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Review Article

On the Abstract Subordinated Exit Equation

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Let $\mathbb{P} = (P_t)_{t>0}$ be a C_0-contraction semigroup on a real Banach space $\mathcal{B}$. A $\mathbb{P}$-*exit law* is a $\mathcal{B}$-valued function $t \in]0,\infty[\to \varphi_t \in \mathcal{B}$ satisfying the functional equation: $P_t\varphi_s = \varphi_{t+s}$, $s,t > 0$. Let β be a Bochner subordinator and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ (in the Bochner sense) by means of β. Under some regularity assumption, it is proved in this paper that each $\mathbb{P}^\beta$-exit law is subordinated to a unique $\mathbb{P}$-exit law.

1. Introduction

Let $\mathbb{P} := (P_t)_{t\geq 0}$ be a C_0-contraction semigroup on a real Banach space $\mathcal{B}$ with generator $(A, D(A))$. A $\mathbb{P}$- *exit law* is a $\mathcal{B}$-valued function $t \in]0,\infty[\to \varphi_t \in \mathcal{B}$ satisfying the functional equation:

$$P_t\varphi_s = \varphi_{t+s}, \quad s,t > 0. \tag{1.1}$$

Exit laws are introduced by Dynkin (cf. [1]). They play an important role in the framework of potential theory without Green function. Indeed, they allow in this case, an integral representation of potentials and explicit energy formulas. Moreover, this notion was investigated in many papers (cf. [2–13]).In particular, the following theorem is proved in our paper [10].

Theorem 1.1. *If a $\mathbb{P}$-exit law φ is Bochner integrable at 0 (shortly zero-integrable), this is equivalent to,*

$$\int_0^1 \|\varphi_t\| dt < \infty \ , \tag{1.2}$$

then φ is of the form

$$\varphi_t = qP_tV_q(\varphi) - AP_tV_q(\varphi), \quad t, q > 0, \tag{1.3}$$

where $V_q(\varphi) := \int_0^\infty e^{-qs}\varphi_s ds$.

The present paper is devoted to investigate the subordinated abstract case where we study the zero-integrable solution of the exit equation (1.1) after Bochner subordination.

More precisely, let $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator, that is, a vaguely continuous convolution semigroup of subprobability measures on $[0, +\infty[$ and let $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ be the subordinated C_0-semigroup of $\mathbb{P}$ in the sense of Bochner by means of β, that is,

$$P_t^\beta f := \int_0^\infty P_s f \beta_t(ds), \quad f \in \mathcal{B}, \quad t > 0. \tag{1.4}$$

It can be seen that, for each exit law $\varphi = (\varphi_t)_{t>0}$ for $\mathbb{P}$, the function φ^β defined by

$$\varphi_t^\beta := \int_0^\infty \varphi_s \beta_t(ds), \quad t > 0, \tag{1.5}$$

is an exit law for $\mathbb{P}^\beta$. The function φ^β is said to be subordinated to φ by means of β.

Conversely, it is natural to ask if any $\mathbb{P}^\beta$-exit law is subordinated to some $\mathbb{P}$-exit law. In general, we do not have a positive answer (see Example 5.3 below or [2, page 1922]). However, this problem was solved (cf. [2, 4–6]) for $\mathcal{B} = L^2(m)$ and positive $\mathbb{P}^\beta$-exit laws ψ, and under some regularity assumptions on $\mathbb{P}$, β, and ψ. Basing on our paper[10, Theorem 1], we consider in this paper the zero-integrable $\mathbb{P}^\beta$-exit laws in the abstract case. Namely, we prove the following.

Theorem 1.2. *Let $\psi := (\psi_t)_{t>0}$ be a zero-integrable $\mathbb{P}^\beta$-exit law satisfying the following conditions: There exist a constant $q > 0$ such that:*

$$\left(P_tV_q(\psi)\right)_{t>0} \subset D\left(A^\beta\right), \tag{1.6}$$

$$\int_0^1 \left\| A^\beta P_s V_q(\psi) \right\| \beta_t(ds) < \infty, \quad t > 0, \tag{1.7}$$

where $V_q(\psi) := \int_0^\infty e^{-qs}\psi_s ds$ and $(A^\beta, D(A^\beta))$ is the associated generator to $\mathbb{P}^\beta$. Then, ψ is subordinated to a unique $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$. Moreover, φ is explicitly given by

$$\varphi_t := qP_tV_q(\psi) - A^\beta P_tV_q(\psi). \tag{1.8}$$

The conditions in Theorem 1.2 are fulfilled for the closed $\mathbb{P}^\beta$-exit laws ψ. This is always the case for the zero-integrable $\mathbb{P}^\beta$-exit laws in the bounded case.

As application, we consider the holomorphic case and we prove the following result:

Theorem 1.3. *We suppose that $\mathbb{P}$ is a C_0-contraction holomorphic semigroup on $\mathcal{B}$ and β be a Bochner subordinator satisfying*

$$\int_0^1 \frac{1}{s}\beta_t(ds) < \infty, \quad t > 0. \tag{1.9}$$

Then each zero-integrable $\mathbb{P}^\beta$-exit law ψ is subordinated to a unique $\mathbb{P}$-exit law φ. Moreover, φ is given by

$$\varphi_t = (q+a)P_t V_q(\psi) - bAP_t V_q(\psi) + \int_0^\infty \left(P_s V_q(\psi) - P_{s+t}V_q(\psi)\right)\nu(ds), \quad t > 0, \tag{1.10}$$

where a, b, and ν are the parameters of β.

