to +\infty$. Clearly there exists $\xi > 0$ such that

$$||x_n(t)|| = ||\hat{x}_n(t) + \overline{x}_n|| \geq ||\overline{x}_n|| - \xi, \quad \forall t \in T, \tag{3.32}$$

and so $\|x_n(t)\| \to +\infty$ uniformly in T as $n \to \infty$.

From $H(j)_3$(iv) and (v) we can apply Fatou's lemma for obtaining

$$\int_0^b j_\infty(t)\,dt \geq \limsup_{n\to\infty} \int_0^b j(t,x_n(t))\,dt, \tag{3.33}$$

and from our assumptions we have also

$$\int_0^b (h(t),x_n(t))\,dt = \int_0^b (h(t),\hat{x}_n(t))\,dt \longrightarrow \int_0^b (h(t),\hat{x}(t))\,dt, \quad \text{as } n \longrightarrow \infty. \tag{3.34}$$

Moreover, we know that

$$m = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \left[\frac{1}{p}||\hat{x}_n'||_p^p - \int_0^b j(t,x_n(t))\,dt - \int_0^b (h(t),x_n(t))\,dt\right]. \tag{3.35}$$

Hence, using (3.33) and (3.34), we obtain

$$\begin{aligned}
m &\geq \liminf_{n\to\infty} \frac{1}{p}\|\widehat{x}_n'\|_p^p + \liminf_{n\to\infty}\left[-\int_0^b j(t,x_n(t))\,dt - \int_0^b (h(t),x_n(t))\,dt\right] \\
&\geq \frac{1}{p}\|\widehat{x}'\|_p^p - \limsup_{n\to\infty}\int_0^b j(t,x_n(t))\,dt - \int_0^b (h(t),\widehat{x}(t))\,dt \\
&= \psi(\widehat{x}) - \int_0^b j_\infty(t)\,dt,
\end{aligned} \tag{3.36}$$

where $\psi : W_{\text{per}}^{1,p}(T,\mathbb{R}^N) \to \mathbb{R}$ is the functional introduced in the proof of Proposition 3.9. According to the same proposition let $v_0 \in V$ be the unique solution of problem (II), therefore there exists $\overline{n} \geq 1$ such that, for all $n \geq \overline{n}$, it follows $\|\overline{x}_n + v_0(t)\| \geq M$, for all $t \in T$. So, using $H(j)_3$(iv), we deduce that

$$j(t,\overline{x}_n + v_0(t)) \geq j_\infty(t), \quad \text{a.e. } t \in T,\ \forall n \geq \overline{n}, \tag{3.37}$$

which used in (3.36) gives us

$$m \geq \psi(\widehat{x}) - \int_0^b j(t,\overline{x}_n + v_0(t))\,dt \geq \varphi(v_0 + \overline{x}_n), \quad \forall n \geq \overline{n}, \tag{3.38}$$

but this implies that $m = \varphi(v_0 + \overline{x}_n)$, for all $n \geq \overline{n}$. Therefore there exists $x \in W_{\text{per}}^{1,p}(T,\mathbb{R}^N)$ for which $\varphi(x) = m$ and so, also in this case, following the proof of Theorem 3.3, we conclude that $x \in C^1(T,\mathbb{R}^N)$ is a solution of problem (I). □

Remark 3.11. If $N = 1$, we may assume in condition $H(j)_3$(iv) different limits $j_\pm \in L^1(T)$, as $x \to \pm\infty$.

Remark 3.12. Consider now the following example.

Example 3.13. Let $j : R^N \to \mathbb{R}$ be defined as

$$j(x) = \begin{cases} \|x\|, & \|x\| \leq 1, \\ \dfrac{2}{\|x\|+1}, & \|x\| > 1, \end{cases} \tag{3.39}$$

in which, for simplicity, we have dropped the t-dependence. This function satisfies hypotheses $H(j)_3$ but it verifies none of the previous conditions and none of the assumptions required in the existence theorems mentioned in Remark 3.5.

Finally, we consider the case $h = 0$. So the problem under consideration is the following:

$$\begin{gathered}
-\left(\|x'(t)\|^{p-2}x'(t)\right)' \in \partial j(t,x(t)), \quad \text{a.e. on } T = [0,b], \\
x(0) = x(b), \quad x'(0) = x'(b), \quad 1 < p < +\infty.
\end{gathered} \tag{III}$$

In Theorem 3.3, we assumed that $j(t,x) \to -\infty$ as $\|x\| \to \infty$, for almost all t in some positive-measure subset of T. It is natural to ask what can be said about the existence of a solution for problem (III) when $j(t,x) \to +\infty$ uniformly for almost all $t \in T$ as $\|x\| \to \infty$. In this case, the corresponding energy functional is indefinite, in contrast to what happened in the previous existence results. So now we look for critical points which are of the saddle-point variety.

