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Research Article

A Viscosity Hybrid Steepest Descent Method for Generalized Mixed Equilibrium Problems and Variational Inequalities for Relaxed Cocoercive Mapping in Hilbert Spaces

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We present an iterative method for fixed point problems, generalized mixed equilibrium problems, and variational inequality problems. Our method is based on the so-called viscosity hybrid steepest descent method. Using this method, we can find the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of generalized mixed equilibrium problems, and the set of solutions of variational inequality problems for a relaxed cocoercive mapping in a real Hilbert space. Then, we prove the strong convergence of the proposed iterative scheme to the unique solution of variational inequality. The results presented in this paper generalize and extend some well-known strong convergence theorems in the literature.

1. Introduction

Throughout this paper, unless otherwise specified, we consider H to be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto the closed convex subset C. Let $S : C \to C$ be a nonexpansive mapping, that is, $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. The fixed point set of S is defined by

$$F(S) = \{x \in C : Sx = x\}. \tag{1.1}$$

If $C \subset H$ is nonempty, bounded, closed, and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; see, for example, [1, 2]. A mapping $f : C \to C$ is a contraction on C if there exists a constant $\eta \in (0,1)$ such that $\|f(x) - f(y)\| \leq \eta\|x - y\|$ for all $x, y \in C$. In addition, let $\Psi : C \to H$ be a nonlinear mapping. Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function and let $\Theta : C \times C \to \mathbb{R}$ be a bifunction such that $C \cap \operatorname{dom}\varphi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers and $\operatorname{dom}\varphi = \{x \in C : \varphi(x) < +\infty\}$.

The generalized mixed equilibrium problem for finding $x \in C$

$$\Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by GMEP(Θ, φ, Ψ), that is,

$$\text{GMEP}(\Theta, \varphi, \Psi) = \{x \in C : \Theta(x, y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C\}. \tag{1.3}$$

We see that if x is a solution of a problem (1.2), then $x \in \operatorname{dom}\varphi$.

Special Examples

(1) If $\Psi = 0$, then the problem (1.2) is reduced into the mixed equilibrium problem for finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by MEP(Θ, φ).

(2) If $\varphi = 0$, then the problem (1.2) is reduced into the generalized equilibrium problem for finding $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by GEP(Θ, Ψ).

(3) If $\Psi = 0$ and $\varphi = 0$, then the problem (1.2) is reduced into the equilibrium problem for finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by EP(Θ).

(4) If $\Theta = 0$, $\varphi = 0$, and $\Psi = B$, then the problem (1.2) is reduced into the variational inequality problem for finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by VI(C, B).

The generalized mixed equilibrium problem is very general in the sense that it includes, as special cases, fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems in noncooperative games, the equilibrium problem, and Numerous problems in physics, economics, and others. Some methods have been proposed to solve problem (1.2); see, for instance, [3, 4] and the references therein.

Let $B : C \to H$ be a nonlinear mapping. Now, we recall the following definitions.

(d1) B is said to be monotone if for each $x, y \in C$

$$\langle Bx - By, x - y \rangle \geq 0. \tag{1.8}$$

(d2) B is said to be ρ-strongly monotone if there exists a positive real number ρ such that

$$\langle Bx - By, x - y \rangle \geq \rho \|x - y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

(d3) B is said to be ω-Lipschitz continuous if there exists a positive real number ω such that

$$\|Bx - By\| \leq \omega \|x - y\|, \quad \forall x, y \in C. \tag{1.10}$$

(d4) B is said to be ξ-inverse-strongly monotone if there exists a constant $\xi > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \xi \|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.11}$$

(d5) B is said to be relaxed (u, v)-cocoercive if there exist positive real numbers u, v such that

$$\langle Bx - By, x - y \rangle \geq (-u) \|Bx - By\|^2 + v \|x - y\|^2, \quad \forall x, y \in C. \tag{1.12}$$

(d6) A set-valued mapping $Q : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g \rangle \geq 0$.

(d7) A monotone mapping $Q : H \to 2^H$ is called maximal if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping Q is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(Q)$ implies $f \in Qx$.

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequalities for a ξ-inverse-strongly monotone mapping, Takahashi and Toyoda [5] introduced the following iterative scheme:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) S P_C (x_n - \alpha_n B x_n), \quad \forall n \geq 0, \end{gathered} \tag{1.13}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\alpha_n\}$ is a sequence in $(0,2\xi)$. They showed that if $F(S)\cap \mathrm{VI}(C,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z\in F(S)\cap \mathrm{VI}(C,B)$.

For finding an element of $\mathrm{VI}(C,B)$, Iiduka et al. [6] introduced the following iterative scheme:

$$\begin{gathered} x_0\in C \text{ chosen arbitrary}, \\ x_{n+1}=P_C\big(\gamma_n x_n+(1-\gamma_n)P_C(x_n-\alpha_n Bx_n)\big),\quad \forall n\geq 0, \end{gathered} \tag{1.14}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\gamma_n\}$ is a sequence in $(-1,1)$, and $\{\alpha_n\}$ is a sequence in $(0,2\xi)$. They showed that if $\mathrm{VI}(C,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.14) converges weakly to some $z\in \mathrm{VI}(C,B)$.

For finding a common element of $F(S)\cap \mathrm{VI}(C,B)$, let $S:H\to H$ be a nonexpansive mapping. Yamada [7] introduced the following iterative scheme called the hybrid steepest descent method:

$$x_{n+1}=Sx_n-\alpha_n\mu BSx_n,\quad \forall n\geq 1, \tag{1.15}$$

where $x_1=x\in H$, $\{\alpha_n\}\subset(0,1)$, $B:H\to H$ is a strongly monotone and Lipschitz continuous mapping, and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (1.15) converges strongly to the unique solution of the $F(S)\cap \mathrm{VI}(C,B)$.

The hybrid steepest descent method is constructed by blending important ideas in the steepest descent method and in the fixed point theory. The remarkable applicability of this method to the convexly constrained generalized pseudoinverse problem as well as to the convex feasibility problem is demonstrated by constructing nonexpansive mappings whose fixed point sets are the feasible sets of the problems.

On the other hand, Shang et al. [8] introduced a new iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequalities for relaxed (u,v)-cocoercive mappings in a real Hilbert space by using viscosity approximation method. Let $S:C\to C$ be a nonexpansive mapping and let $f:C\to C$ be a contraction mapping. Starting with arbitrary initial $x_1\in C$ and define sequences $\{x_n\}$ recursively by

$$x_{n+1}=\epsilon_n f(x_n)+\beta_n x_n+\gamma_n SP_C(x_n-\alpha_n Bx_n),\quad \forall n\geq 1. \tag{1.16}$$

They proved that under certain appropriate conditions imposed on $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n\}$, the sequence $\{x_n\}$ converges strongly to $z\in F(S)\cap \mathrm{VI}(C,B)$, where $z=P_{F(S)\cap \mathrm{VI}(C,B)}f(z)$.