The condition (1.9) is fulfilled for the fractional power subordinator and the Dirac subordinator.

2. C_0-Contraction Semigroup

For the following notions and properties about C_0-contraction semigroups, we will refer essentially to [14, 15] (cf. also [16, 17]).

Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space and let I be the identity operator on $\mathcal{B}$. For a linear operator $T : \mathcal{B} \to \mathcal{B}$, we denote also by $\|T\| := \sup_{\|f\|\le 1}\|Tf\|$ the norm of T. If $\|T\| < \infty$, T is said to be bounded.

We consider $[0,\infty[$ endowed with its Borel field $\mathcal{A}$ and a measure μ on $([0,\infty[,\mathcal{A})$. We say that a property holds μ.a.e. if the set for which this property fails is μ-negligible. A $\mathcal{B}$-valued function $X :]0,\infty[\to \mathcal{B}$ is said simple if there exists a disjoint sequence $\{A_i \in \mathcal{A} : \mu(A_i) < \infty\}_{1\le i\le n}$ and $X_1,\dots,X_n \in \mathcal{B}$ such that $X(t) = \sum_{i=1}^n X_i 1_{A_i}(t)$ for all $t > 0$. A $\mathcal{B}$-valued function $X :]0,\infty[\to \mathcal{B}$ is also denoted by $X := (X_t)_{t>0}$.

In this paper, we consider the integral in Bochner sense for functions $X :]a,b[\subset]0,\infty[\to \mathcal{B}$ which are μ-strongly measurable (i.e., there exists a sequence of simple functions $X_n :]a,b[\to \mathcal{B}$ satisfying $\lim_{n\to\infty}\|X_n - X\| = 0$, μ.a.e.). For such functions X, it is known that X is μ-Bochner integrable if and only if $\int_a^b \|X(s)\| \ \mu(ds) < \infty$ (cf. [15, page 133]). For such functions X, it is also known that for each bounded linear operator $T : \mathcal{B} \to \mathcal{B}$, we have

$$T\left(\int_a^b X(s)\mu(ds)\right) = \int_a^b TX(s)\mu(ds). \tag{2.1}$$

In the sequel of this work, μ is omitted whenever it is the Lebesgue measure.

2.1. C_0-Contraction Semigroups

A *C_0-contraction semigroup* on $\mathcal{B}$ is a family of linear operators $\mathbb{P} := (P_t)_{t\ge 0}$ on $\mathcal{B}$ satisfying $P_0 = I$, $P_{s+t} = P_sP_t$ for all $s,t \ge 0$, $\|P_t\| \le 1$ for all $t \ge 0$ and $\lim_{t\to 0}\|P_tf - f\| = 0$ for all $f \in \mathcal{B}$.

Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$. The associated *generator* A of $\mathbb{P}$ is defined by

$$Af := \lim_{t \to 0} \frac{P_t f - f}{t} \tag{2.2}$$

on its domain $D(A) := \{f \in \mathcal{B} : \text{the limit in (2.2) exists in } \mathcal{B}\}$. It is known that

(1) $A : D(A) \to \mathcal{B}$ is a closed linear operator;

(2) $D(A)$ is dense in the Banach space $\mathcal{B}$;

(3) the resolvent $\mathcal{R}_q := (qI - A)^{-1}$ of A exists for each $q > 0$.

The proof of the following useful classical properties can be found in [14, pages 4 and 108] and in [15, pages 233–240].

Lemma 2.1. *Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$ with generator $(A, D(A))$ and resolvent $\mathcal{R} := (\mathcal{R}_q)_{q>0}$.*

(1) *For $f \in \mathcal{B}$ and $0 \leq a < b < \infty$, the function $t \to P_t f$ is strongly measurable and the Bochner integral $\int_a^b P_r f\, dr$ is well defined.*

(2) *For each $t \geq 0$ and $f \in D(A)$, we have $P_t f \in D(A)$,*

$$AP_t f = P_t A f = \frac{d}{dt} P_t f, \tag{2.3}$$

$$P_t f - f = \int_0^t A(P_r f)\, dr. \tag{2.4}$$

(3) *For each $t, q > 0$, we have $\mathcal{R}_q = \int_0^\infty e^{-qs} P_s\, ds$, $P_t \mathcal{R}_q = \mathcal{R}_q P_t$, $\mathcal{R}_q(\mathcal{B}) \subset D(A)$ and*

$$\lim_{q \to \infty} q \mathcal{R}_q u = u, \quad u \in \mathcal{B}\ . \tag{2.5}$$

Example 2.2. Let $\mathcal{B} = \mathcal{C}_b([0, \infty[)$ be the Banach space of bounded uniformly continuous real-valued functions on $[0, \infty[$ and let

$$P_t f(x) := \ f(x + t), \quad t \geq 0,\ x \geq 0, f \in \mathcal{C}_b([0, \infty[). \tag{2.6}$$

Then $\mathbb{P} := (P_t)_{t \geq 0}$ is C_0-contraction semigroup with generator $Af = f'$ where $D(A) := \{f \in \mathcal{C}_b([0, \infty[) : f' \text{ exist}, f' \in \mathcal{C}_b([0, \infty[)\}$.

3. Exit Equation

3.1. Exit Laws

Definition 3.1. Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$. A $\mathbb{P}$- *exit law* is a $\mathcal{B}$-valued function $t \in]0,\infty[\to u_t \in \mathcal{B}$ which verifies the so-called *exit equation*:

$$P_t u_s = u_{t+s}, \quad s,t > 0. \tag{3.1}$$

We point here that a $\mathbb{P}$-exit law $t \to u_t$ may be also denoted by $u := (u_t)_{t>0}$.