The new hypotheses on the nonsmooth potential are the following:

$H(j)_4$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(\cdot,0) \in L^1(T)$ and

(i) for all $x \in R^N$, $t \mapsto j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$ there exists $a_r \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \le r$ and all $u \in \partial j(t,x)$, we have $\|u\| \le a_r(t)$;

(iv) there exist $\mu \in (0,p)$ and $M > 0$ such that $\mu j(t,x) \ge j^0(t,x;x)$ for almost all $t \in T$ and all $\|x\| \ge M$;

(v) $j(t,x) \to +\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in T$;

(vi) there exists $C \subset T$, $|C| > 0$, such that $0 \notin \partial j(t,0)$ for almost all $t \in C$.

Theorem 3.14. *If hypotheses $H(j)_4$ hold, then problem (III) has a nontrivial solution $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$.*

Proof. Consider again the energy functional $\varphi : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to \mathbb{R}$,

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.40}$$

Claim 1. φ satisfies the C-(PS)-condition.

Let $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ be a sequence for which there exists $M_1 > 0$ such that $|\varphi(x_n)| \le M_1$ for all $n \ge 1$ and $(1+\|x_n\|)m(x_n) \to 0$ as $n \to \infty$. Let also $x_n^* \in \partial\varphi(x_n)$, $n \ge 1$, be such that $m(x_n) = \|x_n^*\|_*$. Following the notations introduced in Theorem 3.3, we have that there exists $u_n \in L^1(T)$, $u_n(t) \in \partial j(t,x_n(t))$, a.e. on T, such that $x_n^* = A(x_n) - u_n$, $n \ge 1$. Recall that $A : W^{1,p}_{\text{per}}(T,\mathbb{R}^N) \to W^{1,p}_{\text{per}}(T,\mathbb{R}^N)^*$ is the nonlinear operator defined by

$$\langle A(x),y\rangle = \int_0^b ||x'(t)||^{p-2}(x'(t),y'(t))\,dt, \quad \forall x,y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N). \tag{3.41}$$

Moreover applying [7, Lemma 3], we have $j(t,x) \ge G(x) - \gamma(t)$, for almost all $t \in T$ and all $x \in \mathbb{R}^N$, where $G \in C(\mathbb{R}^N,\mathbb{R})$ and $\gamma \in L^1(T)$ are as in Theorem 3.3. So if $x_n = \hat{x}_n + \overline{x}_n$, with $\hat{x}_n \in V$ and $\overline{x}_n \in \mathbb{R}^N$, we obtain

$$M_1 \le \varphi(x_n) \le \frac{1}{p}||\hat{x}_n'||_p^p - G(\overline{x}_n)b + c_4||\hat{x}_n'||_p - c_5, \tag{3.42}$$

for some $c_4, c_5 > 0$. From the choice of the sequence $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$, we have

$$\langle x_n^*,x_n\rangle = \langle A(x_n),x_n\rangle - \int_0^b (u_n(t),x_n(t))\,dt = ||x_n'||_p^p - \int_0^b (u_n(t),x_n(t))\,dt \le \varepsilon_n, \tag{3.43}$$

where $\varepsilon_n \downarrow 0$. Therefore, since $u_n(t) \in \partial j(t,x_n(t))$, $n \geq 1$, we deduce

$$||x_n'||_p^p - \int_0^b j^0(t,x_n(t);x_n(t))\,dt \leq \varepsilon_n \tag{3.44}$$

and, since $|\varphi(x_n)| \leq M_1$, we obtain

$$-\frac{\mu}{p}||x_n'||_p^p + \mu\int_0^b j(t,x_n(t))\,dt \leq \mu M_1, \tag{3.45}$$

for all $n \geq 1$. Adding (3.44) and (3.45), we have then

$$\left(1-\frac{\mu}{p}\right)||x_n'||_p^p + \int_0^b \left(\mu j(t,x_n(t)) - j^0(t,x_n(t);x_n(t))\right)dt \leq \varepsilon_n + \mu M_1. \tag{3.46}$$

Now, denoted by $A_n = \{t \in T : \|x_n(t)\| < M\}$ and $B_n = \{t \in T : \|x_n(t)\| \geq M\}$, from $H(j)_4$ (iv), $H(j)_4$(iii), and the properties of j^0 (see [19, page 545]) we obtain

$$\int_{B_n} \left(\mu j(t,x_n(t)) - j^0(t,x_n(t);x_n(t))\right)dt \geq 0 \tag{3.47}$$

while

$$\left|\int_{A_n} \left(\mu j(t,x_n(t)) - j^0(t,x_n(t);x_n(t))\right)dt\right| \leq \int_{A_n} \left(\mu\left[\,|\,j(t,0)\,| + a_M(t)M\right] + c_6M\right)dt \leq c_7, \tag{3.48}$$

where c_6, c_7 are positive constants. Using these two inequalities in (3.46), it follows that

$$\left(1-\frac{\mu}{p}\right)||x_n'||_p^p \leq \varepsilon_n + \mu M_1 + c_7, \quad \forall n \geq 1, \tag{3.49}$$

which, together with the Poincaré-Wirtinger inequality, tells us that $\{\hat{x}_n\}_n$ is bounded in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$.