For finding a common element of $F(S)\cap \mathrm{GEF}(\Theta,\Psi)$, let C be a nonempty closed convex subset of a real Hilbert space H. Let Ψ be a ξ-inverse-strongly monotone mapping of C into

H and let S be a nonexpansive mapping of C into itself. S. Takahashi and W. Takahashi [9] introduced the following iterative scheme:

$$\begin{gathered}\Theta(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x + (1 - \alpha_n)u_n, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n)Sy_n,\end{gathered} \tag{1.17}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\gamma_n\} \subset [0,1]$, and $\{r_n\} \subset [0,2\xi]$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.17) converges strongly to a common element of $F(S) \cap \mathrm{GEF}(\Theta, \Psi)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [7, 10–12] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences.

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping defined on a real Hilbert space H:

$$\min_{x \in F} \frac{1}{2}\langle Ax, x \rangle - \langle x, b \rangle, \tag{1.18}$$

where F is the fixed point set of a nonexpansive mapping S defined on H and b is a given point in H.

A linear bounded operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.19}$$

Recently, Marino and Xu [13] introduced a new iterative scheme by the viscosity approximation method:

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A)Sx_n. \tag{1.20}$$

They proved that the sequence $\{x_n\}$ generated by (1.20) converges strongly to the unique solution of the variational inequality:

$$\langle \gamma f z - Az, x - z \rangle \leq 0, \quad \forall x \in F(S), \tag{1.21}$$

which is the optimality condition for the minimization problem:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x \rangle - h(x), \tag{1.22}$$

where h is a potential function for γf.

In 2008, Qin et al. [14] proposed the following iterative algorithm:

$$\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S P_C(u_n - \alpha_n B u_n), \tag{1.23}$$

where A is a strongly positive linear bounded operator and B is a relaxed cocoercive mapping of C into H. They proved that if the sequences $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequence $\{x_n\}$ defined by (1.23) converges strongly to the unique solution z of the variational inequality:

$$\langle \gamma f z - Az, x - z\rangle \leq 0, \quad \forall x \in F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(\Theta), \tag{1.24}$$

which is the optimality condition for the minimization problem:

$$\min_{x \in F(S) \cap \mathrm{VI}(C,B) \cap \mathrm{EP}(\Theta)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.25}$$

where h is a potential function for γf.

In this paper, we introduce an iterative scheme by using a viscosity hybrid steepest descent method for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a nonexpansive mapping, and the set of solutions of variational inequality problem for a relaxed cocoercive mapping in a real Hilbert space. The results shown in this paper improve and extend the recent ones announced by many others.

2. Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space and C is a nonempty closed convex subset of H. For a sequence $\{x_n\}$, the notation of $x_n \rightharpoonup x$ and $x_n \to x$ means that the sequence $\{x_n\}$ converges weakly and strongly to x, respectively.

The following lemmata give some characterizations and useful properties of the metric projection P_C in a real Hilbert space. The metric (or *nearest point*) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the following property:

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|. \tag{2.1}$$

Lemma 2.1. *It is well known that the metric projection P_C has the following properties:*

(m1) *for each $x \in H$ and $z \in C$,*

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C; \tag{2.2}$$

(m2) $P_C : H \to C$ *is nonexpansive, that is,*

$$\|P_C x - P_C y\| \le \|x - y\|, \quad \forall x, y \in H; \tag{2.3}$$

(m3) P_C *is firmly nonexpansive, that is,*

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y \rangle \quad \forall x, y \in H. \tag{2.4}$$

In order to prove our main results, we also need the following lemmata.

Lemma 2.2 (see [2]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let B be a mapping of C into H. Let $x^* \in C$. Then, for $\lambda > 0$,*

$$x^* \in VI(C, B) \Longleftrightarrow x^* = P_C(x^* - \lambda B x^*), \tag{2.5}$$

that is,

$$x^* \in VI(C, B) \Longleftrightarrow x^* \in F(P_C(I - \lambda B)), \tag{2.6}$$

where P_C is the metric projection of H onto C.

Lemma 2.3 (see [15]). *Let B be a monotone mapping of C into H and let $N_C w_1$ be the normal cone to C at $w_1 \in C$, that is,*

$$N_C w_1 = \{w \in H : \langle w_1 - w_2, w \rangle \ge 0, \forall w_2 \in C\}, \tag{2.7}$$

and define a mapping Q on C by

$$Q w_1 = \begin{cases} B w_1 + N_C w_1, & w_1 \in C, \\ \emptyset, & w_1 \notin C. \end{cases} \tag{2.8}$$

Then Q is maximal monotone and $0 \in Q w_1$ if and only if $\langle B w_1, w_2 - w_1 \rangle \ge 0$ for all $w_2 \in C$.

Lemma 2.4 (see [16]). *Each Hilbert space H satisfies Opials condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.9}$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.5 (see [13]). *Let C be a nonempty closed convex subset of H, let f be a contraction of H into itself with coefficient $\eta \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}/\eta$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\overline{\gamma} - \eta\gamma)\|x - y\|^2, \quad x, y \in H. \tag{2.10}$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \eta\gamma$.

Lemma 2.6 (see [13]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.7 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\gamma_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \tag{2.11}$$

Suppose

$$\begin{gathered} x_{n+1} = (1 - \gamma_n)y_n + \gamma_n x_n, \quad \forall n \geq 0, \\ \limsup_{n \to \infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \end{gathered} \tag{2.12}$$

Then, $\lim_{n \to \infty}\|y_n - x_n\| = 0$.

Lemma 2.8 (see [18]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \varrho_n)a_n + \sigma_n, \quad n \geq 0, \tag{2.13}$$

where $\{\varrho_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \varrho_n = \infty$,

(2) $\limsup_{n \to \infty}(\sigma_n/\varrho_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\sigma_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction Θ, the function φ, and the set C:

(H1) $\Theta(x, x) = 0$, $\forall x \in C$;

(H2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$, $\forall x, y \in C$;

(H3) for each $y \in C$, $x \mapsto \Theta(x, y)$ is weakly upper semicontinuous;

(H4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex;

(H5) for each $x \in C$, $y \mapsto \Theta(x, y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist abounded subsets $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.14}$$

(B2) C is a bounded set.

