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Research Article

On P- and p-Convexity of Banach Spaces

Omar Muñiz-Pérez

Department of Mathematics, Center for Mathematical Reserch, A.C., Apdo. Postal 402, 36000 Guanajuato, GTO, Mexico

Correspondence should be addressed to Omar Muñiz-Pérez, omuniz@cimat.mx

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We show that every U-space and every Banach space X satisfying $\delta_X(1) > 0$ are $P(3)$-convex, and we study the nonuniform version of P-convexity, which we call p-convexity.

1. Introduction

Kottman introduced in 1970 the concept of P-convexity in [1]. He proved that every P-convex space is reflexive and also that P-convexity follows from uniform convexity, as well as from uniform smoothness. In this paper we study conditions which guarantee the P-convexity of a Banach space and generalize the result of Kottman concerning uniform convexity in two different ways: every U-space and every Banach space X satisfying $\delta_X(1) > 0$ are $P(3)$-convex. There are many convexity conditions of Banach spaces which have a uniform and also a nonuniform version, for example, strictly convexity is the nonuniform version of uniform convexity, smoothness is the nonuniform version of uniform smoothness, and a u-space is the nonuniform version of a U-space, among others. We also define the concept of p-convexity, which is the nonuniform version of P-convexity and obtain some interesting results.

2. P-Convex Banach Spaces

Throughout this paper we adopt the following notation. $(X, \|\cdot\|)$ will be a Banach space and when there is no possible confusion, we simply write X. The unit ball $\{x \in X : \|x\| \leq 1\}$ and the unit sphere $\{x \in X : \|x\| = 1\}$ are denoted, respectively, by B_X and S_X. $B(y, r)$ will denote the closed ball with center y and radius r. The topological dual space of X is denoted by X^*.

2.1. P-Convexity

The next concept was given by Kottman in [1].

Definition 2.1. Let X be a Banach space. For each $n \in \mathbb{N}$ let

$$P(n, X) = \sup\{r > 0 : \text{there exist } n \text{ disjoint balls of radius } r \text{ in } B_X\}. \tag{2.1}$$

It is easy to see that $P(n, X) \leq 1/2$ for $n \geq 2$.

Definition 2.2. X is said to be P-convex if $P(n, X) < 1/2$ for some $n \in \mathbb{N}$.

The following lemma was proved in [1].

Lemma 2.3. *Let X be a Banach space and $n \in \mathbb{N}$. Then $P(n, X) < 1/2$ if and only if there exists $\varepsilon > 0$ such that for any $x_1, x_2, \ldots, x_n \in S_X$*

$$\min\{\|x_i - x_j\| : 1 \leq i, j \leq n, \quad i \neq j\} \leq 2 - \varepsilon. \tag{2.2}$$

That is, X is P-convex if and only if X satisfies condition (2.2) *for some $n \in \mathbb{N}$ and some $\varepsilon > 0$.*

Definition 2.4. Given $n \in \mathbb{N}$ and $\varepsilon > 0$ we say that X is $P(\varepsilon, n)$-convex if X satisfies (2.2). For each $n \in \mathbb{N}$, X is said to be $P(n)$-convex if it is $P(\varepsilon, n)$-convex for some $\varepsilon > 0$.

2.2. P-Convexity and the Coefficient of Convexity

In [1], Kottman proved that if X is a Banach space satisfying the condition $\delta_X(2/3) > 0$, then X is P(3)-convex, where δ_X is the modulus of convexity. In this section we give a result which improves this condition, and we show that this assumption is sharp.

We recall the following concepts introduced by J. A. Clarkson in 1936.

Definition 2.5. The modulus of convexity of a Banach space X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}. \tag{2.3}$$

The coefficient of convexity of a Banach space X is the number $\varepsilon_0(X)$ defined as

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}. \tag{2.4}$$

We also need the following definition given by R. C. James in 1964.

Definition 2.6. X is said to be uniformly nonsquare if there exists $\alpha > 0$ such that for all $\xi, \eta \in S_X$

$$\min\{\|\xi - \eta\|, \|\xi + \eta\|\} \leq 2 - \alpha. \tag{2.5}$$

In order to prove our theorem we need two known results which can be found in [2].

Lemma 2.7 (Goebel-Kirk). *Let X be a Banach space. For each $\varepsilon \in [\varepsilon_0(X), 2]$, one has the equality $\delta_X(2 - 2\delta_X(\varepsilon)) = 1 - \varepsilon/2$.*

Lemma 2.8 (Ullán). *Let X be a Banach space. For each $0 \leq \varepsilon_2 \leq \varepsilon_1 < 2$ the following inequality holds: $\delta_X(\varepsilon_1) - \delta_X(\varepsilon_2) \leq (\varepsilon_1 - \varepsilon_2)/(2 - \varepsilon_1)$.*

Using these lemmas we obtain:

Theorem 2.9. *Let X be a Banach space which satisfies $\delta_X(1) > 0$, that is, $\varepsilon_0(X) < 1$. Then X is $P(3)$-convex. Moreover, there exists a Banach space X with $\varepsilon_0(X) = 1$ which is not $P(3)$-convex.*

Proof. Let $t_0 = 2 - \sqrt{2 - \varepsilon_0(X)}$. Clearly $\varepsilon_0(X) < t_0 < 1$. Let $x, y, z \in S_X$, and suppose that $\|x - y\| > 2 - 2\delta_X(t_0)$ and $\|x - z\| > 2 - 2\delta_X(t_0)$. By Lemma 2.7, we have