Proposition 3.2. *Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$ with generator $(A, D(A))$.*

(1) *For each $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$, the function $s \to \varphi_s$ is strongly measurable on $]0,\infty[$.*

(2) *For each $h \in \mathcal{B}$, the $\mathcal{B}$-valued function $t \to P_t h$ is a $\mathbb{P}$-exit law. It is called a closed exit law.*

(3) *Let $h \in \mathcal{B}$ such that $(P_t h)_{t>0} \subset D(A)$, then $t \to AP_t h$ is a $\mathbb{P}$-exit law. It is said to be differentiable.*

Proof.

The function $s \to P_s h$ is strongly measurable for each $h \in \mathcal{B}$; then for each $b > 0$, the function $s \to \varphi_{s+b} = P_s\, \varphi_b$ is strongly measurable on $[b,\infty[$. Since $b > 0$ is arbitrary, then u is strongly measurable on $]0,\infty[$.

It is immediate from the semigroup property.

It is a consequence of the semigroup property and (2.3). □

3.2. Integrable Exit Laws

Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$ with generator $(A, D(A))$. In the sequel, we consider $\mathbb{P}$-exit laws $t \to \varphi_t$ which are Bochner integrable at 0 (shortly zero-integrable). This is equivalent to

$$\int_0^1 \|\varphi_s\|\, ds < \infty. \tag{3.2}$$

Theorem 3.3. *Let φ be a zero-integrable $\mathbb{P}$-exit law. Then φ is of the form*

$$\varphi_t = qP_t V_q(\varphi) - AP_t V_q(\varphi), \quad t > 0, \tag{3.3}$$

where $q > 0$ and $V_q(\varphi) := \int_0^\infty e^{-qs} \varphi_s ds$.

Proof. Let $q > 0$ be fixed. Since (3.2) holds if and only if $s \rightarrow e^{\omega s}\varphi_s$ is zero-integrable for all $\omega \in \mathbb{R}$, then using (3.1) and (3.2), we have

$$\begin{aligned}\int_0^\infty e^{-qs}\|\varphi_s\|\ ds &= \int_0^1 e^{-qs}\|\varphi_s\|ds + \int_1^\infty e^{-qs}\|\varphi_s\|ds \\ &= \int_0^1 e^{-qs}\|\varphi_s\|ds + \int_1^\infty e^{-qs}\|P_{s-1}\varphi_1\|ds \\ &\leq \int_0^1 e^{-qs}\|\varphi_s\|ds + \|\varphi_1\| \int_1^\infty e^{-qs}ds < \infty.\end{aligned} \tag{3.4}$$

This implies that $s \rightarrow e^{-qs}\varphi_s$ is Bochner integrable on $]0,\infty[$. Hence, $V_q(\varphi)$ is well defined and lies in B. Moreover, by (3.1), (2.1), and (2.3), we get

$$P_tV_q(\varphi) = \int_0^\infty e^{-qs}\varphi_{s+t}\ ds = \int_0^\infty e^{-qs}P_s\varphi_t\ \ ds = \mathcal{R}_q(\varphi_t), \quad t > 0. \tag{3.5}$$

Using (3.5) and (3.3) holds since

$$qP_tV_q(\varphi) - AP_tV_q(\varphi) = q\mathcal{R}_q(\varphi_t) - A\mathcal{R}_q(\varphi_t) = (qI - A)\mathcal{R}_q(\varphi_t) = \varphi_t. \tag{3.6}$$

□

Corollary 3.4. *Suppose that the generator A of $\mathbb{P}$ is bounded and let φ be a $\mathbb{P}$-exit law. Then φ is zero-integrable if and only if φ is closed.*

Proof. If $\varphi_t = P_tf$ for some $f \in \mathcal{B}$, then $t \rightarrow \varphi_t$ is zero-integrable by Lemma 2.1. Conversely, let φ be a $\mathbb{P}$-exit law satisfying (3.2). Theorem 3.3 may be applied: φ is of the form

$$\varphi_t = qP_tV_q(\varphi) - AP_tV_q(\varphi), \quad t > 0, \tag{3.7}$$

where $V_q(\varphi) := \int_0^\infty e^{-qs}\varphi_s ds$ for some $q > 0$. Moreover, since A is bounded then $D(A) = \mathcal{B}$ (cf. [16, Corollary 1.5] and therefore by (2.3), we get

$$\varphi_t = qP_tV_q(\varphi) - AP_tV_q(\varphi) = qP_tV_q(\varphi) - P_tAV_q(\varphi) = P_t(qV_q(\varphi) - AV_q(\varphi)). \tag{3.8}$$

Hence, φ is a closed exit law. □

Remark 3.5. Results similar to Theorem 3.3 are proved in our paper [10]. Indeed, the proof given in [10] depends fundamentally on the properties of the rescaled C_0-semigroup; however, in this paper, it is based on the resolvent properties of C_0-contraction semigroup.

For closed exit laws, the condition (3.2) is satisfied. However, this not the case, for differentiable exit laws. Indeed, consider again Example 2.2 and let $u(x) = x\sin(1/x)$. Then $u \in \mathcal{B}$ and $\int_0^1 \|AP_tu\|dt = \infty$.