Lemma 2.9 (see [19]). *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction that satisfies (H1)–(H5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \left\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{2.15}$$

for all $z \in H$. Then, the following properties hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$;*

(ii) $T_r^{(\Theta,\varphi)}$ *is single-valued;*

(iii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{2.16}$$

(iv) $F(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(v) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Remark 2.10. If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

Lemma 2.11 (see [9]). *Let C, H, Θ, and T_r^{Θ} be as in Remark 2.10. Then the following holds:*

$$\left\|T_s^{\Theta}x - T_t^{\Theta}x\right\|^2 \leq \frac{s - t}{s}\left\langle T_s^{\Theta}x - T_t^{\Theta}x, T_s^{\Theta}x - x\right\rangle \tag{2.17}$$

for all $s, t > 0$ and $x \in H$.

The following lemma is an immediate consequence of an inner product.

Lemma 2.12. *Let H be a real Hilbert space, let x and y be elements in H, and let $\lambda \in [0, 1]$. Then*

(1) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$,

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

3. Main Results

In this section, we will introduce an iterative scheme by using a viscosity hybrid steepest descent method for finding a common element of the set of fixed points for nonexpansive mappings, the set of solutions of a generalized mixed equilibrium problem, and the set of solutions of variational inequality problem for a relaxed cocoercive mapping in a real Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets.

In order to prove our main results, we first prove the following lemmata.

Lemma 3.1. *Let C, H, Θ, φ, and $T_r^{(\Theta,\varphi)}$ be as in Lemma 2.9. Then the following holds:*

$$\left\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\right\|^2 \leq \frac{s-t}{s}\left\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\right\rangle \tag{3.1}$$

for all $s,t > 0$ and $x \in H$.

Proof. By similar argument as in the proof of Lemma 2.11 in [9], for $s,t > 0$ and $x \in H$. Observing that $\Im = T_s^{(\Theta,\varphi)}x$ and $\vartheta = T_t^{(\Theta,\varphi)}x$, we have

$$\Theta(\Im, y) + \varphi(y) - \varphi(\Im) + \frac{1}{s}\langle y - \Im, \Im - x\rangle \geq 0, \quad \forall y \in C, \tag{3.2}$$

$$\Theta(\vartheta, y) + \varphi(y) - \varphi(\vartheta) + \frac{1}{t}\langle y - \vartheta, \vartheta - x\rangle \geq 0, \quad \forall y \in C. \tag{3.3}$$

Putting $y = \vartheta$ in (3.2) and $y = \Im$ in (3.3), we obtain

$$\begin{gathered}\Theta(\Im, \vartheta) + \varphi(\vartheta) - \varphi(\Im) + \frac{1}{s}\langle \vartheta - \Im, \Im - x\rangle \geq 0, \\ \Theta(\vartheta, \Im) + \varphi(\Im) - \varphi(\vartheta) + \frac{1}{t}\langle \Im - \vartheta, \vartheta - x\rangle \geq 0.\end{gathered} \tag{3.4}$$

So, summing up these two equalities and using the monotonicity of Θ (H2), we get

$$\frac{1}{s}\langle \vartheta - \Im, \Im - x\rangle + \frac{1}{t}\langle \Im - \vartheta, \vartheta - x\rangle \geq 0, \tag{3.5}$$

and hence

$$\left\langle \Im - \vartheta, \frac{\vartheta - x}{t} - \frac{\Im - x}{s}\right\rangle \geq 0. \tag{3.6}$$

We derive from (3.6) that

$$\left\langle \Im - \vartheta, \vartheta - x - \frac{t}{s}(\Im - x)\right\rangle \geq 0, \tag{3.7}$$

and so

$$-\|\Im-\vartheta\|^2+\left\langle \Im-\vartheta,\left(1-\frac{t}{s}\right)(\Im-x)\right\rangle \geq 0. \tag{3.8}$$

This indicates that

$$\left(1-\frac{t}{s}\right)\langle \Im-\vartheta,\Im-x\rangle \geq \|\Im-\vartheta\|^2. \tag{3.9}$$

In other words,

$$\|\Im-\vartheta\|^2 \leq \frac{s-t}{s}\langle \Im-\vartheta,\Im-x\rangle, \tag{3.10}$$

and thus the claim holds. □

Lemma 3.2. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, let $S : C \to C$ be a nonexpansive mapping, and let $B : C \to H$ be an ω-Lipschitz continuous and relaxed (u, v)-cocoercive mappings with $v > u\omega^2$. If $0 \leq \alpha_n \leq 2(v-u\omega^2)/\omega^2$, then $S-\alpha_n BS$ is a nonexpansive mapping in H.*

Proof. Let $\alpha_n \leq 2(v-u\omega^2)/\omega^2$, $v > u\omega^2$. Then, for every $x, y \in C$, we have

$$\begin{aligned}
&\left\|(S-\alpha_n BS)x-(S-\alpha_n BS)y\right\|^2\\
&\quad=\left\|(Sx-Sy)-\alpha_n(BSx-BSy)\right\|^2\\
&\quad=\left\|Sx-Sy\right\|^2-2\alpha_n\langle Sx-Sy,BSx-BSy\rangle+\alpha_n^2\left\|BSx-BSy\right\|^2\\
&\quad\leq\left\|Sx-Sy\right\|^2-2\alpha_n\left\{-u\left\|BSx-BSy\right\|^2+v\left\|Sx-Sy\right\|^2\right\}+\alpha_n^2\left\|BSx-BSy\right\|^2\\
&\quad\leq\left\|Sx-Sy\right\|^2+2\alpha_n u\omega^2\left\|Sx-Sy\right\|^2-2\alpha_n v\left\|Sx-Sy\right\|^2+\alpha_n^2\omega^2\left\|Sx-Sy\right\|^2\\
&\quad=\left(1+2\alpha_n u\omega^2-2\alpha_n v+\alpha_n^2\omega^2\right)\left\|Sx-Sy\right\|^2\\
&\quad\leq\left(1-\alpha_n\omega^2\left[\frac{2(v-u\omega^2)}{\omega^2}-\alpha_n\right]\right)\left\|x-y\right\|^2.
\end{aligned} \tag{3.11}$$

Now, since $(1-\alpha_n\omega^2[2(v-u\omega^2)/\omega^2-\alpha_n]) < 1$, thus $\|(S-\alpha_n BS)x-(S-\alpha_n BS)y\| \leq \|x-y\|$.

Thus, $S-\alpha_n BS$ is a nonexpansive mapping of C into H. □

Now we can prove that a strong convergence theorem is a real Hilbert space.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ_1 and Θ_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (H1)–(H5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower* *semicontinuous and convex function with assumption (B1) or (B2). Let*

(i) $\Psi_1 : C \to H$ *be a* ξ*-inverse-strongly monotone mapping,*

(ii) $\Psi_2 : C \to H$ *be a* β*-inverse-strongly monotone mapping,*

(iii) $B : C \to H$ *be an* ω*-Lipschitz continuous and relaxed* (u, v)*-cocoercive mappings,*

(iv) $f : C \to C$ *be a contraction mapping with coefficient* $\eta \in (0,1)$ *and let* A *be a strongly positive linear bounded self-adjoint operator with the coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\eta$.