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(2 - 2\delta_X(t_0)) = 1 - \left(1 - \frac{t_0}{2}\right) = \frac{t_0}{2}. \tag{2.6}$$

Similarly $\|(x + z)/2\| \leq t_0/2$. Hence we get

$$\|z - y\| \leq \|z + x\| + \|x + y\| \leq 2t_0. \tag{2.7}$$

Finally, from Lemma 2.8 it follows that

$$\delta_X(t_0) = \delta_X(t_0) - \delta_X(\varepsilon_0(X)) \leq \frac{t_0 - \varepsilon_0(X)}{2 - t_0} = \sqrt{2 - \varepsilon_0(X)} - 1 = 1 - t_0. \tag{2.8}$$

Then $\|y - z\| \leq 2t_0 \leq 2 - 2\delta_X(t_0)$, and thus X is $P(3)$-convex.

Now consider for each $1 < p < \infty$ the space $l_{p,\infty}$ defined as follows. Each element $x = \{x_i\}_i \in l_p$ may be represented as $x = x^+ - x^-$, where the respective ith components of x^+ and x^- are given by $(x^+)_i = \max\{x_i, 0\}$ and $(x^-)_i = \max\{-x_i, 0\}$. Set $\|x\|_{p,\infty} = \max\{\|x^+\|_p, \|x^-\|_p\}$ where $\|\cdot\|_p$ stands for the l_p-norm. The space $l_{p,\infty} = (l_p, \|\cdot\|_{p,\infty})$ satisfies $\varepsilon_0(l_{p,\infty}) = 1$ (see [3]). On the other hand let $x_1 = e_1 - e_3$, $x_2 = -e_1 + e_2$, $x_3 = -e_2 + e_3 \in S_{l_{p,\infty}}$, where $\{e_i\}_i$ is the canonical basis in l_p. These points satisfy that $\|x_i - x_j\|_{p,\infty} = 2$, $i \neq j$. Thus $l_{p,\infty}$ is not $P(3,2)$-convex. □

It is known that if a Banach space X satisfies $\varepsilon_0(X) < 1$, then X has normal structure as well as $P(3)$-convexity. The space $X = l_{p,\infty}$ is an example of a Banach space with $\varepsilon_0(X) = 1$ which does not have normal structure (see [3]) and is not $P(3)$-convex.

Kottman also proved in [1] that every uniformly smooth space is a P-convex space. We obtain a generalization of this fact. Before we show this result we recall the next concept.

Definition 2.10. The modulus of smoothness of a Banach space X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{1}{2}(\|x + ty\| + \|x - ty\| - 2) : x, y \in S_X \right\} \tag{2.9}$$

for each $t \geq 0$. X is called uniformly smooth if $\lim_{t \to 0} \rho_X(t)/t = 0$.

The proofs of the following lemmas can be found in [4, 5].

Lemma 2.11. *For every Banach space X, one has* $\lim_{t\to 0}\rho_X(t)/t = (1/2)\varepsilon_0(X^*)$.

Lemma 2.12. *Let X be a Banach space. X is P(3)-convex if and only if X* is P(3)-convex.*

By Theorem 2.9 and by the previous lemmas we deduce the next result.

Corollary 2.13. *If X is a Banach space satisfying* $\lim_{t\to 0}\rho_X(t)/t < 1/2$, *then X is P(3)-convex.*

With respect to $P(4)$-convex spaces we have this result, which is easy to prove.

Proposition 2.14. *If X is a Banach space* $P(\varepsilon, 4)$*-convex, then* $\varepsilon_0(X) \le 2 - \varepsilon$, *and hence X is uniformly nonsquare.*

In fact, in bidimensional normed spaces, $P(4)$-convexity and uniform nonsquareness coincide. The proof of this involves many calculations and can be seen in [6].

Another technical proof (see [6]) shows that if X is a bidimensional normed space, then X is always $P(1,5)$-convex. Hence the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ is $P(1,5)$-convex and $\varepsilon_0(X) = 2$, and thus $P(5)$-convexity does not imply uniform squareness.

2.3. Relation between U-Spaces and P-Convex Spaces

In this section we show that P-convexity follows from U-convexity. The following concept was introduced by Lau in 1978 [7].

Definition 2.15. A Banach space X is called a U-space if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in S_X, \quad f(x - y) > \varepsilon, \quad \text{for some } f \in \nabla(x) \Longrightarrow \left\| \frac{x + y}{2} \right\| \le 1 - \delta, \tag{2.10}$$

where for each $x \in X$

$$\nabla(x) = \{ f \in S_{X^*} : f(x) = \|x\| \}. \tag{2.11}$$

The modulus of this type of convexity was introduced by Gao in [8] and further studied by Mazcuñán-Navarro [9] and Saejung [10]. The following result is proved in [8].

Lemma 2.16. *Let X be a Banach space. If X is U-space, then X is uniformly nonsquare,*

From the above we obtain the next theorem which is a generalization of Kottman's result, who showed in [1] that $P(3)$-convexity follows from uniform convexity.