We consider Example 2.2 and we define

$$\varphi_t^a(x) := \frac{1}{(x+t)^a}, \quad a>0,\ t>0,\ x\geq 0. \tag{3.9}$$

It is proved in [10] that $\varphi^a := (\varphi_t^a)_{t>0}$ is a $\mathbb{P}$-exit law neither closed nor differentiable and $\int_0^1 \|\varphi_s\| ds < \infty$ if and only if $a \in]0,1[$.

4. Subordination of C_0-Contraction Semigroup

4.1. Bochner Subordinator

We consider $\mathbb{R}$ endowed with its Borel σ-field. We denote by ε_t the Dirac measure at point t. Moreover, for each bounded measure μ on $[0,\infty[$, $\mathcal{L}$ denotes its Laplace transform, that is, $\mathcal{L}(\mu)(r) := \int_0^\infty \exp(-rs)\mu(ds)$ for $r>0$.

For the following classical notions, we refer the reader to [17–19].

A *Bochner subordinator* $\beta := (\beta_t)_{t>0}$ is a vaguely continuous convolution semigroup of subprobability measures on $[0,+\infty[$.

Let β be a Bochner subordinator. The associated *Bernstein function* f is defined by the Laplace transform

$$\mathcal{L}(\beta_t)(r) = \exp(-tf(r)), \quad r,t>0. \tag{4.1}$$

In fact, (4.1) establishes a one-to-one correspondence between convolution semigroups $\beta := (\beta_t)_{t\geq 0}$ and Bernstein functions f (cf. [18, Theorem 9.8]). In fact, f admits the representation

$$f(r) = a + br + \int_0^\infty (1-\exp(-rs))\nu(ds), \quad r>0, \tag{4.2}$$

where $a,b \geq 0$ and ν is a measure on $]0,\infty[$ verifying $\int_0^\infty (s/(s+1))\nu(ds) < \infty$. They are called *parameters* of β or of f.

Example 4.1. The *fractional power subordinator* $\eta^\alpha := (\eta_t^\alpha)_{t\geq 0}$ *of index* $\alpha \in]0,1[$ is defined by its Laplace transform $\mathcal{L}(\eta_t^\alpha)(r) = \exp(-tr^\alpha)$ for all $r,t>0$.

The Γ-*subordinator* $\gamma := (\gamma_t)_{t\geq 0}$ is defined by

$$\gamma_t(ds) := 1_{]0,\infty[}(s)\left(\frac{1}{\Gamma(t)}\right)s^{t-1}\exp(-s)ds, \quad t\geq 0. \tag{4.3}$$

The *Poisson subordinator* $\tau := (\tau_t)_{t\geq 0}$ *of jump* $c>0$ is defined by

$$\tau_t := \exp(-ct)\sum_{n=0}^{\infty}\frac{(ct)^n}{n!}\varepsilon_n, \quad t\geq 0. \tag{4.4}$$

The Dirac subordinator $\varepsilon := (\varepsilon_t)_{t\geq 0}$.

4.2. Bochner Subordination

Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$ and let β be a Bochner subordinator. For every $t > 0$ and for every $u \in \mathcal{B}$, we may define

$$P_t^\beta u := \int_0^\infty P_s u \beta_t(ds). \tag{4.5}$$

Then $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ is a C_0-contraction semigroup on $\mathcal{B}$ (see, e.g., [17, Theorem 4.3.1]). It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of β. In what follows, we index by "β" all entities associated to $\mathbb{P}^\beta$. In particular, A^β is the associated generator and $\mathcal{R}^\beta := (\mathcal{R}_q^\beta)_{q>0}$ its associated resolvent.

Let A^β be the generator of $\mathbb{P}^\beta$. The following two remarks will be used throughout this paper: $D(A)$ is a subset of $D(A^\beta)$ (cf. [17, page 299]) and

$$A^\beta u = -au + bAu + \int_0^\infty (P_t u - u)\nu(dt), \quad u \in D(A), \tag{4.6}$$

where a, b, and ν are given in (4.2).

Lemma 4.2. *There exist some constants $K_1, K_2 > 0$ such that*

$$\int_0^\infty \|P_s u - u\|\nu(ds) \leq K_1\|u\| + K_2\|Au\|, \quad u \in D(A). \tag{4.7}$$

Proof. Let $u \in D(A)$. Using the semigroup property and (4.6), we have

$$\begin{aligned} \int_0^\infty \|P_s u - u\|\nu(ds) &\leq \int_0^1 \|P_s u - u\|\nu(ds) + \int_1^\infty \|P_s u - u\|\nu(ds) \\ &\leq \int_0^1 \left\| \int_0^s P_r Au \, dr \right\| \nu(ds) + 2\|u\| \int_1^\infty \nu(ds) \\ &\leq \left(\int_0^1 s\nu(ds) \right) \|Au\| + 2\|u\| \left(\int_1^\infty \nu(ds) \right). \end{aligned} \tag{4.8}$$

Hence, (4.7) holds for $K_1 := 2\int_1^\infty \nu(ds)$ and $K_2 := \int_0^1 s\nu(ds)$. □

Proposition 4.3. *Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$, β a Bochner subordinator, and $\mathbb{P}^\beta$ be the subordinated to $\mathbb{P}$ by means of β. Then*

$$P_t A^\beta h = A^\beta P_t h, \quad t > 0, \ h \in D\left(A^\beta\right). \tag{4.9}$$

In particular, we have $P_t(D(A^\beta)) \subset D(A^\beta)$ for all $t > 0$.

Proof.