Let $S : C \to C$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$.

Assume that

$$\digamma := F(S) \cap GMEP(\Theta_1, \varphi, \Psi_1) \cap GMEP(\Theta_2, \varphi, \Psi_2) \cap VI(C, B) \neq \emptyset. \tag{3.12}$$

Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{v_n\}$, *and* $\{u_n\}$ *be the sequences generated by*

$$\begin{aligned}
u_n &= T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n \Psi_1 x_n),\\
v_n &= T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n \Psi_2 u_n),\\
z_n &= P_C(S v_n - \alpha_n B S v_n),\\
y_n &= \epsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big) z_n,\\
x_{n+1} &= \gamma_n x_n + (1-\gamma_n) y_n, \quad \forall n \geq 1,
\end{aligned} \tag{3.13}$$

where $\{r_n\} \subset [a,b] \subset [0,2\xi]$, $\{s_n\} \subset [c,d] \subset [0,2\beta]$, $\{\gamma_n\} \subset [h,j] \subset (0,1)$, *and* $\{\gamma_n\}$, $\{\epsilon_n\}$, *and* $\{\beta_n\}$ *are three sequences in* $(0,1)$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, *and* $\lim_{n\to\infty} \beta_n = 0$,

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\xi$, *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4) $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < 2\beta$, *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,

(C5) $\{\alpha_n\} \subset [e, g] \subset (0, 2(v - u\omega^2)/\omega^2)$, $v > u\omega^2$, *and* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ *converges strongly to* $z = P_{\digamma}(\gamma f + (I - A))(z)$, *which is the unique solution of the variational inequality:*

$$\langle \gamma f(z) - Az, x - z \rangle \leq 0, \quad \forall x \in \digamma. \tag{3.14}$$

Proof. From the restrictions on control sequences, we may assume, without loss of generality, that $\epsilon_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \geq 1$. Since A is a strongly positive linear bounded self-adjoint operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \ \|x\| = 1\}. \tag{3.15}$$

Observe that

$$\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle = 1-\beta_n-\epsilon_n\langle Ax,x\rangle \geq 1-\beta_n-\epsilon_n\|A\| \geq 0. \tag{3.16}$$

That is, $(1-\beta_n)I-\epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n A\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle| : x\in H,\ \|x\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n\langle Ax,x\rangle : x\in H,\ \|x\|=1\}\\ &\leq 1-\beta_n-\epsilon_n\overline{\gamma}.\end{aligned} \tag{3.17}$$

We will split the proof of Theorem 3.3 into six steps.

Step 1. We claim that the sequence $\{x_n\}$ is bounded.

Indeed, let $x^*\in\mathcal{F}$, by Lemmas 2.2 and 2.9, we obtain

$$x^* = Sx^* = P_C(x^*-\alpha_n Bx^*) = T_{r_n}^{(\Theta_1,\varphi)}(x^*-r_n\Psi_1 x^*) = T_{s_n}^{(\Theta_2,\varphi)}(x^*-s_n\Psi_2 x^*). \tag{3.18}$$

Since $u_n = T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\in \operatorname{dom}\varphi$, Ψ_1 is ξ-inverse-strongly monotone, and $0\leq r_n\leq 2\xi$, we know that, for any $n\in\mathbb{N}$, we have

$$\begin{aligned}\|u_n-x^*\|^2 &= \left\|T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x^*-r_n\Psi_1 x^*)\right\|^2\\ &\leq \|(x_n-r_n\Psi_1 x_n)-(x^*-r_n\Psi_1 x^*)\|^2\\ &= \|(x_n-x^*)-r_n(\Psi_1 x_n-\Psi_1 x^*)\|^2\\ &= \|x_n-x^*\|^2-2r_n\langle x_n-x^*,\Psi_1 x_n-\Psi_1 x^*\rangle + r_n^2\|\Psi_1 x_n-\Psi_1 x^*\|^2\\ &\leq \|x_n-x^*\|^2-2r_n\xi\|\Psi_1 x_n-\Psi_1 x^*\|^2 + r_n^2\|\Psi_1 x_n-\Psi_1 x^*\|^2\\ &= \|x_n-x^*\|^2+r_n(r_n-2\xi)\|\Psi_1 x_n-\Psi_1 x^*\|^2\\ &\leq \|x_n-x^*\|^2.\end{aligned} \tag{3.19}$$

Similarly, from (3.19), $v_n = T_{s_n}^{(\Theta_2,\varphi)}(u_n-s_n\Psi_2 u_n)\in\operatorname{dom}\varphi$, and $0\leq s_n\leq 2\beta$, we can prove that

$$\begin{aligned}\|v_n-x^*\|^2 &= \left\|T_{s_n}^{(\Theta_2,\varphi)}(u_n-s_n\Psi_2 u_n)-T_{s_n}^{(\Theta_2,\varphi)}(x^*-s_n\Psi_2 x^*)\right\|^2\\ &\leq \|u_n-x^*\|^2 \leq \|x_n-x^*\|^2,\end{aligned} \tag{3.20}$$

and hence

$$\|v_n-x^*\| \leq \|u_{\rm n}-x^*\| \leq \|x_n-x^*\|. \tag{3.21}$$

Let $x^* \in \mathcal{F}$, and form Lemma 3.2 $S - \alpha_n BS$ is a nonexpansive mapping and from Lemma 2.2 $x^* = P_C(x^* - \alpha_n Bx^*)$, we have

$$\begin{aligned}
\|z_n - x^*\| &= \|P_C(Sv_n - \alpha_n BSv_n) - P_C(x^* - \alpha_n Bx^*)\| \\
&\le \|(Sv_n - \alpha_n BSv_n) - (x^* - \alpha_n Bx^*)\| \\
&= \|(Sv_n - \alpha_n BSv_n) - (Sx^* - \alpha_n BSx^*)\| \\
&= \|(S - \alpha_n BS)v_n - (S - \alpha_n BS)x^*\| \\
&\le \|v_n - x^*\| \le \|x_n - x^*\|, \\
\left\|y_n - x^*\right\| &= \left\|\epsilon_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \epsilon_n A)(z_n - x^*)\right\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|z_n - x^*\| + \beta_n\|x_n - x^*\| + \epsilon_n\left\|\gamma f(x_n) - Ax^*\right\| \\
&\le (1 - \beta_n - \epsilon_n\overline{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \epsilon_n\left\|\gamma f(x_n) - Ax^*\right\| \\
&\le (1 - \epsilon_n\overline{\gamma})\|x_n - x^*\| + \epsilon_n\gamma\left\|f(x_n) - f(x^*)\right\| + \epsilon_n\left\|\gamma f(x^*) - Ax^*\right\| \\
&\le (1 - \epsilon_n\overline{\gamma})\|x_n - x^*\| + \epsilon_n\gamma\eta\|x_n - x^*\| + \epsilon_n\left\|\gamma f(x^*) - Ax^*\right\| \\
&= (1 - (\overline{\gamma} - \eta\gamma)\epsilon_n)\|x_n - x^*\| + \epsilon_n\left\|\gamma f(x^*) - Ax^*\right\|,
\end{aligned} \tag{3.22}$$