Theorem 2.17. *If X is a U-space, then X is P(3)-convex.*

Proof. By Lemma 2.16 we have that there exists $\alpha > 0$ such that for all $\xi, \eta \in S_X$

$$\min\{ \|\xi - \eta\|, \|\xi + \eta\| \} \le 2 - \alpha. \tag{2.12}$$

Since X is a U-space, for $\varepsilon = \alpha/2$ there exists $\delta > 0$ such that

$$x, y \in S_X, \quad f(x-y) \geq \frac{\alpha}{2}, \quad \text{for some } f \in \nabla(x) \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2.13}$$

We claim that X is $P(\beta,3)$-convex, where $\beta = \min\{\alpha, \delta\}$. Indeed, proceeding by contradiction, assume that there exist $x, y, z \in S_X$ such that

$$\min\{\|x-y\|, \|x-z\|, \|y-z\|\} > 2-\beta. \tag{2.14}$$

Define $w = -y$ and $u = -z$, and let $f \in \nabla(w)$. If $f(w-x) \geq \alpha/2$, then

$$\left\| \frac{w+x}{2} \right\| < 1-\delta. \tag{2.15}$$

Therefore $2-\delta \leq 2-\beta < \|x-y\| < 2-2\delta$, which is not possible. Hence $f(w-x) < \alpha/2$. Similarly we prove $f(w+u) < \alpha/2$. Also $\|x+u\| = \|x-z\| > 2-\beta \geq 2-\alpha$, and hence, by (2.12) we have $f(x-u) \leq \|x-u\| \leq 2-\alpha$. By the above we have

$$2 = 2f(w) = f(w-x) + f(x-u) + f(u+w) < \frac{\alpha}{2} + 2 - \alpha + \frac{\alpha}{2} = 2 \tag{2.16}$$

which is a contradiction. □

2.4. *The Dual Concept of P-Convexity*

In [1], Kottman introduces a property which turns out to be the dual concept of P-convexity. In this section we characterize the dual of a P-convex space in an easier way. We begin by showing Kottman's characterization.

Definition 2.18. Let X be a Banach space and $\varepsilon > 0$. A convex subset A of B_X is said to be ε-flat if $A \bigcap (1-\varepsilon)B_X = \emptyset$. A collection $\mathfrak{D}$ of ε-flats is called complemented if for each pair of ε-flats A and B in $\mathfrak{D}$ we have that $A \bigcup B$ has a pair of antipodal points. For any $n \in \mathbb{N}$ we define

$$F(n,X) = \inf\{\varepsilon > 0 : B_X \text{ has a complemented collection } \mathfrak{D} \text{ of } \varepsilon\text{-flats such that } \operatorname{Card}(\mathfrak{D}) = n\}. \tag{2.17}$$

Theorem 2.19 (Kottman). *Let X be a Banach space and $n \in \mathbb{N}$. Then*

(a) $F(n, X^*) = 0 \Leftrightarrow P(n, X) = 1/2$.

(b) $P(n, X^*) = 1/2 \Leftrightarrow F(n, X) = 0$.

Now we define P-smoothness and prove that it turns out to be the dual concept of P-convexity. The advantage of this characterization is that it uses only simple concepts, and one does not need ε-flats. Besides in the proof of the duality we do not need Helly's theorem nor the theorem of Hahn-Banach, as Kottman does in Theorem 2.19.

Definition 2.20. Let X be a Banach space and $\delta > 0$. For each $f, g \in X^*$ set $S(f, g, \delta) = \{x \in B_X : f(x) \geq 1 - \delta, g(x) \geq 1 - \delta\}$. Given $\delta > 0$ and $n \in \mathbb{N}$, X is said to be $P(\delta, n)$-smooth if for each $f_1, f_2, \dots, f_n \in S_{X^*}$ there exist $1 \leq i, j \leq n, i \neq j$, such that $S(f_i, -f_j, \delta) = \emptyset$. X is said to be $P(n)$-smooth if it is $P(\delta, n)$-smooth for some $\delta > 0$, and X is said to be P-smooth if it is $P(\delta, n)$-smooth for some $\delta > 0$ and some $n \in \mathbb{N}$.

Proposition 2.21. *Let X be a Banach space. Then*

(a) *X is $P(n)$-convex if and only if X^* is $P(n)$-smooth.*

(b) *X is $P(n)$-smooth if and only if X^* is $P(n)$-convex.*

Proof. (a) Let X be a $P(\varepsilon, n)$-convex space. Let $x_1^{**}, \dots, x_n^{**} \in S_{X^{**}}$. We will show that there exist $1 \leq i, j \leq n, i \neq j$, such that $S(x_i^{**}, -x_j^{**}, \varepsilon/4) = \emptyset$. Since X is P-convex, it is also reflexive. Therefore $x_1^{**} = \jmath(x_1), \dots, x_n^{**} = \jmath\jmath(x_n)$ for some $x_1, \dots, x_n \in S_X$, where $\jmath$ is the canonical injection from X to X^{**}. By hypothesis, there exist $1 \leq i, j \leq n$, $i \neq j$, such that $\|x_i - x_j\| \leq 2 - \varepsilon$. Therefore it is enough to prove that

$$\left\{f \in B_{X^*} : f(x_i) \geq 1 - \frac{\varepsilon}{4}, -f(x_j) \geq 1 - \frac{\varepsilon}{4}\right\} = \emptyset. \tag{2.18}$$

We proceed by contradiction supposing that there exists $f \in B_{X^*}$ such that $f(x_i) \geq 1 - \varepsilon/4$ and $-f(x_j) \geq 1 - \varepsilon/4$. Then

$$2 - \varepsilon \geq \|x_i - x_j\| \geq f(x_i - x_j) \geq 2 - \frac{\varepsilon}{2}, \tag{2.19}$$

which is not possible; consequently X^* is $P(\varepsilon/4, n)$-smooth.