Step 1. First we suppose that $h \in D(A)$. From Lemma 4.2,

$$\int_0^\infty \|P_s h - h\| \nu(ds) < \infty. \tag{4.10}$$

So by using (2.1), we get

$$P_t \int_0^\infty (P_s h - h)\nu(ds) = \int_0^\infty (P_s P_t h - P_t h)\nu(ds), \quad t > 0. \tag{4.11}$$

Combining (2.3), (4.6), and (4.11), we have

$$\begin{aligned} P_t A^\beta h &= -a P_t h + b P_t A h + P_t \int_0^\infty (P_s h - h)\nu(ds) \\ &= -a P_t h + b A P_t h + \int_0^\infty (P_s P_t h - P_t h)\nu(ds) \\ &= A^\beta P_t h. \end{aligned} \tag{4.12}$$

Step 2. Now, we suppose that $h \in D(A^\beta)$. Let $\mathcal{R} := (\mathcal{R}_q)_{q>0}$ be the associated resolvent to $\mathbb{P}$ and let $q, t > 0$. Since $\mathcal{R}_q(\mathcal{B}) \subset D(A)$, then from Step 1 and Lemma 2.1, we have

$$P_t A^\beta \mathcal{R}_q h = A^\beta P_t \mathcal{R}_q h = A^\beta \mathcal{R}_q P_t h. \tag{4.13}$$

Hence, by the contraction property and (4.13), we get

$$\begin{aligned} \|P_t A^\beta h - A^\beta P_t h\| &\le \|P_t A^\beta h - q P_t A^\beta \mathcal{R}_q h\| + \|q A^\beta \mathcal{R}_q P_t h - A^\beta P_t h\| \\ &\le \|A^\beta h - A^\beta q \mathcal{R}_q h\| + \|A^\beta q \mathcal{R}_q P_t h - A^\beta P_t h\|. \end{aligned} \tag{4.14}$$

Finally, since A^β is closed then by (2.5) and by letting $q \uparrow \infty$, (4.9) holds. □

5. Subordinated Exit Law

Let $\mathbb{P}$ be a C_0-contraction semigroup with generator $(A, D(A))$, let β be a Bochner subordinator, and let $\mathbb{P}^\beta$ be the subordinated to $\mathbb{P}$ by means of β with generator $(A^\beta, D(A^\beta))$.

Definition 5.1. Let φ be a $\mathbb{P}$-exit law and define(1.5). If the family of Bochner integrals (1.5) is well defined, it easy to verify that $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ is a $\mathbb{P}^\beta$-exit law which is said to be *subordinated to φ in the Bochner sense by means of β*. Notice that if $\varphi_s = P_s u$ for some $u \in \mathcal{B}$, then (1.5) is just (4.5).

Remark 5.2.

(1) *Subordination problem*: conversely, let ψ be a $\mathbb{P}^{\beta}$-exit law, does there exist a $\mathbb{P}$-exit law φ such that ψ is subordinated to φ?

In this paper, we study this problem of $\mathbb{P}^{\beta}$-exit laws which are Bochner integrable at 0.

(2) The condition of zero-integrability is not necessary. Indeed, if we take $\beta_t = \varepsilon_t$, then $\mathbb{P}^{\beta} = \mathbb{P}$ and the subordination problem is solved for each $\mathbb{P}^{\beta}$-exit law ψ since $\psi^{\beta} = \psi$.

Example 5.3. Let $\mathcal{B} = L^1(\lambda^1)$ where λ^1 is the Lebesgue measure on $\mathbb{R}$ and let $\mathbb{P} := (P_t)_{t>0}$ be the left uniform translation on $\mathcal{B}$, that is, $P_t u := \varepsilon_t * u$ for $t \geq 0$ and $u \in \mathcal{B}$. It can be seen that each $\mathbb{P}$-exit law φ is closed, that is, of the form $\varphi_t = P_t u$ for some fixed $u \in \mathcal{B}$.

On the other hand, let $\eta^{1/2} := (\eta_t^{1/2})_{t>0}$ be the fractional powers subordinator of index 1/2. From [18, page 71], $\eta_t^{1/2}$ is absolutely continuous with density $G_t(s) = (1/\sqrt{4\pi}) t s^{-3/2} \exp(-t^2/4s)$ for all $t, s > 0$. The extension of G_t by 0 on $\mathbb{R}$ is denoted by G_t. So, $P_t^{\eta^{1/2}} u = G_t * u$ for all $t > 0$ and $u \in \mathcal{B}$. In particular,

$$P_t^{\eta^{1/2}} G_s(x) = G_t * G_s(x) = G_{s+t}(x), \quad s, t > 0, \; x \in \mathbb{R}, \tag{5.1}$$

by the convolution semigroup property of $\eta^{1/2}$. Therefore, the family $G := (G_t)_{t>0}$ is a $\mathbb{P}^{\eta^{1/2}}$-exit law. Moreover, G is zero-integrable (By using the change of variables $y = t^{-2}s$, we have $\|G_t\| = \|G_1\|$ for all $t > 0$). But there exists no $u \in \mathcal{B}$ such that $G_t = G_t * u$ for each $t > 0$.

Hence, not every zero-integrable $\mathbb{P}^{\eta^{1/2}}$-exit law is subordinated to a $\mathbb{P}$-exit law, because each $\mathbb{P}$-exit law is closed, while this is not the case for all zero-integrable $\mathbb{P}^{\eta^{1/2}}$-exit law.