If $\{\overline{x}_n\}_n$ is unbounded in $\mathbb{R}^N$, by passing to a subsequence, if necessary, we obtain that $\|\overline{x}_n\| \to \infty$ and so, by the properties of the function G (see proof of Theorem 3.3), we deduce that $G(\overline{x}_n) \to +\infty$. Then, from (3.42), $\varphi(x_n) \to -\infty$ which contradicts the choice of $\{x_n\}_n$. Therefore, we have verified that $\{x_n\}_n$ is bounded in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$. So, by passing to a subsequence, we may assume that $x_n \to x$ weakly in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ and $x_n \to x$ in $C(T,\mathbb{R}^N)$. We have, from $H(j)_4$(iii),

$$\begin{aligned}\langle A(x_n), x_n - x\rangle &= \langle x_n^* + u_n, x_n - x\rangle \\ &\leq ||x_n^*||_*||x_n - x|| + ||u_n||_1||x_n - x||_\infty \\ &\leq c_8\left(1 + ||x_n||\right)||x_n^*||_* + c_9||x_n - x||_\infty,\end{aligned} \tag{3.50}$$

for some $c_8, c_9 > 0$. Hence $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$. But it is easy to check that A is monotone, demicontinuous, thus maximal monotone and so generalized pseudomonotone (see [13, pages 75 and 84]). Therefore it follows that $\lim_{n\to\infty}\langle A(x_n), x_n\rangle = \langle A(x), x\rangle$

that is $\lim_{n\to\infty}\|x_n'\|_p = \|x'\|_p$. Since $x_n' \to x'$ weakly in $L^p(T,\mathbb{R}^N)$ and the latter space is uniformly convex, from the Kadec-Klee property (see [17, page 28]), we have that $x_n' \to x'$ in $L^p(T,\mathbb{R}^N)$. Therefore $x_n \to x$ in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ as $n\to\infty$ and this proves Claim 1.

Claim 2. For almost all $t\in T$, all $\|x\|\geq M$ and all $s\geq 1$, we have $j(t,sx)\leq s^\mu j(t,x)$.

On $\mathbb{R}_+\setminus\{0\}$, the function $s\mapsto 1/s^\mu$ is continuous and convex, thus locally Lipschitz. Hence, for all $x\in\mathbb{R}^N$ and almost all $t\in T$, the function $s\mapsto (1/s^\mu)j(t,sx)$ is locally Lipschitz on $\mathbb{R}_+\setminus\{0\}$ and we have (see [20, page 612])

$$\partial_s\left(\frac{1}{s^\mu}j(t,sx)\right)\subset -\frac{\mu}{s^{\mu+1}}j(t,sx)+\frac{1}{s^\mu}(\partial_x j(t,sx),x). \tag{3.51}$$

Using the mean value theorem for locally Lipschitz functions we can find $\lambda\in(1,s)$ such that

$$\frac{1}{s^\mu}j(t,sx)-j(t,x)\in\left(-\frac{\mu}{\lambda^{\mu+1}}j(t,\lambda x)+\frac{1}{\lambda^\mu}(\partial_x j(t,\lambda x),x)\right)(s-1). \tag{3.52}$$

From $H(j)_4$(iv) we obtain

$$\begin{aligned}\frac{1}{s^\mu}j(t,sx)-j(t,x)&\leq\left(-\frac{\mu}{\lambda^{\mu+1}}j(t,\lambda x)+\frac{1}{\lambda^\mu}j^0(t,\lambda x;x)\right)(s-1)\\&=\frac{1}{\lambda^{\mu+1}}\big(-\mu j(t,\lambda x)+j^0(t,\lambda x;\lambda x)\big)(s-1)\leq 0,\end{aligned} \tag{3.53}$$

provided $\|x\|\geq M$. Therefore, we infer that, for almost all $t\in T$, all $\|x\|\geq M$ and all $s\geq 1$, $j(t,sx)\leq s^\mu j(t,x)$. This proves Claim 2.

Claim 3. φ is coercive on V.

For every $v\in V$ we have

$$\begin{aligned}\varphi(v)&=\frac{1}{p}\|v'\|_p^p-\int_0^b j(t,v(t))\,dt\\&=\frac{1}{p}\|v'\|_p^p-\int_A j(t,v(t))\,dt-\int_B j(t,v(t))\,dt,\end{aligned} \tag{3.54}$$

where $A=\{t\in T:\|v(t)\|<M\}$ and $B=\{t\in T:\|v(t)\|\geq M\}$. Note that, from the mean value theorem and from $H(j)_4$(iii), for $\|x\|\leq M$ and for almost all $t\in T$, it is possible to find $r\in[0,1]$ and $u\in\partial j(t,rx)$ such that

$$j(t,x)\leq j(t,0)+(u,x)\leq j(t,0)+a_M(t)M. \tag{3.55}$$

Therefore we can say that for all $\|x\|\leq M$, it follows that

$$j(t,x)\leq\beta_M(t), \tag{3.56}$$

where $\beta_M\in L^1(T)_+$. Immediately we have

$$\int_A j(t,v(t))\,dt\leq\|\beta_M\|_1. \tag{3.57}$$

Also using Claim 2, we obtain

$$\int_B j(t,v(t))\,dt \le \int_B \frac{||v(t)||^\mu}{M^\mu} j\left(t, \frac{Mv(t)}{||v(t)||}\right) dt \le \frac{\|v\|_\infty^\mu}{M^\mu} \int_B j\left(t, \frac{Mv(t)}{||v(t)||}\right) dt, \tag{3.58}$$

and so

$$\int_B j(t,v(t))\,dt \le \frac{\|v\|_\infty^\mu}{M^\mu} ||\beta_M||_1. \tag{3.59}$$