Now let X be a Banach space such that X^* is $P(\varepsilon, n)$-smooth. Let $x_1, \dots, x_n \in S_X$. By hypothesis, there exist $1 \leq i, j \leq n$, $i \neq j$, such that $S(\jmath(x_i), -\jmath(x_j), \varepsilon) = \emptyset$, that is, for each $f \in B_{X^*}$ we have $f(x_i) < 1 - \varepsilon$ or $-f(x_j) < 1 - \varepsilon$. We will see that $\|x_i - x_j\| \leq 2 - \varepsilon$. We again proceed by contradiction supposing that $\|x_i - x_j\| = \|\jmath(x_i - x_j)\| > 2 - \varepsilon$. There exists $f \in S_{X^*}$ such that $\jmath(x_i - x_j)(f) = f(x_i) - f(x_j) > 2 - \varepsilon$. If $f(x_i) < 1 - \varepsilon$, then

$$1 = \|f\|\|x_j\| \geq -f(x_j) > 2 - \varepsilon - f(x_i) > 1 \tag{2.20}$$

which is not possible. Similarly if $-f(x_j) < 1 - \varepsilon$, we obtain a contradiction. Thus $\|x_i - x_j\| \leq 2 - \varepsilon$, and consequently X is $\mathrm{P}(\varepsilon, n)$-convex. The proof of (b) is analogous to the proof of (a). □

Therefore the conditions X is $P(n)$-smooth and $F(n, X) > 0$ must be equivalent.

3. *p*-Convex Banach Spaces

In this section we introduce the nonuniform version of P-convexity and we call it p-convexity.

Definition 3.1. Let X be a Banach space and $n \in \mathbb{N}$. X is said to be $p\ (n)$-convex if for any $x_1, \ldots, x_n \in S_X$, there exist $1 \le i, j \le n$, $i \ne j$, such that $\|x_i - x_j\| < 2$. X is said to be p-convex if is $p(n)$-convex for some $n \in \mathbb{N}$.

Kottman defined the concept of P-convexity in terms of the intersection of balls. We will do something similar to give an equivalent definition of p-convexity. It is easy to see that in a normed space any two closed balls of radius $1/2$ contained in the unit ball have non empty intersection. If the radius is less than $1/2$, for example, in l_1 for every n and for every $r < 1/2$, then there exist n closed balls of radius r so that no two of them intersect. In fact let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis of l_1. Then the closed balls of radius $r < 1/2$ centered at the points $(1/2)e_i$, $i \in \mathbb{N}$ are disjoint and contained in the unit ball. However, if X is $p(n)$-convex, we will see that for any n points in the unit ball there exists $r < 1/2$ so that if the n closed balls centered at these n points are contained in the unit ball, there are two different balls with non empty intersection. To prove this we need the following lemma, which was shown in [11].

Lemma 3.2. *Let X be a Banach space and $x, y \in X$, $x, y \ne 0$. Then*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ge \frac{1}{\min\{\|x\|, \|y\|\}} \left(\|x - y\| - \big| \|x\| - \|y\| \big| \right). \tag{3.1}$$

Lemma 3.3. *X is a $p(n)$-convex space if and only if for any $y_1, \ldots, y_n \in B_X$ there exists $r \in (0, 1/2)$ such that, if $B(y_i, r) \subset B_X$ for all $i = 1, \ldots, n$, then there are $1 \le i, j \le n$, $i \ne j$, so that*

$$B(y_i, r) \cap B(y_j, r) \ne \emptyset. \tag{3.2}$$

Proof. Assume that X satisfies condition (3.2), and let $x_1, \ldots, x_n \in S_X$. Let $r \in (0, 1/2)$ be the number which satisfies condition (3.2) for $x_1/2, \ldots, x_n/2$. It is easy to see that $B(x_i/2, r) \subset B_X$ for each $i = 1, \ldots, n$. Therefore there exist $1 \le i, j \le n$, $i \ne j$, such that

$$B\left(\frac{x_i}{2}, r\right) \cap B\left(\frac{x_j}{2}, r\right) \ne \emptyset. \tag{3.3}$$

Let

$$y \in B\left(\frac{x_i}{2}, r\right) \cap B\left(\frac{x_j}{2}, r\right). \tag{3.4}$$

We have

$$\left\| \frac{x_i - x_j}{2} \right\| \le \left\| \frac{x_i}{2} - y \right\| + \left\| \frac{x_j}{2} - y \right\| < 2r < 1, \tag{3.5}$$

and thus X is $p(n)$-convex. Now we suppose that there exist $y_1, \ldots, y_n \in B_X$ such that for any $\rho \in (0, 1/2)$ we have

$$B\left(y_i, \frac{1}{2} - \rho\right) \subset B_X \tag{3.6}$$

for all $i = 1, \ldots, n$, and

$$B\left(y_i, \frac{1}{2} - \rho\right) \cap B\left(y_j, \frac{1}{2} - \rho\right) = \emptyset, \tag{3.7}$$

for all $i, j = 1, \ldots, n$, $i \neq j$. We verify that X is not $p(n)$-convex in four steps.

(a) Take $\|y_i - y_j\| > 1 - 2\rho$ for any $i, j = 1, \ldots, n$, $i \neq j$.