Remark 5.4. Example 5.3 proves that we need to add some condition in order to solve the subordination problem. Next, we will suppose that $\psi := (\psi_t)_{t>0}$ satisfies the following conditions.

(H): There exists a constant $q > 0$ such as(1.6) and (1.7) where $V_q(\psi) := \int_0^{\infty} e^{-qs} \psi_s ds$.

Theorem 5.5. *Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$ and let β be a Bochner subordinator. Suppose that ψ is a zero-integrable $\mathbb{P}^{\beta}$-exit law satisfying* (*H*). *Then ψ is subordinated to a unique $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$. Moreover, φ is explicitly given by*

$$\varphi_t := q P_t V_q(\psi) - A^{\beta} P_t V_q(\psi), \quad t > 0, \tag{5.2}$$

where q and V_q are given by (*H*).

Proof. Since $(P_t V_q(\psi))_{t>0} \subset D(A^{\beta})$, then the family $\varphi := (\varphi_t)_{t>0}$ defined by (5.2) is well defined and lies in $\mathcal{B}$. Moreover, by (5.2) and (4.9), we get

$$P_s \varphi_t = P_s \Big(q P_t V_q(\psi) - A^{\beta} P_t V_q(\psi) \Big) = q P_{s+t} V_q(\psi) - A^{\beta} P_{s+t} V_q(\psi) = \varphi_{s+t}, \tag{5.3}$$

which implies that $\varphi := (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law. Now using (1.6) and (4.9), we have

$$\begin{aligned}\int_1^\infty \|A^\beta P_s V_q(\varphi)\|\beta_t(ds) &= \int_1^\infty \|A^\beta P_{s-1} P_1 V_q(\varphi)\|\beta_t(ds)\\ &= \int_1^\infty \|P_{s-1} A^\beta P_1 V_q(\varphi)\|\beta_t(ds)\\ &\le \int_1^\infty \|A^\beta P_1 V_q(\varphi)\|\beta_t(ds)\\ &\le \beta_t([1,\infty[)\|A^\beta P_1 V_q(\varphi)\| < \infty,\end{aligned} \tag{5.4}$$

and by (1.7), we conclude that

$$\int_0^\infty \|A^\beta P_s V_q(\varphi)\|\beta_t(ds) < \infty, \quad t > 0. \tag{5.5}$$

Therefore, from (5.5), we have

$$\begin{aligned}\int_0^\infty \|\varphi_s\|\beta_t(ds) &= \int_0^\infty \|q P_s V_q(\varphi) - A^\beta P_s V_q(\varphi)\|\beta_t(ds)\\ &\le q\int_0^\infty \|P_s V_q(\varphi)\|\beta_t(ds) + \int_0^\infty \|A^\beta P_s V_q(\varphi)\|\beta_t(ds)\\ &\le q\ \ \beta_t([0,\infty[)\|V_q(\varphi)\| + \int_0^\infty \|A^\beta P_s V_q(\varphi)\|\beta_t(ds) < \infty.\end{aligned} \tag{5.6}$$

Hence, the subordinated $\varphi^\beta := (\varphi_t^\beta)_{t>0}$ defined by (1.5) is well defined.

On the other hand, for all $s, t > 0$, we have

$$\begin{aligned} P_s\varphi_t^\beta &\overset{(1.5)}{=} P_s\int_0^\infty \varphi_r\beta_t(dr) \overset{(2.1)}{=} \int_0^\infty P_s\varphi_r\beta_t(dr)\\ &\overset{(3.1)}{=} \int_0^\infty \varphi_{r+t}\beta_t(dr) = \int_0^\infty P_r\varphi_s\beta_t(dr)\\ &= P_t^\beta\varphi_s \overset{(8)}{=} P_t^\beta\Big(P_s V_q(\varphi) - A^\beta P_s V_q(\varphi)\Big)\\ &\overset{(2.3)}{=} P_s P_t^\beta V_q(\varphi) - A^\beta P_s P_t^\beta V_q(\varphi) \overset{(18)}{=} P_s\Big(P_t^\beta V_q(\varphi) - A^\beta P_t^\beta V_q(\varphi)\Big) = P_s\varphi_t\end{aligned} \tag{5.7}$$

by using Theorem 3.3 since φ is Bochner integrable at 0. Therefore,

$$\varphi_t = P_{t/2}^\beta\varphi_{t/2} = \int_0^\infty P_s\varphi_{t/2}\beta_{t/2}(ds) = \int_0^\infty P_s\varphi_{t/2}^\beta\beta_{t/2}(ds) = P_{t/2}^\beta\varphi_{t/2}^\beta = \varphi_t^\beta, \tag{5.8}$$

which implies that $\varphi = \varphi^\beta$.

Finally, let us prove the uniqueness: Let $\phi := (\phi_t)_{t>0}$ be a $\mathbb{P}$-exit law such that $\psi = \phi^\beta$. Since for all $s, t > 0$, we have

$$\begin{aligned} P_s\phi_t^\beta &= P_s\int_0^\infty \phi_r\beta_t(dr) \overset{(2.1)}{=} \int_0^\infty P_s\phi_r\beta_t(dr) \overset{(3.1)}{=} \int_0^\infty \phi_{r+t}\beta_t(dr) \\ &= \int_0^\infty P_r\phi_s\beta_t(dr) \overset{(4.4)}{=} P_t^\beta\phi_s, \end{aligned} \tag{5.9}$$

then for all $t > 0$,

$$P_tV_q(\psi) = \int_0^\infty e^{-sq}P_t\phi_s^\beta ds = \int_0^\infty e^{-sq}P_s^\beta\phi_t ds = R_q^\beta(\phi_t). \tag{5.10}$$