which yields that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\le \gamma_n\|x_n - x^*\| + (1-\gamma_n)\left\|y_n - x^*\right\| \\
&\le \gamma_n\|x_n - x^*\| + (1-\gamma_n)\left\{(1 - (\overline{\gamma} - \eta\gamma)\epsilon_n)\|x_n - x^*\| + \epsilon_n\left\|\gamma f(x^*) - Ax^*\right\|\right\} \\
&= \gamma_n\|x_n - x^*\| + (1-\gamma_n)\|x_n - x^*\| - (1-\gamma_n)(\overline{\gamma} - \eta\gamma)\epsilon_n\|x_n - x^*\| + (1-\gamma_n)\epsilon_n\left\|\gamma f(x^*) - Ax^*\right\| \\
&= (1 - (1-\gamma_n)(\overline{\gamma} - \eta\gamma)\epsilon_n)\|x_n - x^*\| + (1-\gamma_n)\epsilon_n\left\|\gamma f(x^*) - Ax^*\right\|.
\end{aligned} \tag{3.23}$$

By mathematical induction, putting $D = \max\{\|x_1 - x^*\|, \|\gamma f(x^*) - Ax^*\|/\overline{\gamma} - \eta\gamma\}$, we have that $\|x_n - x^*\| \le D$ for all $n \ge 1$. Indeed, we can easily see that $\|x_1 - x^*\| \le D$. Suppose that $\|x_k - x^*\| \le D$ for some positive integral k. Then we have that

$$\begin{aligned}
\|x_{k+1} - x^*\| &\le (1 - (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k)\|x_k - x^*\| + (1-\gamma_k)\epsilon_k\left\|\gamma f(x^*) - Ax^*\right\| \\
&\le (1 - (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k)D + (1-\gamma_k)\epsilon_k\left\|\gamma f(x^*) - Ax^*\right\| \\
&= (1 - (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k)D + (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k\frac{\left\|\gamma f(x^*) - Ax^*\right\|}{(\overline{\gamma} - \eta\gamma)} \\
&\le (1 - (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k)D + (1-\gamma_k)(\overline{\gamma} - \eta\gamma)\epsilon_k D \\
&= D.
\end{aligned} \tag{3.24}$$

This shows that $\{x_n\}$ is bounded in H. From (3.21), we know that $\{u_n\}$ and $\{v_n\}$ are bounded in C and so $\{y_n\}$, $\{\Psi_1 u_n\}$, $\{\Psi_2 x_n\}$, $\{Sv_n\}$, $\{BSv_n\}$, and $\{f(x_n)\}$ are bounded sequence in H.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Since $S-\alpha_n BS$ is nonexpansive, we have

$$
\begin{aligned}
\|z_{n+1}-z_n\| &= \|P_C(Sv_{n+1}-\alpha_{n+1}BSv_{n+1})-P_C(Sv_n-\alpha_n BSv_n)\| \\
&\le \|(Sv_{n+1}-\alpha_{n+1}BSv_{n+1})-(Sv_n-\alpha_n BSv_n)\| \\
&= \|(Sv_{n+1}-\alpha_{n+1}BSv_{n+1})-(Sv_n-\alpha_{n+1}BSv_n)+(\alpha_n-\alpha_{n+1})BSv_n\| \\
&\le \|(Sv_{n+1}-\alpha_{n+1}BSv_{n+1})-(Sv_n-\alpha_{n+1}BSv_n)\|+|\alpha_n-\alpha_{n+1}|\|BSv_n\| \\
&\le \|v_{n+1}-v_n\|+|\alpha_n-\alpha_{n+1}|\|BSv_n\|.
\end{aligned}
\tag{3.25}
$$

Next, we estimate $\|u_{n+1}-u_n\|$. Observing that $u_n=T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)$ and $u_{n+1}=T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_{n+1}-r_{n+1}\Psi_1 x_{n+1})$, we have

$$
\begin{aligned}
\|u_{n+1}-u_n\| &= \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_{n+1}-r_{n+1}\Psi_1 x_{n+1})-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&= \Big\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_{n+1}-r_{n+1}\Psi_1 x_{n+1})-T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n) \\
&\quad + T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\Big\| \\
&\le \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_{n+1}-r_{n+1}\Psi_1 x_{n+1})-T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&\le \|(x_{n+1}-r_{n+1}\Psi_1 x_{n+1})-(x_n-r_n\Psi_1 x_n)\| \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&= \|x_{n+1}-x_n-r_{n+1}(\Psi_1 x_{n+1}-\Psi_1 x_n)+(r_{n+1}-r_n)\Psi_1 x_n\| \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&\le \|x_{n+1}-x_n-r_{n+1}(\Psi_1 x_{n+1}-\Psi_1 x_n)\|+|r_{n+1}-r_n|\|\Psi_1 x_n\| \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\| \\
&\le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|\Psi_1 x_n\| \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)-T_{r_n}^{(\Theta_1,\varphi)}(x_n-r_n\Psi_1 x_n)\right\|.
\end{aligned}
\tag{3.26}
$$

Similarly, we can prove that

$$\|v_{n+1}-v_n\| \le \|u_{n+1}-u_n\| + |s_{n+1}-s_n|\|\Psi_2 u_n\| + \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\|. \tag{3.27}$$

Substitution (3.26) into (3.27), we derive

$$\begin{aligned}\|v_{n+1}-v_n\| \le{}& \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|\Psi_1 x_n\| + |s_{n+1}-s_n|\|\Psi_2 u_n\| \\ &+ \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\ &+ \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\|.\end{aligned} \tag{3.28}$$

Since $\{\Psi_1 x_n\}$, $\{\Psi_2 u_n\}$, and $\{BSv_n\}$ are bounded, K is an appropriate constant such that

$$K \ge \max\left\{\sup_{n\ge1}\{\|\Psi_1 x_n\|\}, \sup_{n\ge1}\{\|\Psi_2 u_n\|\}, \sup_{n\ge1}\{\|BSv_n\|\}\right\}. \tag{3.29}$$