(b) Take $1/2 - 3\rho < \|y_i\| \leq 1/2 + \rho$, for all $i = 1, \ldots, n$. To verify this claim we note that $\|y_i/\|y_i\| - y_i\| \geq 1/2 - \rho$ for all i, because if $\|y_i/\|y_i\| - y_i\| < 1/2 - \rho$ for some i, then $y_i/\|y_i\| \in \operatorname{int} B(y_i, 1/2 - \rho) \subset \operatorname{int} B_X$, which is not possible. Hence, as $\|y_i/\|y_i\| - y_i\| = 1 - \|y_i\|$, it follows that $\|y_i\| = 1 - \|y_i/\|y_i\| - y_i\| \leq 1/2 + \rho$, for each $i = 1, \ldots, n$. Now, if $\|y_i\| \leq 1/2 - 3\rho$ for some i, we have by (a) that for any $j \neq i$, $1 - 2\rho < \|y_i - y_j\| \leq \|y_i\| + \|y_j\| \leq (1/2 - 3\rho) + (1/2 + \rho) = 1 - 2\rho$ which is not possible.

(c) Take $|\|y_i\| - \|y_j\|| < 4\rho$, for any $i, j = 1, \ldots, n$, $i \neq j$. Indeed, by (b) we get $-4\rho = (1/2 - 3\rho) - (1/2 + \rho) < \|y_i\| - \|y_j\| < (1/2 + \rho) - (1/2 - 3\rho) = 4\rho$.

(d) From (a), (b), (c), and by Lemma 3.2, we have

$$\left\| \frac{y_i}{\|y_i\|} - \frac{y_j}{\|y_j\|} \right\| \geq \frac{1}{\|y_i\|} \left(\|y_i - y_j\| - \left| \|y_i\| - \|y_j\| \right| \right) > 2 - \frac{16\rho}{1 + 2\rho} \tag{3.8}$$

for any $i, j = 1, \ldots, n$, $i \neq j$. Since $\rho > 0$ is arbitrary, as $\rho \to 0$, we obtain $\|y_i/\|y_i\| - y_j/\|y_j\|\| = 2$, for all $i, j = 1, \ldots, n$, $i \neq j$, and thus X is not $p(\mathrm{n})$-convex. □

Next we give some examples of spaces which are not p-convex. The first is not reflexive and the last one is superreflexive.

Example 3.4. c_0, and consequently, $C[0,1]$ and l_∞ are not p-convex spaces. Indeed, let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis in c_0. For each $n \in \mathbb{N}$ we define $u_i = \sum_{j=1}^{n} \lambda_{i,j} e_j$, where $\lambda_{i,j} = 1$ if $j \neq i$, $\lambda_{i,i} = -1$, and $i = 1, \ldots, n$. Clearly $u_1, \ldots, u_n \in S_{c_0}$, and for each $i \neq j$ we have $\|u_i - u_j\|_\infty = 2$.

Example 3.5. Let X denote the space obtained by renorming l_2 as follows. For $x = (x_i)_{i \in \mathbb{N}} \in l_2$ set

$$|||x||| = \max\left\{ \sup_{i,j} |x_i - x_j|, \left(\sum_{i=1}^{\infty} x_i^2 \right)^{1/2} \right\}. \tag{3.9}$$

Then $\|x\| \leq |||x||| \leq \sqrt{2}\|x\|$, where $\|\cdot\|$ stands for the l_2-norm and X is superreflexive. On the other hand, the canonical basis $\{e_n\}_n$ in l_2 satisfies $\|e_i - e_j\|_\infty = 2$ for each $i \neq j$. Thus X is not p-convex.

Now we will mention several properties that imply p-convexity.

Recall the following concepts. Let X be a Banach space. X is said to be a *u-space* if it satisfies the following implication:

$$x, y \in S_X, \quad \left\| \frac{x+y}{2} \right\| = 1 \Longrightarrow \nabla(x) = \nabla(y). \tag{3.10}$$

X is said to be smooth if for any $x \in S_X$, there exists a unique $f \in S_{X^*}$ such that $f(x) = 1$. That is, for each $x \in S_X$, $\nabla(x)$ contains a single point. X is called *strictly convex* if the following implication holds:

$$\forall x, y \in B_X : x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1. \tag{3.11}$$

Proposition 3.6. *Every smooth space, every strictly convex space and every u-space are p(3)-convex space.*

Proof. Every smooth space and every strictly convex space are u-space. It suffices to show that $p(3)$-convexity follows from being u-space. If X is a u-space, then for any $x, y \in S_X$ the following inequality holds: $\min\{\|x-y\|, \|x+y\|\} < 2$. Indeed, if we suppose that there exist $x, y \in S_X$ such that $\|x+y\| = \|x-y\| = 2$, then $\nabla(x) = \nabla(y)$ and $\nabla(x) = \nabla(-y)$, which is not possible. Suppose that X is not $p(3)$-convex, and there exist $x, y, z \in S_X$ so that $\|x-y\| = \|y-z\| = \|z-x\| = 2$. Since $(1/2)\|x-y\| = (1/2)\|y-z\| = 1$,we have $\nabla(x) = \nabla(-y) = \nabla(z)$. Let $f \in \nabla(-y)$; then $f(x+z) \leq \|x+z\| < 2$, and

$$2 = f(x) + f(-y) = f(x+z) - f(z) + f(-y) = f(x+z) < 2. \tag{3.12}$$

Thus X is p(3)-convex. □

Obviously P-convexity implies p-convexity; however, a p-convex space is not necessarily P-convex, even if the space is reflexive as the following example shows.

