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Research Article

A New Method to Prove and Find Analytic Inequalities

Xiao-Ming Zhang,[1] Bo-Yan Xi,[2] and Yu-Ming Chu[1]

[1] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*
[2] *Department of Mathematics, Inner Mongolia University for the Nationalities, Tongliao 028000, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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We present a new method to study analytic inequalities. As for its applications, we prove the well-known Hölder inequality and establish several new analytic inequalities.

1. Monotonicity Theorem

Throughout the paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ denotes the set of strictly positive real numbers. Let $n \geq 2$, $n \in \mathbb{N}$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$; the arithmetic mean $A(\mathbf{x})$ and the power mean $M_r(\mathbf{x})$ of order r with respect to the positive real numbers $x_1, x_2, \ldots, x_n$ are defined by $A(\mathbf{x}) = (1/n)\sum_{i=1}^{n} x_i$, $M_r(\mathbf{x}) = ((1/n)\sum_{i=1}^{n} x_i^r)^{1/r}$ for $r \neq 0$, and $M_0(\mathbf{x}) = (\prod_{i=1}^{n} x_i)^{1/n}$, respectively.

In [1], Pachpatte gave many basic methods and tools for researchers working in inequalities. In this section, we present a monotonicity theorem which can be used as powerful tool to prove and find analytic inequalities.

Lemma 1.1. *Suppose that $m < M$, $D = \{(x_1, x_2) \mid m \leq x_2 \leq x_1 \leq M\}$. If $f : D \to \mathbb{R}$ has continuous partial derivatives, then $\partial f/\partial x_1 \geq (\leq) \partial f/\partial x_2$ holds in D if and only if $f(a,b) \geq (\leq)\ f(a-l, b+l)$ holds for all $(a,b) \in D$ and $l > 0$ with $b < b+l \leq a-l < a$.*

Proof. We only prove the case of $\partial f/\partial x_1 \geq \partial f/\partial x_2$.

Necessity. For all $(x_1, x_2) \in D$ and $l \in \mathbb{R}_+$ with $m \leq x_2 < x_2 + l \leq x_1 - l < x_1 \leq M$, by the assumption we have $f(x_1 - l, x_2 + l) - f(x_1, x_2) \leq 0$. Then from the Langrange's mean value theorem we know that there exists $\xi_l \in (0,l)$ such that

$$l\left(-\frac{\partial f(x_1-\xi_l,x_2+\xi_l)}{\partial x_1}+\frac{\partial f(x_1-\xi_l,x_2+\xi_l)}{\partial x_2}\right)\le 0,$$
$$-\frac{\partial f(x_1-\xi_l,x_2+\xi_l)}{\partial x_1}+\frac{\partial f(x_1-\xi_l,x_2+\xi_l)}{\partial x_2}\le 0. \tag{1.1}$$

Letting $l \to 0+$, we get

$$\frac{\partial f(x_1,x_2)}{\partial x_1}\ge\frac{\partial f(x_1,x_2)}{\partial x_2}. \tag{1.2}$$

According to the continuity of partial derivatives, we know that

$$\frac{\partial f(x_1,x_1)}{\partial x_1}\ge\frac{\partial f(x_1,x_1)}{\partial x_2} \tag{1.3}$$

holds also.

Sufficiency. For all $(a,b)\in D$ and $l>0$ with $b<b+l\le a-l<a$, from the assumption and the Langrange's mean value theorem we know that there exists $\xi_l\in(0,l)$ such that

$$\begin{aligned}f(a,b)-f(a-l,b+l)&=-\big(f(a-l,b+l)-f(a,b)\big)\\&=-l\left(-\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_1}+\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_2}\right)\\&=l\left(\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_1}-\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_2}\right)\\&\ge 0.\end{aligned} \tag{1.4}$$

Therefore the proof of Lemma 1.1 is completed. □

Theorem 1.2. *Suppose that $D\subset\mathbb{R}^n$ is a symmetric convex set with nonempty interior, $f: D\to\mathbb{R}$ has continuous partial derivatives, and*

$$\breve{D}_i=\left\{\mathbf{x}\in D\mid x_i=\max_{1\le j\le n}\{x_j\}\right\}-\{\mathbf{x}\in D\mid x_1=x_2=\cdots=x_n\},$$
$$\widehat{D}_i=\left\{\mathbf{x}\in D\mid x_i=\min_{1\le j\le n}\{x_j\}\right\}-\{x\in D\mid x_1=x_2=\cdots=x_n\}, \tag{1.5}$$

$i=1,2,\ldots,n$. If for all $i,j=1,2,\ldots,n$ with $i\ne j$,

$$\frac{\partial f}{\partial x_i}>(<)\frac{\partial f}{\partial x_j} \tag{1.6}$$

holds in $\breve{D}_i \cap \widehat{D}_j$*, then*

$$f(a_1, a_2, \ldots, a_n) \geq (\leq) f(A(\mathbf{a}), A(\mathbf{a}), \ldots, A(\mathbf{a})) \tag{1.7}$$

for all $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in D$*, with equality if only if* $a_1 = a_2 = \cdots = a_n$.

Proof. If $n = 2$, then Theorem 1.2 follows from Lemma 1.1 and $l = |a_1 - a_2|/2$. We assume that $n \geq 3$ in the next discussion. Without loss of generality, we only prove the case of $\partial f/\partial x_i > \partial f/\partial x_j$ with $i \neq j$.

If $a_1 = a_2 = \cdots = a_n$, then inequality (1.7) is clearly true. If $\max_{1\leq j\leq n}\{a_j\} \neq \min_{1\leq j\leq n}\{a_j\}$, then without loss of generality we assume that $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$.

(1) If $a_1 > \max_{2\leq j\leq n}\{a_j\}$ and $a_n < \min_{1\leq j\leq n-1}\{a_j\}$, then $(a_1, a_2, \ldots, a_n) \in \breve{D}_1 \cap \widehat{D}_n$. From Lemma 1.1 and the conditions in Theorem 1.2 we know that there exist $a_1^{(1)}$ and $a_n^{(1)}$ such that $l = a_1 - a_1^{(1)} = a_n^{(1)} - a_n > 0$, $a_1^{(1)} = a_2$ or $a_n^{(1)} = a_{n-1}$, and

$$f(a_1, a_2, a_3, \ldots, a_n) \geq f\left(a_1^{(1)}, a_2, a_3, \ldots, a_n^{(1)}\right). \tag{1.8}$$

For the sake of convenience, we denote $a_i^{(1)} = a_i,\ 2 \leq i \leq n-1$. Consequently,

$$f(a_1, a_2, a_3, \ldots, a_n) \geq f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_n^{(1)}\right). \tag{1.9}$$