Substitution (3.28) into (3.25), we obtain

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|v_{n+1}-v_n\| + |\alpha_n-\alpha_{n+1}|\|BSv_n\| \\ &\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|\Psi_1 x_n\| + |s_{n+1}-s_n|\|\Psi_2 u_n\| + |\alpha_n-\alpha_{n+1}|\|BSv_n\| \\ &\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\ &\quad + \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\| \\ &\le \|x_{n+1}-x_n\| + K(|r_{n+1}-r_n| + |s_{n+1}-s_n| + |\alpha_n-\alpha_{n+1}|) \\ &\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\ &\quad + \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\|.\end{aligned} \tag{3.30}$$

From (3.13), we have

$$\begin{aligned} y_n &= \epsilon_n\gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \epsilon_n A\big)z_n, \\ y_{n+1} &= \epsilon_{n+1}\gamma f(x_{n+1}) + \beta_{n+1}x_{n+1} + \big((1-\beta_{n+1})I - \epsilon_{n+1}A\big)z_{n+1}.\end{aligned} \tag{3.31}$$

Simple calculations show that

$$\begin{aligned} y_{n+1} - y_n &= \epsilon_{n+1}\gamma\big(f(x_{n+1}) - f(x_n)\big) + (\epsilon_{n+1} - \epsilon_n)\gamma f(x_n) + \beta_{n+1}(x_{n+1} - x_n) \\ &\quad + (\beta_{n+1} - \beta_n)x_n + \big[(1-\beta_{n+1})I - \epsilon_{n+1}A\big](z_{n+1} - z_n) \\ &\quad - (\beta_{n+1} - \beta_n)z_n - (\epsilon_{n+1} - \epsilon_n)Az_n \\ &= \epsilon_{n+1}\gamma\big(f(x_{n+1}) - f(x_n)\big) + (\epsilon_{n+1} - \epsilon_n)\big(\gamma f(x_n) - Az_n\big) \\ &\quad + \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)(x_n - z_n) \\ &\quad + \big[(1-\beta_{n+1})I - \epsilon_{n+1}A\big](z_{n+1} - z_n), \end{aligned} \tag{3.32}$$

which yields that

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \epsilon_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\epsilon_{n+1} - \epsilon_n|\|\gamma f(x_n) - Az_n\| \\ &\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n - z_n\| + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma})\|z_{n+1} - z_n\| \\ &\le \epsilon_{n+1}\gamma\eta\|x_{n+1} - x_n\| + |\epsilon_{n+1} - \epsilon_n|\|\gamma f(x_n) - Az_n\| \\ &\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n - z_n\| + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma})\|z_{n+1} - z_n\|. \end{aligned} \tag{3.33}$$

Substitution (3.30) into (3.33) yields that

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \epsilon_{n+1}\gamma\eta\|x_{n+1} - x_n\| + |\epsilon_{n+1} - \epsilon_n|\|\gamma f(x_n) - Az_n\| \\ &\quad + \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|x_n - z_n\| + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma}) \\ &\quad \times \Big\{\|x_{n+1} - x_n\| + K(|r_{n+1} - r_n| + |s_{n+1} - s_n| + |\alpha_n - \alpha_{n+1}|) \\ &\qquad + \Big\|T^{(\Theta_1,\varphi)}_{r_{n+1}}(x_n - r_n\Psi_1 x_n) - T^{(\Theta_1,\varphi)}_{r_n}(x_n - r_n\Psi_1 x_n)\Big\| \\ &\qquad + \Big\|T^{(\Theta_2,\varphi)}_{s_{n+1}}(u_n - s_n\Psi_2 u_n) - T^{(\Theta_2,\varphi)}_{s_n}(u_n - s_n\Psi_2 u_n)\Big\|\Big\} \\ &= \big(1 - \epsilon_{n+1}(\overline{\gamma} - \gamma\eta)\big)\|x_{n+1} - x_n\| + |\epsilon_{n+1} - \epsilon_n|\|\gamma f(x_n) - Az_n\| \\ &\quad + |\beta_{n+1} - \beta_n|\|x_n - z_n\| + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma})K \\ &\quad \times (|r_{n+1} - r_n| + |s_{n+1} - s_n| + |\alpha_n - \alpha_{n+1}|) + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma}) \\ &\quad \times \Big\|T^{(\Theta_1,\varphi)}_{r_{n+1}}(x_n - r_n\Psi_1 x_n) - T^{(\Theta_1,\varphi)}_{r_n}(x_n - r_n\Psi_1 x_n)\Big\| \\ &\quad + (1 - \beta_{n+1} - \epsilon_{n+1}\overline{\gamma})\Big\|T^{(\Theta_2,\varphi)}_{s_{n+1}}(u_n - s_n\Psi_2 u_n) - T^{(\Theta_2,\varphi)}_{s_n}(u_n - s_n\Psi_2 u_n)\Big\| \end{aligned}$$

$$\begin{aligned}
&\leq \left(1-\epsilon_{n+1}\left(\overline{\gamma}-\gamma\eta\right)\right)\|x_{n+1}-x_n\| \\
&\quad + M\left(|\epsilon_{n+1}-\epsilon_n| + \left|\beta_{n+1}-\beta_n\right| + |r_{n+1}-r_n| + |s_{n+1}-s_n| + |\alpha_n-\alpha_{n+1}|\right) \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\
&\quad + \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\|,
\end{aligned} \tag{3.34}$$

where M is an appropriate constant such that

$$M \geq \max\left\{\sup_{n\geq 1}\left\{\left\|\gamma f(x_n) - Az_n\right\|\right\}, \sup_{n\geq 1}\{\|x_n - z_n\|\}, K\right\}. \tag{3.35}$$

Since $x_{n+1} = \gamma_n x_n + (1-\gamma_n)y_n$ and

$$\begin{aligned}
\left\|y_{n+1}-y_n\right\| - \|x_{n+1}-x_n\| &\leq M\left(|\epsilon_{n+1}-\epsilon_n| + \left|\beta_{n+1}-\beta_n\right| + |r_{n+1}-r_n| + |s_{n+1}-s_n| + |\alpha_n-\alpha_{n+1}|\right) \\
&\quad + \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\
&\quad + \left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\|,
\end{aligned} \tag{3.36}$$

next, we estimate

$$\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_{\mathrm{n}} - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\|. \tag{3.37}$$