Now, from (3.57) and (3.59), using the Poincaré-Wirtinger inequality, we obtain

$$\varphi(v) \ge \frac{1}{p}\|v'\|_p^p - \|v'\|_p^\mu c_{10} - c_{11}, \tag{3.60}$$

for some $c_{10}, c_{11} > 0$. Because $\mu < p$, we conclude that φ is coercive on V as claimed.

Claim 4. φ is anticoercive on $\mathbb{R}^N$.

Since for $y \in \mathbb{R}^N$, $\varphi(y) = -\int_0^b j(t,y)\,dt$, the claim is a direct consequence of hypothesis $H(j)_4$(v).

From the claims proved we are in the position of applying a saddle-point theorem for nonsmooth functionals (see [13, Theorem 2.1.4]) and obtaining the existence of a point $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ such that $0 \in \partial\varphi(x)$. From $H(j)_4$(vi) $x \ne 0$ and following the proof of Theorem 3.3, we have that $x \in C^1_{\text{per}}(T,\mathbb{R}^N)$ and it is a solution of problem (III). □

Remark 3.15. Hypotheses $H(j)_4$(vi) is assumed only to avoid the possibility that the solution x is trivial.

Remark 3.16. If $j(t,\cdot) \in C^1(\mathbb{R}^N,\mathbb{R})$, then $j^0(t,x;x) = (j'_x(t,x),x)$, for all $x \in \mathbb{R}^N$. So hypothesis $H(j)_4$(iv) is an Ambrosetti-Rabinowitz-type condition.

Moreover, we observe that if $H(j)_4$ hold, following the proof of Claim 3 we have that

$$j(t,x) \le j(t,0) + a_M(t)M := \beta_M(t), \quad \text{a.e. on } T,\ \forall \|x\| \le M, \tag{3.61}$$

while, from Claim 2, for all $\|x\| \ge M$, we obtain

$$j(t,x) \le \frac{\|x\|^\mu}{M^\mu} j\left(t, \frac{Mx}{\|x\|}\right) \le \frac{\|x\|^\mu}{M^\mu}\beta_M(t), \quad \text{a.e. on } T. \tag{3.62}$$

Therefore, we deduce that

$$j(t,x) \le \hat{a}(t)\|x\|^\mu + \hat{b}(t), \quad \text{a.e. on } T,\ \forall x \in \mathbb{R}^N, \tag{3.63}$$

for some $\hat{a}, \hat{b} \in L^1(T)_+$.

This shows that j satisfies a growth condition strictly less then p.

Remark 3.17. Theorem 3.14 extends Theorem 3.4.4 stated in [13] since the inequality required in our condition $H(j)_4$(iv) is, in a certain sense, opposite to that assumed in $H(j)_4$(iv) of that theorem. This is also evident by considering the following two examples of functions satisfying our conditions but not those of [13, Theorem 3.4.4].

Example 3.18. Let $j : T \times \mathbb{R} \to \mathbb{R}$ be defined as

$$j(t,x) = \begin{cases} 3 - x, & |x| \le 1, \\ x^{\mu} + 1, & x > 1, \\ |x|^{\mu} + 3, & x < -1, \end{cases} \tag{3.64}$$

where $\mu \in (0, p)$.

Example 3.19. Let $j : T \times \mathbb{R}^2 \to \mathbb{R}$ be so defined,

$$j(t,x) = \begin{cases} u^2 + u, & \|x\| \le 1, \\ \|x\|^2 (u^2 + u), & \|x\| > 1, \end{cases} \tag{3.65}$$

where $x = (u, v)$ and $4 < \mu < p$.

4. A multiplicity result

In this section, we present a result in which we produce more than one solution (multiple periodic solutions). The hypotheses on the nonsmooth potential are the following:

$H(j)_5$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

(i) for all $x \in R^N$, $t \mapsto j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$ there exists $a_r \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| \le r$ and all $u \in \partial j(t,x)$, we have $\|u\| \le a_r(t)$;

(iv) there exist $\sigma > p$ and $\delta > 0$ such that $j(t,x) \le c_0 \|x\|^{\sigma}$ for almost all $t \in T$ and all $x \in R^N$ with $c_0 > 0$ and $j(t,x) \ge 0$ for almost all $t \in T$ and all $\|x\| \le \delta$;

(v) there exists $C \subset T$, $|C| > 0$, such that $j(t,x) \to -\infty$ as $\|x\| \to \infty$ for almost all $t \in C$;

(vi) there exists $\beta \in L^1(T)_+$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, we have $j(t,x) \le \beta(t)$;

(vii) $\exists\, x_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^b j(t,x_0)dt > 0$ and $j(t,0) = 0$, a.e. on T.