If $a_1^{(1)} = a_2^{(1)} = \cdots = a_n^{(1)}$, then Theorem 1.2 holds. Otherwise, for the case of $a_1^{(1)} = a_2^{(1)} > a_n^{(1)}$, $(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_n^{(1)}) \in \breve{D}_1 \cap \widehat{D}_n$ and

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=(a_1^{(1)},a_2^{(1)},a_3^{(1)},\ldots,a_n^{(1)})} > \left.\frac{\partial f(\mathbf{x})}{\partial x_n}\right|_{\mathbf{x}=(a_1^{(1)},a_2^{(1)},a_3^{(1)},\ldots,a_n^{(1)})}. \tag{1.10}$$

From the continuity of partial derivatives we know that there exists $\varepsilon > 0$ such that

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=(s,a_2^{(1)},a_3^{(1)},\ldots,t)} > \left.\frac{\partial f(\mathbf{x})}{\partial x_n}\right|_{\mathbf{x}=(s,a_2^{(1)},a_3^{(1)},\ldots,t)}, \tag{1.11}$$

where $s \in [a_1^{(1)} - \varepsilon, a_1^{(1)}]$ and $t \in [a_n^{(1)}, a_n^{(1)} + \varepsilon]$. Denote $a_1^{(2)} = a_1^{(1)} - \varepsilon$, $a_n^{(2)} = a_n^{(1)} + \varepsilon$, $a_i^{(2)} = a_i^{(1)} (2 \leq i \leq n-1)$. By Lemma 1.1, we get

$$f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_n^{(1)}\right) \geq f\left(a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots, a_n^{(2)}\right), \tag{1.12}$$

and $a_2^{(2)} = \max_{1\leq i\leq n}\{a_i^{(2)}\}$. For the case of $a_1^{(1)} > a_{n-1}^{(1)} = a_n^{(1)}$, after a similar argument, we get inequality (1.12) with $a_{n-1}^{(2)} = \min_{1\leq i\leq n}\{a_i^{(2)}\}$.

Repeating the above steps, we get $\{a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}\}$ $(i = 1, 2, \ldots)$ such that $\sum_{j=1}^{n} a_j^{(i)}$ is a constant and $\{a_j^{(i)}\}$ $(i = 1, 2, \ldots)$ are monotone increasing (decreasing) sequences if $a_j \geq (\leq)\ A(\mathbf{a}), j = 1, 2, 3, \ldots, n$, and

$$f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_n^{(1)}\right) \geq f\left(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, \ldots, a_n^{(i)}\right). \tag{1.13}$$

If there exists $i \in \mathbb{N}$ such that $a_1^{(i)} = a_2^{(i)} = \cdots = a_n^{(i)}$, then the proof of Theorem 1.2 is completed. Otherwise, we denote $\alpha = \inf_{i \in \mathbb{N}}\{\max\{a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}\}\}$; without loss of generality, we assume that

$$\begin{gathered} \max\left\{a_1^{(i_j)}, a_2^{(i_j)}, \ldots, a_n^{(i_j)}\right\} = a_1^{(i_j)} \longrightarrow \alpha, \\ \lim_{j \to +\infty}\left(a_1^{(i_j)}, a_2^{(i_j)}, \ldots, a_n^{(i_j)}\right) = (\alpha, b_2, b_3, \ldots, b_n), \end{gathered} \tag{1.14}$$

where $\{i_j\}_{j=1}^{+\infty}$ is a subsequence of $\mathbb{N}$. Then from the continuity of function f, we get

$$f(a_1, a_2, a_3, \ldots, a_n) \geq f(\alpha, b_2, b_3, \ldots, b_n). \tag{1.15}$$

If $\alpha \neq \min\{b_2, b_3, \ldots, b_n\}$, then we repeat the above arguments and get a contradiction to the definition of α. Hence $\alpha = b_2 = b_3 = \cdots = b_n$. From $\alpha + \sum_{i=2}^{n} b_i = \sum_{i=1}^{n} a_i$ we get $\alpha = b_2 = b_3 = \cdots = b_n = A(\mathbf{a})$; the proof of Theorem 1.2 is completed.

(2) The proof for the case of $a_1 = \max_{2 \leq j \leq n}\{a_j\}$ or $a_n = \min_{1 \leq j \leq n-1}\{a_j\}$ is implied in the proof of (1). □

In particular, according to Theorem 1.2 the following corollary holds.

Corollary 1.3. *Suppose that $D \subset \mathbb{R}^n$ is a symmetric convex set with nonempty interior, $f : D \to \mathbb{R}$ is a symmetric function with continuous partial derivatives, and*

$$\begin{gathered} \breve{D}_1 = \left\{\mathbf{x} \in D \mid x_1 = \max_{1 \leq j \leq n}\{x_j\}\right\} - \{\mathbf{x} \in D \mid x_1 = x_2 = \cdots = x_n\}, \\ \widehat{D}_2 = \left\{\mathbf{x} \in D \mid x_2 = \min_{1 \leq j \leq n}\{x_j\}\right\} - \{\mathbf{x} \in D \mid x_1 = x_2 = \cdots = x_n\}, \\ D^* = \breve{D}_1 \cap \widehat{D}_2. \end{gathered} \tag{1.16}$$

If $\partial f / \partial x_1 > (<) \partial f / \partial x_2$ holds in D^*, then

$$f(a_1, a_2, \ldots, a_n) \geq (\leq) f(A(\mathbf{a}), A(\mathbf{a}), \ldots, A(\mathbf{a})) \tag{1.17}$$

for all $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in D$, and equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

2. Comparing with Schur's Condition

The Schur convexity was introduced by I. Schur [2] in 1923; the following Definitions 2.1 and 2.2 can be found in [2, 3].

Definition 2.1. For $\mathbf{u} = (u_1, u_2, \ldots u_n), \mathbf{v} = (v_1, v_2, \ldots v_n) \in \mathbb{R}^n$, without loss of generality one assumes that $u_1 \geq u_2 \geq \cdots \geq u_n$ and $v_1 \geq v_2 \geq \cdots \geq v_n$. Then $\mathbf{u}$ is said to be majorized by $\mathbf{v}$ (in symbols $\mathbf{u} \prec \mathbf{v}$) if $\sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} v_i$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$.

Definition 2.2. Suppose that $\Omega \subset \mathbb{R}^n$. A real-valued function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur convex (Schur concave) if $\mathbf{u} \prec \mathbf{v}$ implies that $\varphi(\mathbf{u}) \leq (\geq)\, \varphi(\mathbf{v})$.

Recall that the following so-called Schur's condition is very useful for determining whether or not a given function is Schur convex or concave.