Note that $\liminf_{n\to\infty} r_n > 0$; there exists a constant $\tilde{r} > 0$ such that $r_n \geq \tilde{r} > 0$ for all $n \geq 1$. From Lemma 3.1, we get

$$\begin{aligned}
&\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\|^2 \\
&\quad \leq \frac{r_{n+1}-r_n}{r_{n+1}}\Big\langle T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n), \\
&\qquad\qquad T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - (x_n - r_n\Psi_1 x_n)\Big\rangle \\
&\quad = \Big\langle \frac{r_{n+1}-r_n}{r_{n+1}}\left(T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right), \\
&\qquad T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - (x_n - r_n\Psi_1 x_n)\Big\rangle \\
&\quad \leq \frac{|r_{n+1}-r_n|}{r_{n+1}}\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\
&\qquad \left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - (x_n - r_n\Psi_1 x_n)\right\|.
\end{aligned} \tag{3.38}$$

It follows that

$$\begin{aligned}
&\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| \\
&\quad \le \frac{|r_{n+1} - r_n|}{\widetilde{r}}\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - (x_n - r_n\Psi_1 x_n)\right\|.
\end{aligned} \tag{3.39}$$

Since $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$, we obtain

$$\lim_{n\to\infty}\left\|T_{r_{n+1}}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)\right\| = 0. \tag{3.40}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\left\|T_{s_{n+1}}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n)\right\| = 0. \tag{3.41}$$

Consequently, from (3.40), (3.41), and conditions in Theorem 3.3, we obtain

$$\lim_{n\to\infty}\left(\left\|y_{n+1} - y_n\right\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.42}$$

It follows from Lemma 2.7 that

$$\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{3.43}$$

In view of (3.13), we see that

$$\|x_{n+1} - x_n\| = \left(1 - \gamma_n\right)\left\|x_n - y_n\right\|, \quad \forall n \ge 1, \tag{3.44}$$

which, combining with (3.43) and $0 < h \le \gamma_n \le j < 1$, yields that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.45}$$

Step 3. We claim that $\lim_{n\to\infty}\|Sz_n - z_n\| = 0$.

Observing that $y_n = \epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)z_n$, we have

$$\left\|y_n - z_n\right\| \le \epsilon_n\left\|\gamma f(x_n) - Az_n\right\| + \beta_n\|x_n - z_n\|, \tag{3.46}$$

which, combining with the conditions (C1) and (C2), gives

$$\lim_{n\to\infty}\left\|y_n - z_n\right\| = 0. \tag{3.47}$$

From (3.43) and (3.47), we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.48}$$

For any $x^* \in \mathcal{F}$, we see that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(Sv_n - \alpha_n BSv_n) - P_C(x^* - \alpha_n Bx^*)\|^2 \\
&\le \|(Sv_n - \alpha_n BSv_n) - (x^* - \alpha_n Bx^*)\|^2 \\
&= \|(Sv_n - x^*) - \alpha_n (BSv_n - Bx^*)\|^2 \\
&= \|Sv_n - x^*\|^2 - 2\alpha_n \langle Sv_n - x^*, BSv_n - Bx^* \rangle + \alpha_n^2 \|BSv_n - Bx^*\|^2 \\
&\le \|v_n - x^*\|^2 - 2\alpha_n \langle Sv_n - x^*, BSv_n - Bx^* \rangle + \alpha_n^2 \|BSv_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 - 2\alpha_n \left\{ -u\|BSv_n - Bx^*\|^2 + v\|Sv_n - x^*\|^2 \right\} + \alpha_n^2 \|BSv_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 + 2\alpha_n u\|BSv_n - Bx^*\|^2 - 2\alpha_n v\|Sv_n - x^*\|^2 + \alpha_n^2 \|BSv_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 + 2\alpha_n u\|BSv_n - Bx^*\|^2 - \frac{2\alpha_n v}{\omega^2}\|BSv_n - Bx^*\|^2 + \alpha_n^2 \|BSv_n - Bx^*\|^2 \\
&= \|x_n - x^*\|^2 + \left(2\alpha_n u + \alpha_n^2 - \frac{2\alpha_n v}{\omega^2} \right) \|BSv_n - Bx^*\|^2,
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
&\left\| y_n - x^* \right\|^2 \\
&= \left\| \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*) + \beta_n (x_n - x^*) + \epsilon_n \left(\gamma f(x_n) - Ax^* \right) \right\|^2 \\
&= \left\| \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*) + \beta_n (x_n - x^*) \right\|^2 \\
&\quad + \epsilon_n^2 \left\| \gamma f(x_n) - Ax^* \right\|^2 + 2\beta_n \epsilon_n \left\langle x_n - x^*, \gamma f(x_n) - Ax^* \right\rangle \\
&\quad + 2\epsilon_n \left\langle \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*), \gamma f(x_n) - Ax^* \right\rangle \\
&\le \left((1-\beta_n - \epsilon_n \overline{\gamma}) \|z_n - x^*\| + \beta_n \|x_n - x^*\| \right)^2 + \epsilon_n^2 \left\| \gamma f(x_n) - Ax^* \right\|^2 \\
&\quad + 2\beta_n \epsilon_n \langle x_n - x^*, \gamma f(x_n) - Ax^* \rangle + 2\epsilon_n \left\langle \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*), \gamma f(x_n) - Ax^* \right\rangle \\
&= (1-\beta_n - \epsilon_n \overline{\gamma})^2 \|z_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + 2(1-\beta_n - \epsilon_n \overline{\gamma}) \beta_n \|z_n - x^*\| \|x_n - x^*\| \\
&\quad + \epsilon_n^2 \left\| \gamma f(x_n) - Ax^* \right\|^2 + 2\beta_n \epsilon_n \left\langle x_n - x^*, \gamma f(x_n) - Ax^* \right\rangle \\
&\quad + 2\epsilon_n \left\langle \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*), \gamma f(x_n) - Ax^* \right\rangle \\
&\le (1-\beta_n - \epsilon_n \overline{\gamma})^2 \|z_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2 + (1-\beta_n - \epsilon_n \overline{\gamma}) \beta_n \left\{ \|z_n - x^*\|^2 + \|x_n - x^*\|^2 \right\} \\
&\quad + \epsilon_n^2 \left\| \gamma f(x_n) - Ax^* \right\|^2 + 2\beta_n \epsilon_n \left\langle x_n - x^*, \gamma f(x_n) - Ax^* \right\rangle \\
&\quad + 2\epsilon_n \left\langle \left((1-\beta_n) I - \epsilon_n A \right)(z_n - x^*), \gamma f(x_n) - Ax^* \right\rangle \\
&= \left[(1-\epsilon_n \overline{\gamma})^2 - 2(1-\epsilon_n \overline{\gamma}) \beta_n + \beta_n^2 \right] \|z_n - x^*\|^2 + \beta_n^2 \|x_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\left((1-\epsilon_n\overline{\gamma})\beta_n-\beta_n^2\right)\left(\|z_n-x^*\|^2+\|x_n-x^*\|^2\right)+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2\\
&+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&=\left[(1-\epsilon_n\overline{\gamma})^2-(1-\epsilon_n\overline{\gamma})\beta_n\right]\|z_n-x^*\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-x^*\|^2\\
&+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&=(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|z_n-x^*\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-x^*\|^2\\
&+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&\le(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n-x^*\|^2+\left(2\alpha_nu+\alpha_n^2-\frac{2\alpha_nv}{\omega^2}\right)\|BSv_n-Bx^*\|\right\}\\
&+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&=(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|x_n-x^*\|^2+(1-\epsilon_n\overline{\gamma})\beta_n\|x_n-x^*\|^2\\
&+(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(2\alpha_nu+\alpha_n^2-\frac{2\alpha_nv}{\omega^2}\right)\|BSv_n-Bx^*\|+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2\\
&+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&=(1-\epsilon_n\overline{\gamma})^2\|x_n-x^*\|^2+(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(2\alpha_nu+\alpha_n^2-\frac{2\alpha_nv}{\omega^2}\right)\|BSv_n-Bx^*\|\\
&+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\varepsilon_n\langle((1-\beta_n)I-\varepsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&\le\|x_n-x^*\|^2+(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma})\left(2\alpha_nu+\alpha_n^2-\frac{2\alpha_nv}{\omega^2}\right)\|BSv_n-Bx^*\|\\
&+\varepsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\varepsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&+2\varepsilon_n\langle((1-\beta_n)I-\varepsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle.
\end{aligned}
\tag{3.50}
$$