Example 3.7. Let $\{r_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $r_k > 1$ for each $k \in \mathbb{N}$ and $r_k \downarrow 1$, when $k \to \infty$. Consider the space $X = \sum_{k=1}^{\infty} \oplus_2 l_{r_k}$. It is known that this space is strictly convex, hence it is also *p(3)*-convex. It is also known that X is reflexive. However X is not P-convex. Indeed, let $\varepsilon > 0$. We choose $k \in \mathbb{N}$ such that $2 - \varepsilon < 2^{1/r_k}$. *If* $\{e_i\}_{i=1}^{\infty}$ is the canonical basis of l_{r_k}, we have that $\|e_i - e_j\|_{r_k} = 2^{1/r_k} > 2 - \varepsilon$ for all $i, j \in \mathbb{N}$, $i \neq j$, and hence X is not a P-convex space.

We have obtained a result which shows a strong relation between P-convexity and p-convexity with respect to the ultrapower of Banach spaces. We recall the definition and some results regarding ultrapowers which can be found in [4].

A filter $\mathfrak{U}$ on I is called an ultrafilter on I if $\mathfrak{U}$ is a maximal element from $\wp$ with respect to the set inclusion. $\mathfrak{U}$ is an ultrafilter on I if and only if for all $A \subset I$ either $A \in \mathfrak{U}$ or $I \setminus A \in \mathfrak{U}$. Let $\{X_i\}_{i \in I}$ be a family of Banach spaces, and let

$$l_\infty(X_i) = \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} X_i : \sup\{\|x_i\|_{X_i} : i \in I\} < \infty \right\}. \tag{3.13}$$

If we define $\|\{x_i\}_{i\in I}\|_\infty = \sup\{\|x_i\|_{X_i} : i \in I\}$ for each $\{x_i\}_{i\in I} \in l_\infty(X_i)$, then $\|\cdot\|_\infty$ defines a norm in $l_\infty(X_i)$, and $(l_\infty(X_i), \|\cdot\|_\infty)$ is a Banach space. If $\mathfrak{U}$ is a free ultrafilter on I, then for each $\{x_i\}_{i\in I} \in l_\infty(X_i)$ we have $\lim_{\mathfrak{U}} x_i$ always exists and is unique. Let $\mathfrak{U}$ be an ultrafilter on I, and define

$$\mathcal{N}_{\mathfrak{U}} = \left\{\{x_i\} \in l_\infty(X_i) : \lim_{\mathfrak{U}}\|x_i\| = 0\right\}. \tag{3.14}$$

$\mathcal{N}_{\mathfrak{U}}$ is a closed subspace of $l_\infty(X_i)$. The *ultraproduct* of $\{X_i\}_{i\in I}$ with respect to the ultrafilter $\mathfrak{U}$ on I is the quotient space $l_\infty(X_i)/\mathcal{N}_{\mathfrak{U}}$ equipped with the quotient norm, which is denoted by $\{X_i\}_{\mathfrak{U}}$ and its elements by $\{x_i\}_{\mathfrak{U}}$. If $X_i = X$ for all $i \in I$, then $\{X\}_{\mathfrak{U}} = \{X_i\}_{\mathfrak{U}}$ is called the *ultrapower* of X. The quotient norm in $\{X_i\}_{\mathfrak{U}}$,

$$\|\{x_i\}_{\mathfrak{U}}\| = \inf\left\{\left\|\{x_i + y_i\}_i\right\|_\infty : \{y_i\}_i \in \mathcal{N}_{\mathfrak{U}}\right\}, \tag{3.15}$$

satisfies the equality

$$\|\{x_i\}_{\mathfrak{U}}\| = \lim_{\mathfrak{U}}\|x_i\|_{X_i}, \quad \text{for each}\{x_i\}_{\mathfrak{U}} \in \{X_i\}_{\mathfrak{U}}. \tag{3.16}$$

If $\mathfrak{U}$ is nontrivial, then X can be embedded into $\{X\}_{\mathfrak{U}}$ isometrically. We will write $\widetilde{X}_i$ instead of $\{X_i\}_{\mathfrak{U}}$ and $\tilde{x}$ instead of $\{x_i\}_{\mathfrak{U}}$ unless we need to specify the ultrafilter we are talking about.

It is known that X is uniformly convex if and only if $\widetilde{X}$ is strictly convex, X is uniformly smooth if and only if $\widetilde{X}$ is smooth, and X is a U-space if and only if $\widetilde{X}$ is a u-space (see [12]). Similarly we obtain the following result.

Theorem 3.8. *Let X be a Banach space and $m \in \mathbb{N}$. The following are equivalent:*

(a) $\widetilde{X}$ is $P(m)$-convex.

(b) X is $P(m)$-convex,

(c) $\widetilde{X}$ is $p(m)$-convex,

Proof. $(a) \Rightarrow (b)$. Let $\{x_i^{(n)}\}_n \in \tilde{x}_i$, $\tilde{x}_i \in S_{\widetilde{X}}$, $i = 1, \ldots, m$. Since $\lim_{\mathfrak{U}}\|x_i^{(n)}\|_X = \|\tilde{x}_i\|_{\widetilde{X}} = 1$ for all i, there exists a subsequence $\{x_i^{(n_k)}\}_k$ of $\{x_i^{(n)}\}_n$ such that $\lim_{k\to\infty}\|x_i^{(n_k)}\|_X = 1$ and $\|x_i^{(n_k)}\|_X > 0$, for all $k \in \mathbb{N}$. Define