Theorem 2.3 (see [2, page 57]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a symmetric convex set with nonempty interior* $\operatorname{int}\Omega$*. If $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in* $\operatorname{int}\Omega$*, then φ is Schur convex (Schur concave) on Ω if and only if it is symmetric and*

$$(u_1 - u_2)\left(\frac{\partial \varphi}{\partial u_1} - \frac{\partial \varphi}{\partial u_2}\right) \geq (\leq)\, 0 \tag{2.1}$$

holds for any $u = (u_1, u_2, \ldots, u_n) \in \operatorname{int}\Omega$.

It is well known that a convex function is not necessarily a Schur convex function, and a Schur convex function need not be convex in the ordinary sense either. However, under the assumption of ordinary convexity, f is Schur convex if and only if it is symmetric [4].

Although the Schur convexity is an important tool in researching analytic inequalities, but the restriction of symmetry cannot be used in dealing with nonsymmetric functions. Obviously, Theorem 1.2 is the generalization and development of Theorem 2.3; the following results in Sections 3–5 show that a large number of inequalities can be proved, improved, and found by Theorem 1.2.

3. A Proof for the Hölder Inequality

Using Theorem 1.2 and Corollary 1.3, we can prove some well-known inequalities, for example, power mean inequality, Hölder inequality, and Minkowski inequality. In this section, we only prove the Hölder inequality.

Proposition 3.1 (Hölder inequality). *Suppose that*

$$(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n. \tag{3.1}$$

If $p, q > 1$ and $1/p + 1/q = 1$, then

$$\left(\sum_{k=1}^{n} x_k^p\right)^{1/p} \left(\sum_{k=1}^{n} y_k^q\right)^{1/q} \geq \sum_{k=1}^{n} x_k y_k. \tag{3.2}$$

Proof. Let $(a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$ and

$$f : \mathbf{b} \longrightarrow \left(\sum_{k=1}^{n} a_k\right)^{1/p} \left(\sum_{k=1}^{n} a_k b_k\right)^{1/q} - \sum_{k=1}^{n} a_k b_k^{1/q}, \quad b \in \mathbb{R}_+^n. \tag{3.3}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial b_i} &= \frac{1}{q} \cdot \left(\sum_{k=1}^{n} a_k\right)^{1/p} \left(\sum_{k=1}^{n} a_k b_k\right)^{1/q-1} a_i - \frac{1}{q} \cdot a_i b_i^{1/q-1}, \\ \frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} &= \frac{1}{q} \left(\frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right). \end{aligned} \tag{3.4}$$

Let $\mathbf{b} \in \breve{D}_i \cap \widehat{D}_j$ (see (1.5)).

(1) If $a_i \geq a_j$, then

$$\begin{aligned} \frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} &\geq \frac{1}{q} \left(\frac{\sum_{k=1}^{n} a_k}{b_i \sum_{k=1}^{n} a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right) \\ &= \frac{1}{q} a_j \left(b_j^{-1/p} - b_i^{-1/p}\right) \\ &> 0. \end{aligned} \tag{3.5}$$

(2) If $a_i \leq a_j$, then

$$\begin{aligned} \frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} &\geq \frac{1}{q} \left(\frac{\sum_{k=1}^{n} a_k}{b_j \sum_{k=1}^{n} a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right) \\ &= \frac{1}{q} a_i \left(b_j^{-1/p} - b_i^{-1/p}\right) \\ &> 0. \end{aligned} \tag{3.6}$$

From Theorem 1.2 we get

$$f(\mathbf{b}) \geq f(A(\mathbf{b}), A(\mathbf{b}), \ldots, A(\mathbf{b})), \tag{3.7}$$

that is,

$$\left(\sum_{k=1}^{n} a_k\right)^{1/p} \left(\sum_{k=1}^{n} a_k b_k\right)^{1/q} - \sum_{k=1}^{n} a_k b_k^{1/q} \geq 0. \tag{3.8}$$

Therefore, the Hölder inequality follows from (3.8) with $a_k = x_k^p$ and $b_k = y_k^q / x_k^p$. □

4. Improvement of the Sierpiński Inequality

In the section, we give some improvements of the well-known Sierpiński inequality:

$$[M_{-1}(\mathbf{a})]^{(n-1)/n}[A(\mathbf{a})]^{1/n} \leq M_0(\mathbf{a}) \leq [M_{-1}(\mathbf{a})]^{1/n}[A(\mathbf{a})]^{(n-1)/n}. \tag{4.1}$$

Theorem 4.1. *Suppose that $n \geq 3$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, $\beta > 0 > \alpha$. If $\lambda = -2\alpha/n(\beta - \alpha)$ for $\beta + \alpha > 0$ and $\lambda = 1/n$ for $\beta + \alpha \leq 0$, then*

$$[M_\alpha(\mathbf{a})]^{1-\lambda} \cdot \left[M_\beta(\mathbf{a})\right]^\lambda \leq M_0(\mathbf{a}). \tag{4.2}$$

Proof. Let $f(\mathbf{x}) = (1/n\beta)\ln(\prod_{i=1}^n x_i) - ((1-\lambda)/\alpha)\ln((1/n)\sum_{i=1}^n x_i^{\alpha/\beta})$, $\mathbf{x} \in \mathbb{R}_+^n$. Then

$$\begin{gathered}
\frac{\partial f(\mathbf{x})}{\partial x_j} = \frac{1}{n\beta x_j} - \frac{1-\lambda}{\beta}\frac{x_j^{\alpha/\beta-1}}{\sum_{i=1}^n x_i^{\alpha/\beta}}, \quad j = 1, 2, \\
\frac{\partial f(\mathbf{x})}{\partial x_1} - \frac{\partial f(\mathbf{x})}{\partial x_2} = \frac{x_2 - x_1}{n\beta x_1 x_2} - \frac{1-\lambda}{\beta}\frac{x_1^{\alpha/\beta-1} - x_2^{\alpha/\beta-1}}{\sum_{i=1}^n x_i^{\alpha/\beta}}.
\end{gathered} \tag{4.3}$$

Case 1. $\alpha + \beta > 0$. Let

$$g(t) = \frac{\beta+\alpha}{\beta-\alpha}t^{\beta-\alpha} - t^\beta + t^{-\alpha} - \frac{\beta+\alpha}{\beta-\alpha}, \quad t \in (1, +\infty). \tag{4.4}$$

Then

$$\begin{gathered}
t^{\alpha+1}g'(t) = (\beta+\alpha)t^\beta - \beta t^{\beta+\alpha} - \alpha, \\
\left[t^{\alpha+1}g'(t)\right]' = (\beta+\alpha)\beta t^{\beta+\alpha-1}(t^{-\alpha} - 1) > 0.
\end{gathered} \tag{4.5}$$