Therefore, from (5.2), we have

$$\varphi_t = qR_q^\beta(\phi_t) - A^\beta R_q^\beta(\phi_t) = \left(qI - A^\beta\right)R_q^\beta(\phi_t) = \phi_t, \quad t > 0. \tag{5.11}$$

□

Remark 5.6. In addition, if $P_t(V_q(\psi))_{t>0} \subset D(A)$, then from (4.6), φ is of the form

$$\varphi_t = (q+a)P_tV_q(\psi) - bAP_tV_q(\psi) + \int_0^\infty (P_sV_q(\psi) - P_{s+t}V_q(\psi))\nu(ds), \tag{5.12}$$

where a, b, and ν are the parameters associated to β.

In particular, this is the case of each ψ satisfying (H) whenever the parameter b of β is not zero or A is bounded. Indeed, from [17, Theorem 5.3.8], we have $D(A) = D(A^\beta)$.

If ψ satisfies (H) for some $q > 0$, then it satisfies (H) for all $\varepsilon > 0$. Indeed, by Theorem 5.5, ψ is subordinated to some $\mathbb{P}$-exit law φ. Moreover, exactly as (5.10), we have

$$P_tV_\varepsilon(\psi) = \int_0^\infty e^{-s\varepsilon}P_t\psi_s ds = \int_0^\infty e^{-s\varepsilon}P_s^\beta\varphi_t ds = \mathcal{R}_\varepsilon(\varphi_t), \quad t, \varepsilon > 0. \tag{5.13}$$

So, $P_tV_\varepsilon(\psi) \in D(A^\beta)$ and

$$\begin{aligned} A^\beta P_tV_\varepsilon(\psi) &= A^\beta\mathcal{R}_\varepsilon^\beta(\varphi_t) = \left(\varepsilon I - A^\beta\right)\mathcal{R}_\varepsilon^\beta(\varphi_t) - \varepsilon\mathcal{R}_\varepsilon^\beta(\varphi_t) \\ &= \varphi_t - \varepsilon P_tV_\varepsilon(\psi). \end{aligned} \tag{5.14}$$

Therefore, from the proof of Theorem 5.5, we have

$$\begin{aligned}\int_0^1 \|A^\beta P_s V_q(\psi)\| \beta_t(ds) &= \int_0^1 \|\varphi_s - \varepsilon P_s V_\varepsilon(\psi)\| \beta_t(ds) \\ &\leq \int_0^1 (\|\varphi_s\| + \|\varepsilon P_s V_\varepsilon(\psi)\|) \beta_t(ds) \\ &\leq \int_0^1 \|\varphi_s\| \beta_t(ds) + \varepsilon \|V_\varepsilon(\psi)\| \int_0^1 \beta_t(ds).\end{aligned} \tag{5.15}$$

Hence, (1.6) and (1.7) hold for each $\varepsilon > 0$.

The conditions of Theorem 5.5 are fulfilled for the natural example of $\mathbb{P}^\beta$-exit law. Indeed, we have the following result.

Corollary 5.7. *Each closed $\mathbb{P}^\beta$-exit law ψ, that is, $\psi_t = P_t^\beta u$ for some $u \in \mathcal{B}$, is subordinated to unique exit law $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$. Moreover, φ is explicitly given by $\varphi_t = P_t u$ for all $t > 0$.*

Proof. Let ψ be a closed $\mathbb{P}^\beta$-exit law. It is easy to see that ψ is zero-integrable. Moreover, for all $q > 0$, we have

$$\begin{aligned}V_q(\psi) &:= \int_0^\infty e^{-qs} \psi_s ds = \int_0^\infty e^{-qs} P_s^\beta u \, ds = \mathcal{R}_q^\beta(u), \\ P_t V_q(u) &= \int_0^\infty e^{-qs} P_s^\beta P_t u ds = \mathcal{R}_q^\beta(P_t u), \quad t > 0,\end{aligned} \tag{5.16}$$

which implies that $V_q(\psi) \in D(A^\beta)$ and $(P_t V_q(\psi))_{t>0} \subset D(A^\beta)$. Moreover,

$$\begin{aligned}\int_0^1 \|A^\beta P_s V_q(\psi)\| \beta_t(ds) &= \int_0^1 \|P_s A^\beta V_q(\psi)\| \beta_t(ds) \\ &\leq \int_0^1 \|A^\beta V_q(\psi)\| \beta_t(ds) \leq \|A^\beta V_q(\psi)\|.\end{aligned} \tag{5.17}$$

So, ψ satisfies (H) and by Theorem 5.5, we get

$$\varphi_t = \left(qI - A^\beta\right) P_t V_q(\psi) = \left(qI - A^\beta\right) \mathcal{R}^\beta(P_t u)) = P_t u, \quad t > 0. \tag{5.18}$$

□

Corollary 5.8. *Suppose that A is bounded or f is bounded, then Theorem 5.5 may be applied for each zero-integrable $\mathbb{P}^\beta$-exit law.*

Proof. According to [17, Theorem 4.3.8, page 303], A^β is bounded if and only if A is bounded or f is bounded. So the proof is an immediate consequence of Corollaries 3.4 and 5.7. □

6. Application to Holomorphic Case

Definition 6.1. Let $\mathbb{P}$ be a C_0-contraction semigroup on $\mathcal{B}$. $\mathbb{P}$ is said to be *holomorphic* if there exists a holomorphic extension $z \to P_z$ to $S := \{z \in \mathbb{C}^* : |\arg z| < \theta\}$; $0 < \theta < \pi/2$.