$$y_i^{(n_k)} = \frac{x_i^{(n_k)}}{\left\|x_i^{(n_k)}\right\|_X}, \qquad \Gamma_{i,j} = \left\{k \in \mathbb{N} : \left\|y_i^{(n_k)} - y_j^{(n_k)}\right\|_X \le 2 - \varepsilon\right\}, \tag{3.17}$$

for each $i, j = 1, \ldots, m$, $i \neq j$. We verify that there exist $1 \le i, j \le m$, $i \neq j$, such that $\Gamma_{i,j} \in \mathfrak{U}$. We proceed by contradiction assuming that, $\Gamma_{i,j} \notin \mathfrak{U}$ for all $i \neq j$. Hence $\mathbb{N} \setminus \Gamma_{i,j} \in \mathfrak{U}$ for all $i \neq j$, and consequently $\mathbb{N} \setminus (\bigcup_{i\neq j}\Gamma_{i,j}) \neq \emptyset$, therefore there exists $k_0 \in \mathbb{N} \setminus (\bigcup_{i\neq j}\Gamma_{i,j})$. Thus we have $\|y_i^{(n_{k_0})} - y_j^{(n_{k_0})}\| > 2 - \varepsilon$ for each $i \neq j$, and X is not $P(m)$-convex, which is a contradiction.

Therefore there exist $1 \le i, j \le m, i \ne j$, such that $\Gamma_{i,j} \in \mathfrak{U}$, and hence $\lim_{\mathfrak{U}} \|y_i^{(n_k)} - y_j^{(n_k)}\|_X \le 2 - \varepsilon$. Finally, note that

$$\begin{aligned}
\left\|x_i^{(n_k)} - x_j^{(n_k)}\right\|_X &\le \left\|x_i^{(n_k)} - y_i^{(n_k)}\right\|_X + \left\|x_j^{(n_k)} - y_j^{(n_k)}\right\|_X + \left\|y_i^{(n_k)} - y_j^{(n_k)}\right\|_X \\
&= \left|1 - \left\|x_i^{(n_k)}\right\|_X\right| + \left|1 - \left\|x_i^{(n_k)}\right\|_X\right| + \left\|y_i^{(n_k)} - y_j^{(n_k)}\right\|_X, \\
\|\tilde{x}_i - \tilde{x}_j\|_{\tilde{X}} &= \lim_{\mathfrak{U}} \left\|x_i^{(n)} - x_j^{(n)}\right\|_X = \lim_{\mathfrak{U}} \left\|x_i^{(n_k)} - x_j^{(n_k)}\right\|_X \\
&\le \lim_{\mathfrak{U}} \left|1 - \left\|x_i^{(n_k)}\right\|_X\right| + \lim_{\mathfrak{U}} \left|1 - \left\|x_i^{(n_k)}\right\|_X\right| + \lim_{\mathfrak{U}} \left\|y_i^{(n_k)} - y_j^{(n_k)}\right\|_X \le 2 - \varepsilon.
\end{aligned} \tag{3.18}$$

Therefore $\tilde{X}$ is $P(m)$-convex.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$. Suppose that X is not $P(m)$-convex. Hence for any $n \in \mathbb{N}$ there exist $x_1^{(n)}, \dots, x_m^{(n)} \in S_X$ such that $\|x_i^{(n)} - x_j^{(n)}\|_X > 2 - 1/n$ for all $i, j = 1, \dots, m$, $i \ne j$. Define $\tilde{x}_i = \{x_i^{(n)}\}_{\mathfrak{U}}$ for each $i = 1, \dots, m$. Clearly $\tilde{x}_i \in S_{\tilde{X}}$ for all i, because $\|\tilde{x}_i\|_{\tilde{X}} = \lim_{\mathfrak{U}} \|x_i^{(n)}\|_X = 1$, and also,

$$\|\tilde{x}_i - \tilde{x}_j\|_{\tilde{X}} = \lim_{\mathfrak{U}} \left\|x_i^{(n)} - x_j^{(n)}\right\|_X = \lim_{n \to \infty} \left\|x_i^{(n)} - x_j^{(n)}\right\|_X = 2, \tag{3.19}$$

for each $i \ne j$. Hence $\tilde{X}$ is not $p(m)$-convex. □

By the above theorem we can deduce the following known result.

Corollary 3.9. *If X is P -convex, then X is superreflexive.*

Proof. If X is P-convex, then $\tilde{X}$ is P-convex and therefore is reflexive. However in ultrapower reflexivity and superreflexivity are equivalent, hence $\tilde{X}$ is superreflexive, and consequently X is superreflexive. □

Now we turn our attention to some results regarding the p-convexity and the P-convexity of quotient spaces. To prove them we need the following concept.

Definition 3.10. A subspace Y of a normed space X is said to be proximinal if for all $x \in X$ there exists $y \in Y$ such that $d(x, Y) = \|x - y\|$.

It is easy to see that every proximinal subspace Y of a Banach space X is closed.