Therefore, $t^{\alpha+1}g'(t)$ is monotone increasing in $(1, +\infty)$. From

$$\lim_{t\to 1+} t^{\alpha+1}g'(t) = \lim_{t\to 1+}\left[(\beta+\alpha)t^\beta - \beta t^{\beta+\alpha} - \alpha\right] = 0, \tag{4.6}$$

we know that $t^{\alpha+1}g'(t) > 0$, $g'(t) > 0$. Then $\lim_{t\to 1+} g(t) = 0$ leads to $g(t) > 0$ and

$$\begin{gathered}
\frac{\beta+\alpha}{\beta-\alpha}t^{\beta-\alpha} - t^{\beta} + t^{-\alpha} - \frac{\beta+\alpha}{\beta-\alpha} > 0, \\
\frac{\beta+\alpha}{\beta-\alpha}t^{\beta} - \left(1 + \frac{2\alpha}{\beta-\alpha}\right)t^{\alpha} - t^{\alpha+\beta} + 1 > 0, \\
\frac{\beta+\alpha}{\beta-\alpha}t^{\beta} - \left(n - 1 + \frac{2\alpha}{\beta-\alpha}\right)t^{\alpha} - t^{\alpha+\beta} + (n-1) > 0, \\
(1-n\lambda)t^{\beta} - (n-1-n\lambda)t^{\alpha} - t^{\alpha+\beta} + (n-1) > 0,
\end{gathered} \tag{4.7}$$

$$(1-\lambda)\frac{1-t^{\alpha-\beta}}{t^{\alpha}+(n-1)} > \frac{t^{\beta}-1}{nt^{\beta}}. \tag{4.8}$$

We assume that $\mathbf{x} \in D^*$ (see (1.16)). Let $t = (x_1/x_2)^{1/\beta}$. Then inequality (4.8) becomes

$$\begin{gathered}
(1-\lambda)\frac{x_2^{\alpha/\beta-1} - x_1^{\alpha/\beta-1}}{x_1^{\alpha/\beta} + (n-1)x_2^{\alpha/\beta}} > \frac{x_1 - x_2}{nx_1x_2}, \\
\frac{1-\lambda}{\beta}\frac{x_2^{\alpha/\beta-1} - x_1^{\alpha/\beta-1}}{\sum_{i=1}^{n} x_i^{\alpha/\beta}} > \frac{x_1 - x_2}{n\beta x_1x_2}.
\end{gathered} \tag{4.9}$$

Combining inequalities (4.3) and (4.9) yields that $\partial f(\mathbf{x})/\partial x_1 - \partial f(\mathbf{x})/\partial x_2 > 0$. Using Corollary 1.3 we have

$$\begin{gathered}
f(x_1, x_2, \ldots, x_n) \geq f(A(\mathbf{x}), A(\mathbf{x}), \ldots, A(\mathbf{x})), \\
\frac{1}{n\beta}\ln\left(\prod_{i=1}^{n} x_i\right) - \frac{1-\lambda}{\alpha}\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha/\beta}\right) \geq \frac{\lambda}{\beta}\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).
\end{gathered} \tag{4.10}$$

Letting $a_i = x_i^{1/\beta}$, $i = 1, 2, \ldots, n$, we get

$$[M_{\alpha}(\mathbf{a})]^{1-\lambda} \cdot \left[M_{\beta}(\mathbf{a})\right]^{\lambda} \leq M_0(\mathbf{a}). \tag{4.11}$$

Case 2. $\alpha + \beta < 0$. Let $t > 1$. Then from $\alpha < 0$ and $\alpha + \beta < 0$, one has

$$(n-1) > (n-2)t^{\alpha} + t^{\alpha+\beta}. \tag{4.12}$$

Hence inequality (4.8) holds. The rest is similar to above, so we omit it.

The proof of Theorem 4.2 is similar to the proof of Theorem 4.1, and so we omit it. □

Theorem 4.2. *Suppose that $n \geq 3$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, $\beta > 0 > \alpha$. If $\theta = (n-1)/n$ for $\beta + \alpha > 0$ and $\theta = 1 - 2\beta/n(\beta - \alpha)$ for $\beta + \alpha \leq 0$, then*

$$M_0(\mathbf{a}) \leq [M_\alpha(\mathbf{a})]^{1-\theta} \cdot \left[M_\beta(\mathbf{a})\right]^\theta. \tag{4.13}$$

Theorem 4.3. *Suppose that $n \geq 3$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$. If $r = -\ln n/(n-1)[\ln n - \ln(n-1)]$, then $r < -1$ and*

$$[M_{1/r}(\mathbf{a})]^{(n-1)/n}[A(\mathbf{a})]^{1/n} \leq M_0(\mathbf{a}) \leq [M_r(\mathbf{a})]^{1/n}[A(\mathbf{a})]^{(n-1)/n}. \tag{4.14}$$

Proof. Let $n \geq 3$ and

$$f : \mathbf{x} \in (0, +\infty)^n \longrightarrow \frac{\sum_{k=1}^n x_k^{1/r}}{n}\left(\frac{\sum_{k=1}^n x_k}{n}\right)^{1/r(n-1)} - \left(\prod_{k=1}^n x_k\right)^{1/r(n-1)}. \tag{4.15}$$

Then

$$\begin{aligned}
r &= -\frac{\ln n}{\ln(1 + 1/(n-1))^{n-1}} < -\frac{\ln n}{\ln e} = -\ln n < -1, \\
\frac{\partial f}{\partial x_1} &= \frac{x_1^{(1-r)/r}}{rn}\left(\frac{\sum_{k=1}^n x_k}{n}\right)^{1/r(n-1)} + \frac{\sum_{k=1}^n x_k^{1/r}}{rn^2(n-1)}\left(\frac{\sum_{k=1}^n x_k}{n}\right)^{(1/r(n-1))-1} \\
&\quad - \frac{1}{r(n-1)x_1}\left(\prod_{k=1}^n x_k\right)^{1/r(n-1)}.
\end{aligned} \tag{4.16}$$