Theorem 4.1. *If hypotheses $H(j)_5$ hold, then problem (III) has two nontrivial solutions $x_1, x_2 \in C^1_{\text{per}}(T, \mathbb{R}^N)$.*

Proof. We consider the locally Lipschitz energy functional $\varphi : W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p} \|x'\|_p^p - \int_0^b j(t, x(t))\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T, \mathbb{R}^N). \tag{4.1}$$

Because of hypotheses $H(j)_5$(v) and (vi), from the proof of Theorem 3.3, we have that φ is coercive on $W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$ and so it is bounded from below. Now we are going to prove that φ satisfies the (PS)-condition. To this end let $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$ be a sequence for which there exists $M_1 > 0$ such that $|\varphi(x_n)| \le M_1$ for all $n \ge 1$ and $m(x_n) \to 0$ as $n \to \infty$. Let also $x_n^* \in \partial\varphi(x_n), n \ge 1$, be such that $m(x_n) = \|x_n^*\|_*$, therefore there exists $u_n \in L^1(T)$, $u_n(t) \in \partial j(t, x_n(t))$, a.e. on T, such that $x_n^* = A(x_n) - u_n$, $n \ge 1$. From the coercivity of φ we deduce that $\{x_n\}_n$ is bounded in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$. So, by passing to a subsequence, we may assume that $x_n \to x$ weakly in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ and $x_n \to x$ in $C(T,\mathbb{R}^N)$. Then, since

$$\langle A(x_n), x_n - x\rangle \le \|x_n^*\|_* \|x_n - x\| + \|u_n\|_1 \|x_n - x\|_\infty, \tag{4.2}$$

we deduce that $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \le 0$. But A is a generalized pseudomonotone operator, therefore $\lim_{n\to\infty}\|x_n'\|_p = \|x'\|_p$. Since $x_n' \to x'$ weakly in $L^p(T,\mathbb{R}^N)$ and the latter space is uniformly convex, from the Kadec-Klee property, we have that $x_n' \to x'$ in $L^p(T,\mathbb{R}^N)$. Therefore $x_n \to x$ in $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ as $n \to \infty$, as we want to show.

Observe then that from $H(j)_5$(vii) and from the fact that φ is bounded below, we have

$$-\infty < \inf_{W^{1,p}_{\text{per}}(T,\mathbb{R}^N)} \varphi \le \varphi(x_0) < 0 = \varphi(0). \tag{4.3}$$

Moreover, considering for $W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ the usual decomposition, if $v \in V$, from $H(j)_5$(iv) it follows that

$$\varphi(v) = \frac{1}{p}\|v'\|_p^p - \int_0^b j(t, v(t))\,dt \ge \frac{1}{p}\|v'\|_p^p - c_0\int_0^b \|v(t)\|^\sigma dt, \tag{4.4}$$

and so, using again the Poincaré-Wirtinger inequality, we obtain that

$$\varphi(v) \ge \frac{1}{p}\|v'\|_p^p - c_{12}\|v'\|_p^\sigma, \tag{4.5}$$

for some $c_{12} > 0$. Therefore, since $p < \sigma$, there exists $\rho > 0$ such that if $\|v'\|_p < \rho$, then $\varphi(v) \ge 0$.

On the other hand, always from $H(j)_5$(iv), if $y \in R^N$ is such that $\|y\| \le \delta$, then $\varphi(y) = -\int_0^b j(t,y)\,dt \le 0$. Using, therefore, [13, Theorem 2.4.1], there exist two nontrivial critical points $x_1, x_2 \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ of φ. As before it follows that, for $i = 1,2$, $x_i \in C^1_{\text{per}}(T,\mathbb{R}^N)$ and solves problem (III). □

Remark 4.2. Our multiplicity result extends some other analogous theorems proved in [14, Theorem 10], [13, Theorems 3.4.10 and 3.4.11], in the sense that there exists functions satisfying the assumptions of our theorem, but not those of the mentioned theorems as we can see with the following examples.

Example 4.3. Let $j : T \times \mathbb{R}^N \to \mathbb{R}$ be so defined,

$$j(t,x) = \begin{cases} \dfrac{1}{\sigma}\|x\|^\sigma, & \|x\| \le 1, \\ -\chi_E(t)\log\|x\| + \dfrac{1}{\sigma}, & \|x\| > 1, \end{cases} \tag{4.6}$$

where $\sigma > p > 1$ and E is a subset of T with positive measure. This function satisfies our conditions $H(j)_5$ but it does not verify the hypotheses of [13, Theorem 3.4.10].

Example 4.4. Let $j : T \times \mathbb{R}^N \to \mathbb{R}$ be so defined,

$$j(t,x) = \begin{cases} \frac{1}{\sigma}\|x\|^{\sigma}, & \|x\| \le 1, \\ -\|x\|^{p} + \frac{1}{\sigma} + 1, & \|x\| > 1, \end{cases} \tag{4.7}$$

where $\sigma > p > 1$. This function satisfies $H(j)_5$ but note the hypotheses of [13, Theorem 3.4.11] and of [14, Theorem 10].