Proposition 3.11. *If X is p(n)-convex and Y is a proximinal subspace of X, then X/Y is p (n)-convex.*

Proof. Let $q : X \to X/Y$ be the quotient function. By the proximinality of Y we have $q(B_X) = B_{X/Y}$. Let $\tilde{x}_1, \dots, \tilde{x}_n \in S_{X/Y}$ and $x_1, \dots, x_n \in S_X$ such that $\tilde{x}_i = q(x_i)$. Since X is $p(n)$-convex, there exist $1 \le i, j \le n$, $i \ne j$, such that $\|x_i - x_j\| < 2$, and consequently $\|\tilde{x}_i - \tilde{x}_j\| < 2$. □

Corollary 3.12. *Let X be p(n)-convex and reflexive. If Y is a closed subspace of X, then X/Y is p(n)-convex.*

Proof. It is shown in [13] that a Banach space X is reflexive if and only if each closed subspace of X is proximinal, and thus the corollary is a consequence of Proposition 3.11. □

Similarly we can prove that if X is $P(\varepsilon, n)$-convex and Y is a closed subspace of X, then X/Y is $P(\varepsilon, n)$-convex.

We obtain two results involving ψ-direct sums of p-convex spaces. Next we will define these sums as in [14] by Saito , et al.

Definition 3.13. Set $\Psi = \{\psi : [0,1] \to \mathbb{R} \mid \psi$ is a continuous convex function, $\max\{1-t, t\} \leq \psi(t) \leq 1,$ for all $0 \leq t \leq 1.\}$

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. For each $\psi \in \Psi$, one defines the norm $\|\cdot\|_\psi$ in $X \oplus Y$ as $\|(0,0)\|_\psi = 0$ and for each $(x,y) \neq (0,0)$

$$\|(x,y)\|_\psi = (\|x\|_X + \|y\|_Y)\psi\left(\frac{\|y\|_Y}{\|x\|_X + \|y\|_Y}\right). \tag{3.20}$$

In [15] it is shown that $(X \oplus Y, \|\cdot\|_\psi)$ is a Banach space, denoted by $X \oplus_\psi Y$ called the *ψ-direct and sum* of X and Y.

The proof of the following theorem is similar to the proof of Theorem 3.5 in [16], which shows the corresponding result for P-convex spaces.

Theorem 3.14. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is p-convex if and only if X and Y are p-convex.*

In [17] there is a theorem stating several equivalent conditions for strict convexity. We prove a similar result for p-convexity.

Lemma 3.15. *Let X be a Banach space. The next assertions are equivalent.*

(a) *X is $p(n)$-convex.*

(b) *For any $q \in (1,\infty)$ and for any $x_1, \ldots, x_n \in X$, not all zero, there exist $1 \leq i, j \leq n$, $i \neq j$, such that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$.*

(c) *For some $q \in (1,\infty)$ and for any $x_1, \ldots, x_n \in X$, not all zero, there exist $1 \leq i, j \leq n$, $i \neq j$, such that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$.*

Proof. The implications $(b) \Rightarrow (c) \Rightarrow (a)$ are immediate. We verify $(a) \Rightarrow (b)$. Let $q \in (1,\infty)$ and $x_1, \ldots, x_n \in X$, not all zero. If $x_j = 0$ and $x_i \neq 0$ for some $1 \leq i, j \leq n$, then it is clear that $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$. Suppose that $x_1, \ldots, x_n \in X \setminus \{0\}$. There exist $1 \leq i, j \leq n$, $i \neq j$, such that

$$\left\|\frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|}\right\| < 2. \tag{3.21}$$

If $\|x_j\| \leq \|x_i\|$ by Lemma 3.2 we get

$$\|x_i - x_j\| \leq \|x_j\|\left\|\frac{x_i}{\|x_i\|} - \frac{x_j}{\|x_j\|}\right\| + \|x_i\| + \|x_j\| < \|x_i\| + \|x_j\|. \tag{3.22}$$

As the function $t \mapsto t^q$ is convex we obtain that

$$\left\|\frac{x_i - x_j}{2}\right\|^q < \left(\frac{\|x_i\| + \|x_j\|}{2}\right)^q \leq \frac{1}{2}\left(\|x_i\|^q + \|x_j\|^q\right). \tag{3.23}$$

Thus $\|x_i - x_j\| < 2^{(q-1)/q}(\|x_i\|^q + \|x_j\|^q)^{1/q}$. □

Proposition 3.16. *Let $\{X_i\}_{i\in I}$ be a family of p(n)-convex spaces, where the index set $I \neq \emptyset$ has any cardinality. Then the space $X = l_q(X_i)$ $(1 < q < \infty)$ is p(n)-convex.*

Proof. Let $x^{(k)} = \{x_i^{(k)}\}_{i\in I} \in X$, $1 \leq k \leq n$, not all zero. Let $i_0 \in I$ be such that $x_{i_0}^{(k)} \neq 0$, for some $k \in \{1, \ldots, n\}$. As X_{i_0} is a *p(n)*-convex space, we have by the preceding lemma that there exist $1 \leq l, m \leq n$ such that

$$\left\|x_{i_0}^{(l)} - x_{i_0}^{(m)}\right\|^q < 2^{q-1}\left(\left\|x_{i_0}^{(l)}\right\|^q + \left\|x_{i_0}^{(m)}\right\|^q\right). \tag{3.24}$$

By the above we obtain

$$\begin{aligned} \left\|x^{(l)} - x^{(m)}\right\|_q^q &= \sum_{i\in I}\left\|x_i^{(l)} - x_i^{(m)}\right\|^q \\ &< \sum_{i\in I} 2^{q-1}\left(\left\|x_i^{(l)}\right\|^q + \left\|x_i^{(m)}\right\|^q\right) = 2^{q-1}\left(\left\|x^{(l)}\right\|_q^q + \left\|x^{(m)}\right\|_q^q\right). \end{aligned} \tag{3.25}$$

Therefore, by the previous lemma, X is *p(n)*-convex. □

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