Therefore, we get

$$\begin{aligned}
\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} &= \frac{1}{rn}\left(x_1^{(1-r)/r} - x_2^{(1-r)/r}\right)\left(\frac{\sum_{k=1}^n x_k}{n}\right)^{1/r(n-1)} \\
&\quad - \frac{1}{r(n-1)}\left(\prod_{k=1}^n x_k\right)^{1/r(n-1)}\left(\frac{1}{x_1} - \frac{1}{x_2}\right),
\end{aligned}$$

$$\left(\prod_{k=1}^{n}x_k\right)^{1/(-r(n-1))}\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_2}\right)=\frac{x_1^{(1-r)/(-r)}-x_2^{(1-r)/(-r)}}{-rnx_1^{(1-r)/(-r)}x_2^{(1-r)/(-r)}}\left(\frac{n\prod_{k=1}^{n}x_k}{\sum_{k=1}^{n}x_k}\right)^{1/(-r)(n-1)}+\frac{x_1-x_2}{r(n-1)x_1x_2}$$
$$=\frac{x_1^{(1-r)/(-r)}-x_2^{(1-r)/(-r)}}{-rnx_1^{(1-r)/(-r)}x_2^{(1-r)/(-r)}}\left(\frac{n}{\sum_{k=1}^{n}\prod_{i=1,\neq k}^{n}x_i^{-1}}\right)^{1/(-r)(n-1)}+\frac{x_1-x_2}{r(n-1)x_1x_2}. \tag{4.17}$$

We assume that $\mathbf{x}\in D^*$ (see (1.16)). Then we have

$$\left(\prod_{k=1}^{n}x_k\right)^{1/-r(n-1)}\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_2}\right)$$
$$\geq\frac{x_1^{(1-r)/(-r)}-x_2^{(1-r)/(-r)}}{-rnx_1^{(1-r)/(-r)}x_2^{(1-r)/(-r)}}\left(\frac{n}{x_2^{-(n-1)}+(n-1)x_1^{-1}x_2^{-(n-2)}}\right)^{1/(-r)(n-1)}+\frac{x_1-x_2}{r(n-1)x_1x_2}. \tag{4.18}$$

Letting $x_1/x_2=t>1$, from $n^{1+1/r(n-1)}=n^{1-(\ln n-\ln(n-1))/\ln n}=n-1$, we get

$$\left(\prod_{k=1}^{n}x_k\right)^{1/-r(n-1)}\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_2}\right)$$
$$\geq\frac{1}{n-1}\cdot\frac{x_1^{(1-r)/(-r)}-x_2^{(1-r)/(-r)}}{-rx_1^{(1-r)/(-r)}x_2^{(1-r)/(-r)}}\left(\frac{x_1x_2^{n-1}}{x_1+(n-1)x_2}\right)^{(\ln n-\ln(n-1))/\ln n}+\frac{x_1-x_2}{r(n-1)x_1x_2}$$
$$=\frac{1}{-r(n-1)x_2}\left[\frac{t^{(1-r)/(-r)}-1}{t^{(1-r)/(-r)}}\left(\frac{t}{t+n-1}\right)^{(\ln n-\ln(n-1))/\ln n}-\frac{t-1}{t}\right]$$
$$=\frac{1}{-r(n-1)x_2}\left[\frac{t^{(1-r)/(-r)}-1}{t^{(1-r)/(-r)}}\cdot\left(1+\frac{n-1}{t}\right)^{-(\ln n-\ln(n-1))/\ln n}-\frac{t-1}{t}\right]. \tag{4.19}$$

According to Bernoulli's inequality $(1+x)^{\alpha} < 1+\alpha x$ with $x \ge -1$, $x \ne 0$, and $0<\alpha<1$, one has

$$\begin{aligned}
&\left(\prod_{k=1}^{n} x_k\right)^{1/-r(n-1)} \left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \\
&\quad > \frac{1}{-r(n-1)x_2}\left[\frac{t^{(1-r)/(-r)}-1}{t^{(1-r)/(-r)}} \cdot \frac{1}{1+(\ln n - \ln(n-1))/\ln n \cdot (n-1)/t} - \frac{t-1}{t}\right] \\
&\quad = \frac{1}{-r(n-1)x_2}\left[\frac{t^{(1-r)/(-r)}-1}{t^{(1-r)/(-r)} - t^{1/(-r)}/r} - \frac{t-1}{t}\right] \\
&\quad = \frac{1}{-r(n-1)x_2} \cdot \frac{(1+1/r)t^{1/(-r)} - 1/r \cdot t^{(1+r)/(-r)} - 1}{t^{(1-r)/(-r)} - t^{1/(-r)}/r}.
\end{aligned} \tag{4.20}$$

For $0<s<1$ and $t>1$, it is not difficult to verify that $(1-s)t^{s} + st^{s-1} - 1 > 0$. Letting $s=-1/r$, we have

$$\begin{gathered}
\left(1+\frac{1}{r}\right)t^{1/(-r)} - \frac{1}{r} \cdot t^{(1+r)/(-r)} - 1 > 0, \\
\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} > 0.
\end{gathered} \tag{4.21}$$

Using Corollary 1.3, we know that

$$\begin{gathered}
f(x_1, x_2, \ldots, x_n) \ge f(A(\mathbf{x}), A(\mathbf{x}), \ldots, A(\mathbf{x})), \\
\frac{\sum_{k=1}^{n} x_k^{1/r}}{n}\left(\frac{\sum_{k=1}^{n} x_k}{n}\right)^{1/r(n-1)} \ge \left(\prod_{k=1}^{n} x_k\right)^{1/r(n-1)}.
\end{gathered} \tag{4.22}$$

Letting $a_i = x_i^{1/r}$ $(i=1,2,\ldots,n)$, we get

$$\frac{\sum_{k=1}^{n} a_k}{n}\left(\frac{\sum_{k=1}^{n} a_k^{r}}{n}\right)^{1/r(n-1)} \ge \left(\prod_{k=1}^{n} a_k\right)^{1/(n-1)}, \tag{4.23}$$

$$[A(\mathbf{a})]^{(n-1)/n}[M_r(\mathbf{a})]^{1/n} \ge M_0(\mathbf{a}). \tag{4.24}$$

From (4.23), we get

$$\left(\prod_{k=1}^{n} a_k\right)^{r} \ge \left(\frac{\sum_{k=1}^{n} a_k}{n}\right)^{(n-1)r} \cdot \frac{\sum_{k=1}^{n} a_k^{r}}{n}. \tag{4.25}$$

Letting $a_i \to a_i^{1/r}$ $(i = 1, 2, \ldots, n)$, we have

$$\prod_{k=1}^{n} a_k \geq \left(\frac{\sum_{k=1}^{n} a_k^{1/r}}{n} \right)^{(n-1)r} \cdot \frac{\sum_{k=1}^{n} a_k}{n},$$
$$M_0(\mathbf{a}) \geq [M_{1/r}(\mathbf{a})]^{(n-1)/n} [A(\mathbf{a})]^{1/n}. \tag{4.26}$$

Inequality (4.14) is proved. □

5. Five New Inequalities

Let $n \geq 3$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$. Then

$$\prod_{n}^{k}(\mathbf{a}) = \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k} \sum_{j=1}^{k} a_{i_j} \right)^{1/\binom{n}{k}} \tag{5.1}$$

References [5, 6] is the third symmetric mean of $\mathbf{a}$.