5. The scalar case

We conclude with an existence result for the scalar problem (i.e., $N = 1$) with an indefinite energy functional. The problem under consideration is the following:

$$\begin{gathered} -\left(|x'(t)|^{p-2}x'(t)\right)' \in \partial j(t,x(t)), \quad \text{a.e. on } T = [0,b], \\ x(0) = x(b), \quad x'(0) = x'(b), \quad 1 < p < +\infty. \end{gathered} \tag{IV}$$

We denote by λ_0 and λ_1 the first two eigenvalues of the negative scalar p-Laplacian with periodic boundary conditions (see [13, Section 1.5]). Our aim is to prove the existence of a solution for problem (IV) where, asymptotically at ∞, we permit a partial interaction with λ_0 and λ_1. In fact, our hypotheses on the nonsmooth potential are the following:

$H(j)_6$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in$, $t \mapsto j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \mapsto j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$ there exists $a_r \in L^q(T)_+$ $(1/p + 1/q = 1)$ such that for almost all $t \in T$, all $|x| \le r$ and all $u \in \partial j(t,x)$, we have $|u| \le a_r(t)$;

(iv) there exist $\vartheta_1, \vartheta_2 \in L^\infty(T)_+$ such that $0 \le \vartheta_1(t) \le \vartheta_2(t) \le \lambda_1$ for almost all $t \in T$ and the first and the third inequalities are strict on a set (not necessarily the same) of positive measure and

$$\vartheta_1(t) \le \liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \vartheta_2(t), \tag{5.1}$$

uniformly for almost all $t \in T$ and for all $u \in \partial j(t,x)$;

(v) there exists $M > 0$ such that for almost all $t \in T$ and all $|x| \ge M$ we have

$$j(t,x) \le \frac{\lambda_1}{p}|x|^p; \tag{5.2}$$

(vi) $j(t,x) \to +\infty$ as $\|x\| \to \infty$ uniformly for almost all $t \in T$.

Theorem 5.1. *If hypotheses $H(j)_6$ hold, then problem (IV) has a nontrivial solution $x \in C^1_{\text{per}}(T,\mathbb{R})$.*

Proof. We start by proving that the energy functional

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}), \tag{5.3}$$

satisfies the (PS)-condition. So let $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R})$ be a sequence for which there exists $M_1 > 0$ such that $|\varphi(x_n)| \le M_1$ for all $n \ge 1$ and $m(x_n) \to 0$ as $n \to \infty$. Let also $x_n^* \in \partial\varphi(x_n)$, $n \ge 1$, such that $m(x_n) = \|x_n^*\|_*$. As before, we can find $u_n \in L^q(T), u_n(t) \in \partial j(t,x_n(t))$, a.e. on T, such that $x_n^* = A(x_n) - u_n$, $n \ge 1$. Therefore, for all $w \in W^{1,p}_{\text{per}}(T,\mathbb{R})$ it follows that

$$\left| \langle A(x_n), w\rangle - \int_0^b u_n(t)w(t)dt \right| \le \varepsilon_n \|w\|, \tag{5.4}$$

where $\varepsilon_n \downarrow 0$.

We claim that $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R})$ is bounded. We proceed by contradiction. Suppose that $\{x_n\}_n$ is unbounded. Then, by passing to a subsequence, if necessary, we may assume that $\|x_n\| \to \infty$ and $x_n \ne 0$, for all $n \ge 1$. Set $y_n = x_n/\|x_n\|$, $n \ge 1$. We may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W^{1,p}_{\text{per}}(T,\mathbb{R}), \qquad y_n \longrightarrow y \quad \text{in } C_{\text{per}}(T,\mathbb{R}). \tag{5.5}$$

In (5.4), let $w = y_n - y \in W^{1,p}_{\text{per}}(T,\mathbb{R})$, $n \ge 1$. Dividing with $\|x_n\|^{p-1}$, we obtain

$$\left| \langle A(y_n), y_n - y\rangle - \int_0^b \frac{u_n(t)}{||x_n||^{p-1}}(y_n(t) - y(t))dt \right| \le \frac{\varepsilon_n}{||x_n||^{p-1}}||y_n - y||. \tag{5.6}$$

Fix $\varepsilon > 0$. By virtue of hypothesis $H(j)_6$(iv), there exists $M_2 > 0$ such that

$$\begin{aligned} &\left.\begin{aligned} u &\le (\vartheta_2(t)+\varepsilon)x^{p-1} \\ u &\ge (\vartheta_1(t)-\varepsilon)x^{p-1} \end{aligned}\right\} \text{ a.e. } t \in T,\ \forall x > M_2 \text{ and all } u \in \partial j(t,x), \\ &\left.\begin{aligned} u &\ge (\vartheta_2(t)+\varepsilon)|x|^{p-2}x \\ u &\le (\vartheta_1(t)-\varepsilon)|x|^{p-2}x \end{aligned}\right\} \text{ a.e. } t \in T,\ \forall x < -M_2 \text{ and all } u \in \partial j(t,x). \end{aligned} \tag{5.7}$$