Remark 6.2 (Construction by Bochner subordination). Let F be the Banach algebra of complex Borel measures on $[0, \infty[$, with convolution as multiplication, and normed by the total variation $\|\cdot\|_F$. A Bochner subordinator $\beta = (\beta_t)_{t>0}$ is said to be of type *Carasso-Kato* if:

The associated parameters $a = b = 0$ and the mapping $t \to \beta_t$ is continuously differentiable from $]0, \infty[$ to F such that $\|(\partial/\partial t)\beta_t\|_F \le c/t$ as $t \to 0$ and for some constant $c > 0$.

It is proved in [19] that for each C_0-contraction semigroup $\mathbb{Q}$ and each subordinator β of type Carasso-Kato, the subordinated $\mathbb{P} := \mathbb{Q}^\beta$ is a C_0-contraction holomorphic semigroup.

Note that the fractional power subordinator, Γ-subordinator, and Poisson subordinator are of type Carasso-Kato.

The proof of the following useful classical properties can be found in [14, pages 4 and 108] and in [15, pages 233–240].

Lemma 6.3. *Let $\mathbb{P}$ be a C_0-contraction holomorphic semigroup on $\mathcal{B}$ with generator $(A, D(A))$. Then there exists a constant $t > 0$ such that*

$$\|AP_t h\| \le \frac{C\|h\|}{t} \quad \text{as } t \longrightarrow 0. \tag{6.1}$$

Note that the condition (1.6) *from* (H) *is fulfilled for all C_0-contraction holomorphic semigroup $\mathbb{P}$. Indeed, using the above Lemma, the range of P_t is contained in $D(A)$, hence also in $D(A^\beta)$.*

Proposition 6.4. *Let β be a Bochner subordinator such as* (1.9) *Then,*

$$\int_0^1 \|A^\beta P_s h\| \beta_t(ds) < \infty, \quad t > 0, h \in \mathcal{B}. \tag{6.2}$$

Proof. Let $t > 0$ and $h \in \mathcal{B}$. Since $P_t h \in D(A)$, then from (4.6), we have

$$A^\beta P_t h = -aP_t h + bAP_t h + \int_0^\infty (P_s P_t h - P_t h)\nu(ds), \tag{6.3}$$

where a, b, and ν are given in (4.2). By using Lemma 4.2, we have

$$\int_0^\infty \|P_{s+t} h - P_t h\| \nu(ds) \le K_1 \|P_t h\| + K_2 \|AP_t h\| \le K_1 \|h\| + K_2 \|AP_t h\| \tag{6.4}$$

for some $K_1, K_2 > 0$. Therefore, by (6.3), we conclude that

$$\|A^\beta P_t h\| \le K_3 \|h\| + K_4 \|AP_t h\|, \tag{6.5}$$

where $K_3 := a + K_1$ and $K_4 := b + K_2$. Moreover, combining Lemma 6.3, (1.9), and (6.5), we have

$$\begin{aligned}\int_0^1 \|A^\beta P_s h\|\beta_t(ds) &\leq K_3\|h\|\int_0^1 \beta_t(ds) + K_4\int_0^1 \|AP_s h\|\beta_t(ds)\\ &\leq K_3\|h\| + K_4 C\|h\|\int_0^1 \frac{1}{s}\beta_t(ds) < \infty.\end{aligned} \tag{6.6}$$

Hence, (6.2) holds. □

Remark 6.5. The condition (1.9) holds as soon as the associated Bernstein function f satisfies

$$\int_0^\infty e^{-tf(r)}dr < \infty, \quad t > 0. \tag{6.7}$$

Indeed by using the Fubini's theorem, we have

$$\begin{aligned}\int_0^1 \frac{1}{s}\beta_t(ds) &\leq \int_0^\infty \left(\int_0^\infty e^{-sr}dr\right)\beta_t(ds) \leq \int_0^\infty \left(\int_0^\infty e^{-sr}\beta_t(ds)\right)dr\\ &\leq \int_0^\infty \mathcal{L}(\beta_t)(r)dr \leq \int_0^\infty e^{-tf(r)}dr.\end{aligned} \tag{6.8}$$

Hence, (1.9) holds for the Dirac and the fractional power subordinators.

Theorem 6.6. *Let $\mathbb{P}$ be a C_0-contraction holomorphic semigroup on $\mathcal{B}$ and let β be a Bochner subordinator satisfying* (1.9)*. Then each zero-integrable $\mathbb{P}^\beta$-exit law ψ is subordinated to a unique $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$. Moreover, φ is explicitly given by*

$$\varphi_t = (q+a)P_tV_q(\psi) - bAP_tV_q(\psi) + \int_0^\infty (P_sV_q(\psi) - P_{s+t}V_q(\psi))\nu(ds), \quad t > 0, \tag{6.9}$$

where a, b, and ν are the parameters of β and $V_q(\psi) := \int_0^\infty e^{-qs}\psi_s ds$ for some $q > 0$.

Proof. Let $q > 0$. Since $V_q(\psi) := \int_0^\infty e^{-qs}\psi_s ds \in \mathcal{B}$, then from Proposition 6.4, (1.6) and (1.7) hold. Therefore, ψ satisfies (H). So the proof is an immediate consequence of Theorem 5.5 and (4.6). □

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