Theorem 5.1. *If* $2 \leq k \leq n-1$, $p = (k-1)/(n-1)$, *then*

$$\prod_{n}^{k}(\mathbf{a}) \geq [A(\mathbf{a})]^{p} [M_0(\mathbf{a})]^{1-p} \tag{5.2}$$

with the best possible constant $p = (k-1)/(n-1)$.

Proof. Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$ and

$$f(\mathbf{a}) = \left[\prod_{i=1}^{n} a_i \right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)} \cdot \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k} \sum_{j=1}^{k} a_{i_j}. \tag{5.3}$$

Then

$$\frac{\partial f}{\partial a_1} = -\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1} \left[\prod_{i=1}^{n} a_i \right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)} \cdot \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k} \sum_{j=1}^{k} a_{i_j}$$
$$+ \left[\prod_{i=1}^{n} a_i \right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)} \cdot \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k} \sum_{j=1}^{k} a_{i_j} \cdot \left(\sum_{2 \leq i_1 < \cdots < i_{k-1} \leq n} \frac{1}{a_1 + \sum_{i=1}^{k-1} a_{i_j}} \right), \tag{5.4}$$

$$\begin{aligned}\frac{\partial f}{\partial a_1}-\frac{\partial f}{\partial a_2}=&-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}\left[\prod_{i=1}^{n}a_i\right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)}\cdot\prod_{1\le i_1<\cdots<i_k\le n}\frac{1}{k}\sum_{j=1}^{k}a_{i_j}\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\\&+\left[\prod_{i=1}^{n}a_i\right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)}\cdot\prod_{1\le i_1<\cdots<i_k\le n}\frac{1}{k}\sum_{j=1}^{k}a_{i_j}\\&\cdot\left(\sum_{3\le i_1<\cdots i_{k-1}\le n}\left(\frac{1}{a_1+\sum_{i=1}^{k-1}a_{i_j}}-\frac{1}{a_2+\sum_{i=1}^{k-1}a_{i_j}}\right)\right)\\=&\,(a_1-a_2)\cdot\left[\prod_{i=1}^{n}a_i\right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)}\cdot\prod_{1\le i_1<\cdots<i_k\le n}\frac{1}{k}\sum_{j=1}^{k}a_{i_j}\\&\cdot\left[\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2}-\sum_{3\le i_1<\cdots i_{k-1}\le n}\frac{1}{\left(a_1+\sum_{i=1}^{k-1}a_{i_j}\right)\left(a_2+\sum_{i=1}^{k-1}a_{i_j}\right)}\right].\end{aligned}\tag{5.5}$$

If $\mathbf{a}\in D^*$ (see (1.16)), then

$$\begin{gathered}a_1+(k-1)a_2>a_1,\\\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2}>\frac{\binom{n-2}{k-1}}{ka_2(a_1+(k-1)a_2)},\\\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2}>\sum_{3\le i_1<\cdots i_{k-1}\le n}\frac{1}{(a_1+(k-1)a_2)ka_2},\\\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2}>\sum_{3\le i_1<\cdots i_{k-1}\le n}\frac{1}{\left(a_1+\sum_{i=1}^{k-1}a_{i_j}\right)\left(a_2+\sum_{i=1}^{k-1}a_{i_j}\right)}.\end{gathered}\tag{5.6}$$

Combining inequalities (5.5) and (5.6) yields that $\partial f/\partial a_1-\partial f/\partial a_2>0$. Then from Corollary 1.3 we have

$$f(a_1,a_2,\ldots,a_n)\ge f(A(\mathbf{a}),A(\mathbf{a}),\ldots,A(\mathbf{a}))\tag{5.7}$$

for all $\mathbf{a}=(a_1,a_2,\ldots,a_n)\in\mathbb{R}_+^n$, which implies that

$$\left[\prod_{i=1}^{n}a_i\right]^{-(n-k)\cdot\binom{n}{k}/n(n-1)}\cdot\prod_{1\le i_1<\cdots<i_k\le n}k^{-1}\sum_{j=1}^{k}a_{i_j}\ge[A(\mathbf{a})]^{(k-1)\cdot\binom{n}{k}/(n-1)}.\tag{5.8}$$

Therefore, inequality (5.2) is proved. □

Taking $a_1 = a_2 = \cdots = a_{n-1} = 1$ and $a_n = x$ in inequality (5.2), we get

$$\left(\frac{x+k-1}{k}\right)^{\binom{n-1}{k-1}/\binom{n}{k}} \geq \left(\frac{x+n-1}{n}\right)^{p} \left(\sqrt[n]{x}\right)^{1-p},$$
$$p \leq \frac{(k/n)\ln((x+k-1)/k) - (1/n)\ln x}{\ln(x+n-1) - \ln\left(n\sqrt[n]{x}\right)}. \tag{5.9}$$

Letting $x \to +\infty$, we get

$$p \leq \lim_{x\to+\infty} \frac{k/n \cdot 1/(x+k-1) - 1/nx}{1/(x+n-1) - 1/nx} = \lim_{x\to+\infty} \frac{kx/(x+k-1) - 1}{nx/(x+n-1) - 1} = \frac{k-1}{n-1}. \tag{5.10}$$

So $p = (k-1)/(n-1)$ is the best possible constant.

For $n \geq 2$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$, Alzer [7] established the following inequality:

$$\frac{n-1}{n}A(\mathbf{a}) + \frac{1}{n}M_{-1}(\mathbf{a}) \geq M_0(\mathbf{a}). \tag{5.11}$$

Theorems 5.2 and 5.3 are the improvements of Alzer's inequality.

Theorem 5.2. *If $p = n^2/(n^2+4n-4)$, then*

$$pA(\mathbf{a}) + (1-p)M_{-1}(\mathbf{a}) \geq M_0(\mathbf{a}). \tag{5.12}$$

Proof. Firstly, let $p > n^2/(n^2+4n-4)$, and

$$f(\mathbf{x}) = p/n \cdot \sum_{i=1}^{n} e^{x_i} + (1-p)n \cdot \left(\sum_{i=1}^{n} e^{-x_i}\right)^{-1}, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \tag{5.13}$$

Then

$$\frac{\partial f}{\partial x_1} = \frac{p}{n}e^{x_1} + (1-p)\frac{n}{\left(\sum_{i=1}^{n} e^{-x_i}\right)^2}e^{-x_1},$$
$$\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} = \frac{p}{n}\left(e^{x_1} - e^{x_2}\right) - (1-p)\frac{n}{\left(\sum_{i=1}^{n} e^{-x_i}\right)^2}\left(e^{-x_2} - e^{-x_1}\right). \tag{5.14}$$