Therefore, using also hypothesis $H(j)_6$(iii), there exists $\xi \in L^\infty(T)_+$ such that

$$|u| \le a_{M_1}(t) + \xi(t)|x|^{p-1}, \quad \text{a.e. } t \in T,\ \forall x \in \mathbb{R} \text{ and all } u \in \partial j(t,x). \tag{5.8}$$

We deduce that $\{u_n/\|x_n\|^{p-1}\}_n$ is bounded in $L^q(T)$. So

$$\int_0^b \frac{u_n(t)}{||x_n||^{p-1}}(y_n(t) - y(t))dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{5.9}$$

and then, from (5.6),

$$\limsup_{n\to\infty} \langle A(y_n), y_n - y\rangle \le 0. \tag{5.10}$$

As before, exploiting the pseudomonotonicity of A and the Kadec-Klee property of $L^p(T)$, we infer that

$$y_n \longrightarrow y \quad \text{in } W^{1,p}_{\text{per}}(T,\mathbb{R}). \tag{5.11}$$

Moreover, passing to a subsequence, if necessary, we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \longrightarrow h \quad \text{weakly in } L^q(T). \tag{5.12}$$

Consider now the following two sets:

$$\begin{aligned} C^+_{\varepsilon,n} &= \left\{t \in T : x_n(t) > 0,\ \vartheta_1(t) - \varepsilon \le \frac{u_n(t)}{|x_n(t)|^{p-1}} \le \vartheta_2(t) + \varepsilon\right\}, \\ C^-_{\varepsilon,n} &= \left\{t \in T : x_n(t) < 0,\ \vartheta_1(t) - \varepsilon \le \frac{u_n(t)}{|x_n(t)|^{p-2}x_n(t)} \le \vartheta_2(t) + \varepsilon\right\}. \end{aligned} \tag{5.13}$$

Since $\|x_n\| \to \infty$, we may assume, passing to a subsequence if necessary, that $\|x_n\|_\infty \to \infty$ (otherwise y must be 0 which contradicts the fact that $\|y_n\| = 1$), so it is possible to find $\overline{n} \ge 1$ such that $T = C^+_{\varepsilon,n} \cup C^-_{\varepsilon,n}$, for all $n \ge \overline{n}$. On the other hand, denoted by $C^+ = \{t \in T : y(t) > 0\}$ and by $C^- = \{t \in T : y(t) < 0\}$, it is simple to see that

$$\begin{aligned} \chi_{C^+_{\varepsilon,n}}(t) &\longrightarrow 1, \quad \text{a.e. on } C^+, \\ \chi_{C^-_{\varepsilon,n}}(t) &\longrightarrow 1, \quad \text{a.e. on } C^-, \end{aligned} \tag{5.14}$$

where $\chi_{C^+_{\varepsilon,n}}$ and $\chi_{C^-_{\varepsilon,n}}$ are the characteristic functions of the sets $C^+_{\varepsilon,n}$ and $C^-_{\varepsilon,n}$, respectively.

Now putting $h_n = u_n/\|x_n\|^{p-1}$ from (5.12) and (5.14), we have that

$$\begin{aligned} \|\chi_{C^+_{\varepsilon,n}} h_n - h_n\| &\longrightarrow 0, \quad \text{in } L^1(C^+), \\ \|\chi_{C^-_{\varepsilon,n}} h_n - h_n\| &\longrightarrow 0, \quad \text{in } L^1(C^-), \end{aligned} \tag{5.15}$$

and so

$$\begin{aligned} \chi_{C^+_{\varepsilon,n}} h_n &\longrightarrow h \quad \text{weakly in } L^1(C^+), \\ \chi_{C^-_{\varepsilon,n}} h_n &\longrightarrow h \quad \text{weakly in } L^1(C^-). \end{aligned} \tag{5.16}$$

Moreover, for a.e. $t \in C^+$, there exists $\hat{n} \ge 1$ such that, for all $n \ge \hat{n}$, $t \in C^+_{\varepsilon,n}$ and so

$$(\vartheta_1(t) - \varepsilon)\,|y_n(t)|^{p-1} \le h_n(t) \le (\vartheta_2(t) + \varepsilon)\,|y_n(t)|^{p-1}. \tag{5.17}$$

Taking the weak limit in $L^1(C^+)$, we obtain

$$(\vartheta_1(t) - \varepsilon)\,|y(t)|^{p-1} \le h(t) \le (\vartheta_2(t) + \varepsilon)\,|y(t)|^{p-1}. \tag{5.18}$$

Analogously, for a.e. $t \in C^-$,

$$(\vartheta_2(t) + \varepsilon)\,|y(t)|^{p-2}y(t) \le h(t) \le (\vartheta_1(t) - \varepsilon)\,|y(t)|^{p-2}y(t). \tag{5.19}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\begin{gathered}\vartheta_1(t)\,|\,y(t)\,|^{\,p-1} \le h(t) \le \vartheta_2(t)\,|\,y(t)\,|^{\,p-1}, \quad \text{a.e. on } C^+,\\ \vartheta_2(t)\,|\,y(t)\,|^{\,p-2}y(t) \le h(t) \le \vartheta_1(t)\,|\,y(t)\,|^{\,p-2}y(t), \quad \text{a.e. on } C^-.\end{gathered} \tag{5.20}$$

Since from (5.8) it is obvious that $h(t) = 0$ for a.e. $t \in T$ such that $y(t) = 0$, from (5.20), there exists $g \in L^\infty(T)_+$ such that

$$h(t) = g(t)\,|\,y(t)\,|^{\,p-2}y(t), \quad \text{a.e. } t \in T, \tag{5.21}$$

with $\vartheta_1(t) \le g(t) \le \vartheta_2(t)$, a.e. $t \in T$.