If $x_1 = \max_{1\le i\le n}\{x_i\} > x_2 = \min_{1\le i\le n}\{x_i\}$, $t = e^{x_1-x_2} > 1$, then

$$\begin{aligned}
\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} &\ge \frac{p}{n}\left(e^{x_1} - e^{x_2}\right) - (1-p)\frac{n}{\left((n-1)e^{-x_1} + e^{-x_2}\right)^2}\left(e^{-x_2} - e^{-x_1}\right) \\
&= \frac{\left(e^{x_1} - e^{x_2}\right)}{n\left((n-1)e^{x_2} + e^{x_1}\right)^2}\left[p\left((n-1)e^{x_2} + e^{x_1}\right)^2 - (1-p)n^2 e^{x_1} e^{x_2}\right] \\
&= \frac{e^{3x_2}(t-1)}{n\left((n-1)e^{x_2} + e^{x_1}\right)^2}\left[p(n-1+t)^2 - n^2 t + pn^2 t\right] \\
&> \frac{e^{3x_2}(t-1)}{n\left((n-1)e^{x_2} + e^{x_1}\right)^2}\left[\frac{n^2}{n^2+4n-4}(n-1+t)^2 - n^2 t + \frac{n^2}{n^2+4n-4}n^2 t\right] \\
&= \frac{n e^{3x_2}(t-1)(t-n+1)^2}{\left(n^2+4n-4\right)\left((n-1)e^{x_2} + e^{x_1}\right)^2} \ge 0.
\end{aligned} \tag{5.15}$$

Then from Corollary 1.3, we get

$$\begin{gathered}
f(\mathbf{x}) \ge f(A(\mathbf{x}), A(\mathbf{x}), \ldots, A(\mathbf{x})), \\
\frac{p}{n}\sum_{i=1}^{n} e^{x_i} + (1-p)\frac{n}{\sum_{i=1}^{n} e^{-x_i}} \ge e^{A(\mathbf{x})} = \sqrt[n]{\prod_{i=1}^{n} e^{x_i}}.
\end{gathered} \tag{5.16}$$

Let $e^{x_i} = a_i$ in above inequality. Then we know that inequality (5.12) holds. From continuity we know that inequality (5.12) holds also for $p = n^2/(n^2+4n-4)$. □

Theorem 5.3. *If $p = (1 - n - \sqrt{5n^2 - 6n + 1})/(2n)$, then*

$$\frac{n-1}{n}A(\mathbf{a}) + \frac{1}{n}M_p(\mathbf{a}) \ge M_0(\mathbf{a}). \tag{5.17}$$

Proof. Let

$$f(\mathbf{a}) = \sqrt[n]{\prod_{i=1}^{n} a_i^{1/p}} - \frac{(n-1)}{n^2} \cdot \sum_{i=1}^{n} a_i^{1/p}, \quad \mathbf{a} \in \mathbb{R}_+^n. \tag{5.18}$$

Then

$$\begin{gathered}
\frac{\partial f}{\partial a_1} = \frac{1}{npa_1}\sqrt[n]{\prod_{i=1}^{n} a_i^{1/p}} - \frac{n-1}{n^2 p} a_1^{1/p-1}, \\
\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} = -\frac{a_1 - a_2}{npa_1 a_2}\prod_{i=1}^{n} a_i^{1/np} - \frac{n-1}{n^2 p}\left(a_1^{1/p-1} - a_2^{1/p-1}\right).
\end{gathered} \tag{5.19}$$

If $a_1 = \max_{1\le i\le n}\{a_i\} > a_2 = \min_{1\le i\le n}\{a_i\} > 0$ and $a_1/a_2 = t > 1$, then from $p < 0$ and $-(a_1-a_2)/npa_1a_2 > 0$ we get

$$\begin{aligned}\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} &\le -\frac{a_1-a_2}{npa_1a_2}a_1^{1/(np)}a_2^{(n-1)/(np)} - \frac{n-1}{n^2p}\left(a_1^{1/p-1}-a_2^{1/p-1}\right)\\ &= \frac{a_1^{1/p-1}}{n^2p}\left[-n\frac{t-1}{t}t^{1-(n-1)/(np)} - (n-1)\left(1-t^{1-1/p}\right)\right].\end{aligned}\tag{5.20}$$

Let $g(t) = -nt^{1-(n-1)/(np)} + nt^{-(n-1)/(np)} + (n-1)t^{1-1/p} - (n-1),\ t > 1$. Then

$$\begin{aligned}g'(t) &= \left(-n+\frac{n-1}{p}\right)t^{-(n-1)/(np)} - \frac{n-1}{p}t^{-1-(n-1)/(np)} + (n-1)\left(1-\frac{1}{p}\right)t^{-1/p},\\ t^{1+(n-1)/(np)}g'(t) &= \left(-n+\frac{n-1}{p}\right)t - \frac{n-1}{p} + (n-1)\left(1-\frac{1}{p}\right)t^{1-1/(np)},\\ \left(t^{1+(n-1)/(np)}g'(t)\right)' &= \left(-n+\frac{n-1}{p}\right) + (n-1)\left(1-\frac{1}{p}\right)\left(1-\frac{1}{np}\right)t^{-1/(np)}\\ &> \left(-n+\frac{n-1}{p}\right) + (n-1)\left(1-\frac{1}{p}\right)\left(1-\frac{1}{np}\right)\\ &= -\frac{1}{p^2}\left[p^2+\left(1-\frac{1}{n}\right)p-1+\frac{1}{n}\right]\\ &= 0.\end{aligned}\tag{5.21}$$

Thus $t^{1+(n-1)/(np)}g'(t)$ is a monotone increasing function. This monotonicity and

$$\begin{aligned}\lim_{t\to 1+}t^{1+(n-1)/(np)}g'(t) &= \lim_{t\to 1+}\left[\left(-n+\frac{n-1}{p}\right)t - \frac{n-1}{p} + (n-1)\left(1-\frac{1}{p}\right)t^{1-(1/np)}\right]\\ &= -1-\frac{n-1}{p}\\ &\ge 0\end{aligned}\tag{5.22}$$

lead to $t^{1+(n-1)/(np)}g'(t) > 0$. Therefore $g'(t) > 0$ and $g(t)$ is a monotone increasing function. From $\lim_{t\to 1+}g(t) = 0$ and the monotonicity of $g(t)$ we know that $g(t) > 0$. By (5.20), we know that $\partial f/\partial a_1 - \partial f/\partial a_2 < 0$. According to Corollary 1.3 we get

$$\begin{aligned}&f(\mathbf{a}) \le f(A(\mathbf{a}), A(\mathbf{a}), \ldots, A(\mathbf{a})),\\ &\sqrt[n]{\prod_{i=1}^{n}a_i^{1/p}} - \frac{n-1}{n^2}\cdot\sum_{i=1}^{n}a_i^{1/p} \le \frac{1}{n}\cdot A^{1/p}(\mathbf{a}).\end{aligned}\tag{5.23}$$

Finally, let $a_i \to a_i^p$ $(i = 1, 2, \ldots, n)$ in the above inequality. Then we know that Theorem 5.3 holds.