We return to (5.4), divide with $\|x_n\|^{p-1}$, and pass to the limit as $n \to \infty$. We obtain

$$\langle A(y), w\rangle = \int_0^b g(t)\,|\,y(t)\,|^{\,p-2}y(t)w(t)\,dt, \quad \forall w \in W^{1,p}_{\text{per}}(T,\mathbb{R}). \tag{5.22}$$

For this, as before, via Green's identity, we have

$$\begin{gathered}-\left(|\,y(t)\,|^{\,p-2}y(t)\right)' = g(t)\,|\,y(t)\,|^{\,p-2}y(t), \quad \text{a.e. on } T = [0,b]\\ y(0) = y(b), \qquad y'(0) = y'(b),\end{gathered} \tag{5.23}$$

and, as we have noticed before, $y \in C^1_{\text{per}}(T,\mathbb{R})$ is nontrivial. But according to [19, page 800], from the strict monotonicity of the eigenvalue on the weight, since $0 \le g(t) \le \lambda_1$, a.e. $t \in T$ (these inequalities are strict on sets of positive measure), we have that

$$\overline{\lambda}_0(g) > \overline{\lambda}_0(\lambda_1). \tag{5.24}$$

Comparing this with (5.23), we reach the contradiction $0 > 0$. This proves that $\{x_n\}_n \subset W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$ is bounded, and then as in the proof of Theorem 3.14, we can verify that φ satisfies the (PS)-condition.

Now let $D = \{x \in W^{1,p}_{\text{per}}(T,\mathbb{R}) : \int_0^b |x(t)|^{p-2}x(t)\,dt = 0\}$. By virtue of hypotheses $H(j)_6$(v) and $H(j)_6$(iii) we have

$$j(t,x) \le \overline{a}(t) + \frac{\lambda_1}{p}|x|^p, \quad \text{a.e. on } T,\ \forall x \in \mathbb{R}. \tag{5.25}$$

So from the properties $\lambda_1 > 0$ (see [13, Corollary 1.5.1]), there exists $\beta_0 > 0$ such that

$$\varphi(x) \ge -\beta_0, \quad \forall x \in D. \tag{5.26}$$

On the other hand, by virtue of hypothesis $H(j)_6$(vi), we can find $\xi > 0$ with the property

$$\varphi(\pm\xi) < -\beta_0. \tag{5.27}$$

Let $\psi : W^{1,p}_{\text{per}}(T,\mathbb{R}) \to \mathbb{R}$ be defined by

$$\psi(x) = \int_0^b |\,x(t)\,|^{\,p-2}x(t)\,dt, \quad \forall x \in W^{1,p}_{\text{per}}(T,\mathbb{R}). \tag{5.28}$$

Consider, moreover, the two sets $C_0 = \{\pm\xi\}$ and $C = \{x \in W^{1,p}_{\text{per}}(T, \mathbb{R}^N) : -\xi \le x(t) \le \xi$, for all $t \in T\}$. We will show that C_0 and D link via C (see [13, Definition 2.1.4 and Remark 2.1.4]). Indeed if η is a continuous function from C to $W^{1,p}_{\text{per}}(T, \mathbb{R})$ such that $\eta(\pm\xi) = \pm\xi$, then $(\psi \circ \eta)(-\xi) < 0 < (\psi \circ \eta)(+\xi)$ and so by the intermediate value theorem we can find $x_0 \in C$ such that $(\psi \circ \eta)(x_0) = 0$, hence $\eta(x) \in D$. Therefore $\eta(C) \cap D \neq \emptyset$, as we want to prove. Apply the minimax principle (see [13, Theorem 2.1.2]) to obtain $x \in W^{1,p}_{\text{per}}(T, \mathbb{R})$ such that $0 \in \partial\varphi(x)$. As before we have that $x \in C^1_{\text{per}}(T, \mathbb{R})$ and it solves problem (IV). $\square$

Remark 5.2. A simple nonsmooth locally Lipschitz function satisfying $H(j)_6$ is the function $j : T \times \mathbb{R} \to \mathbb{R}$ defined by

$$j(t,x) = \max\left\{\frac{\theta(t)}{p}|x|^p, \frac{\theta(t)}{r}|x|^r\right\}, \tag{5.29}$$

where $1 < r < p$ and $\theta \in L^\infty(T)$, $0 \le \theta(t) \le \lambda_1$, a.e. on T and the inequalities are strict on a set (not necessarily the same) of positive measure.

Moreover, we observe that this function does not verify the assumptions required in the analogous results of [16] and in [13, Theorem 3.4.9].

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