If $n \geq 2$ and $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then the following inequalities can be found in [8–10]:

$$\frac{\sum_{1\leq i<j\leq n}(a_i - a_j)^2}{2n^2 a_n} \leq A(\mathbf{a}) - M_0(\mathbf{a}) \leq \frac{\sum_{1\leq i<j\leq n}(a_i - a_j)^2}{2n^2 a_1},$$
$$\frac{a_1^3}{2n^2 a_n^4}\sum_{1\leq i<j\leq n}(a_i - a_j)^2 \leq M_0(\mathbf{a}) - M_{-1}(\mathbf{a}) \leq \frac{a_n^3}{2n^2 a_1^4}\sum_{1\leq i<j\leq n}(a_i - a_j)^2. \tag{5.24}$$

Theorems 5.4 and 5.5 are the improvements of inequalities (5.24). □

Theorem 5.4. *If $n \geq 2$ and $0 < m \leq a_1, a_2, \ldots, a_n \leq M$, then*

$$\frac{\sum_{1\leq i<j\leq n}(a_i - a_j)^2}{2n^2 \cdot M^{(n-1)/n} \cdot A^{1/n}(\mathbf{a})} \leq A(\mathbf{a}) - M_0(\mathbf{a}) \leq \frac{\sum_{1\leq i<j\leq n}(a_i - a_j)^2}{2n^2 \cdot m^{(n-1)/n} \cdot A^{1/n}(\mathbf{a})}. \tag{5.25}$$

Proof. Let

$$f : \mathbf{a} \in [m, M]^n \longrightarrow \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{1/n}\left(\frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k}\right) - \frac{\sum_{1\leq i<j\leq n}(a_i - a_j)^2}{2n^2 M^{(n-1)/n}}. \tag{5.26}$$

Then

$$\frac{\partial f}{\partial a_1} = \frac{1}{n^2}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{1/n-1}\left(\frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k}\right)$$
$$+ \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{1/n}\left(\frac{1}{n} - \frac{1}{n a_1}\sqrt[n]{\prod_{k=1}^{n} a_k}\right) - \frac{\sum_{2\leq i\leq n}(a_1 - a_i)}{n^2 M^{(n-1)/n}},$$
$$\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} = \frac{a_1 - a_2}{n a_1 a_2}\sqrt[n]{\prod_{k=1}^{n} a_k} \cdot \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{1/n} - \frac{a_1 - a_2}{n M^{(n-1)/n}}$$
$$= \frac{a_1 - a_2}{n a_1 a_2 M^{(n-1)/n}}\left[M^{(n-1)/n}\sqrt[n]{\prod_{k=1}^{n} a_k} \cdot \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{1/n} - a_1 a_2\right]. \tag{5.27}$$

We assume that $\mathbf{a} \in D^*$ (see (1.16)). Then

$$\begin{aligned}\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} &> \frac{a_1 - a_2}{n a_1 a_2 M^{(n-1)/n}}\left[M^{(n-1)/n} a_1^{1/n} a_2^{(n-1)/n} \cdot a_2^{1/n} - a_1 a_2\right] \\ &\geq \frac{a_1 - a_2}{n a_1 a_2 M^{(n-1)/n}}\left[a_1^{(n-1)/n} \cdot a_1^{1/n} a_2^{(n-1)/n} \cdot a_2^{1/n} - a_1 a_2\right] \\ &= 0.\end{aligned} \tag{5.28}$$

According to Corollary 1.3, we get

$$\begin{gathered} f(\mathbf{a}) \geq f(A(\mathbf{a}), A(\mathbf{a}), \ldots, A(\mathrm{a})), \\ A(\mathbf{a}) - M_0(\mathbf{a}) \geq \frac{\sum_{1\leq i<j\leq n}\left(a_i - a_j\right)^2}{2n^2 M^{(n-1)/n} A^{1/n}(\mathbf{a})}. \end{gathered} \tag{5.29}$$

Let

$$\begin{aligned} g : \mathbf{a} \in [m, M] \longrightarrow & \frac{\sum_{1\leq i<j\leq n}\left(a_i - a_j\right)^2}{2n^2 m^{(n-1)/n}} \\ & - \left[\frac{1}{n}\sum_{k=1}^{n} a_k\right]^{1/n}\left[\frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k}\right]. \end{aligned} \tag{5.30}$$

A similar argument as above leads to

$$A(\mathbf{a}) - M_0(\mathbf{a}) \leq \frac{\sum_{1\leq i<j\leq n}\left(a_i - a_j\right)^2}{2n^2 m^{(n-1)/n} A^{1/n}(\mathbf{a})}. \tag{5.31}$$

The proof of Theorem 5.4 is completed. □

Let

$$\begin{aligned} f : \mathbf{x} \in \left[\frac{1}{M}, \frac{1}{m}\right] &\longrightarrow \frac{1}{2n^2} \cdot \frac{M^{(n-3)/n}}{m^{(2n-3)/n}} \sum_{1\leq i<j\leq n}\left(\frac{1}{x_i} - \frac{1}{x_j}\right)^2 - \frac{1}{M_0(\mathbf{x})}, \\ g : \mathbf{x} \in \left[\frac{1}{M}, \frac{1}{m}\right] &\longrightarrow \frac{1}{M_0(\mathbf{x})} - \frac{m^{(n-1)/n}}{2n^2 M^{(2n-1)/n}} \sum_{1\leq i<j\leq n}\left(\frac{1}{x_i} - \frac{1}{x_j}\right)^2. \end{aligned} \tag{5.32}$$

The proof of Theorem 5.5 is similar to the proof of Theorem 5.4, and so we omit it.

Theorem 5.5. *Let* $n \geq 2$, $0 < m \leq a_1, a_2, \ldots, a_n \leq M$. *Then*

$$\frac{m^{(n-1)/n}}{2n^2 M^{(2n-1)/n}} \sum_{1\leq i<j\leq n} (a_i - a_j)^2 \leq M_0(\mathbf{a}) - M_{-1}(\mathbf{a}) \leq \frac{M^{(n-3)/n}}{2n^2 m^{(2n-3)/n}} \sum_{1\leq i<j\leq n} (a_i - a_j)^2. \tag{5.33}$$

Remark 5.6. More applications for Theorem 1.2 and Corollary 1.3 were shown in [11].

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