Furthermore, from (3.13) and Lemma 2.12(1), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2&=\left\|\gamma_nx_n+(1-\gamma_n)y_n-x^*\right\|^2\\
&\le\gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\left\|y_n-x^*\right\|^2\\
&\le\gamma_n\|x_n-x^*\|^2+(1-\gamma_n)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big\{ \|x_n - x^*\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(2\alpha_n u + \alpha_n^2 - \frac{2\alpha_n v}{\omega^2}\right)\|BSv_n - Bx^*\| \\
&\quad + \epsilon_n^2 \big\|\gamma f(x_n) - Ax^*\big\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\big\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\big\rangle \Big\} \\
&\le \|x_n - x^*\|^2 + (1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(2\alpha_n u + \alpha_n^2 - \frac{2\alpha_n v}{\omega^2}\right)\|BSv_n - Bx^*\| \\
&\quad + (1-\gamma_n)\epsilon_n^2\big\|\gamma f(x_n) - Ax^*\big\|^2 + 2(1-\gamma_n)\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2(1-\gamma_n)\epsilon_n\big\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\big\rangle.
\end{aligned}
\tag{3.51}
$$

It follows that

$$
\begin{aligned}
&(1-j)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(\frac{2ev}{\omega^2} - 2gu - g^2\right)\|BSv_n - Bx^*\| \\
&\quad \le (1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left(\frac{2\alpha_n v}{\omega^2} - 2\alpha_n u - \alpha_n^2\right)\|BSv_n - Bx^*\| \\
&\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1-\gamma_n)\epsilon_n^2\big\|\gamma f(x_n) - Ax^*\big\|^2 \\
&\qquad + 2(1-\gamma_n)\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\qquad + 2(1-\gamma_n)\epsilon_n\big\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\big\rangle \\
&\quad \le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\qquad + (1-\gamma_n)\epsilon_n^2\big\|\gamma f(x_n) - Ax^*\big\|^2 + 2(1-\gamma_n)\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\qquad + 2(1-\gamma_n)\epsilon_n\big\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\big\rangle.
\end{aligned}
\tag{3.52}
$$

From the condition (C1) and (3.45), we arrive at

$$
\lim_{n\to\infty}\|BSv_n - Bx^*\| = 0. \tag{3.53}
$$

On the other hand, we have

$$
\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(Sv_n - \alpha_n BSv_n) - P_C(x^* - \alpha_n Bx^*)\|^2 \\
&\le \langle (Sv_n - \alpha_n BSv_n) - (x^* - \alpha_n Bx^*), z_n - x^* \rangle \\
&= \frac{1}{2}\Big\{ \|(Sv_n - \alpha_n BSv_n) - (x^* - \alpha_n Bx^*)\|^2 + \|z_n - x^*\|^2 \\
&\qquad - \|(Sv_n - \alpha_n BSv_n) - (x^* - \alpha_n Bx^*) - (z_n - x^*)\|^2 \Big\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{2}\left\{\|v_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(Sv_n - z_n) - \alpha_n(BSv_n - Bx^*)\|^2\right\} \\
&\leq \frac{1}{2}\left\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|Sv_n - z_n\|^2 \right. \\
&\qquad \left. - \alpha_n^2\|BSv_n - Bx^*\|^2 + 2\alpha_n\langle Sv_n - z_n, BSv_n - Bx^*\rangle\right\},
\end{aligned} \tag{3.54}$$

which yields that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|Sv_n - z_n\|^2 + 2\alpha_n\|Sv_n - z_n\|\|BSv_n - Bx^*\|. \tag{3.55}$$

Substituting (3.55) into (3.50), we have

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|z_n - x^*\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\leq (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\left\{\|x_n - x^*\|^2 - \|Sv_n - z_n\|^2 + 2\alpha_n\|Sv_n - z_n\|\|BSv_n - Bx^*\|\right\} \\
&\quad + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&= (1 - \epsilon_n\overline{\gamma})^2\|x_n - x^*\|^2 - (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|Sv_n - z_n\|^2 \\
&\quad + (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})2\alpha_n\|Sv_n - z_n\|\|BSx_n - Bx^*\| \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\leq \|x_n - x^*\|^2 - (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|Sv_n - z_n\|^2 \\
&\quad + (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})2\alpha_n\|Sv_n - z_n\|\|BSv_n - Bx^*\| \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle.
\end{aligned} \tag{3.56}$$

Using (3.51) and (3.56), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\left\|y_n-x^*\right\|^2\\&\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\\&\quad\times\Big\{\|x_n-x^*\|^2-(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|Sv_n-z_n\|^2\\&\qquad+(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})2\alpha_n\|Sv_n-z_n\|\|BSv_n-Bx^*\|\\&\qquad+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\&\qquad+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(Sz_n-x^*),\gamma f(x_n)-Ax^*\rangle\Big\}\\&=\|x_n-x^*\|^2-(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|Sv_n-z_n\|^2\\&\quad+2(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|Sv_n-z_n\|\|BSv_n-Bx^*\|\\&\quad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\&\quad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(Sv_n-x^*),\gamma f(x_n)-Ax^*\rangle.\end{aligned}\tag{3.57}$$

It follows that

$$\begin{aligned}&(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|Sv_n-z_n\|^2\\&\quad\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\qquad+2(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|Sv_n-z_n\|\|BSv_n-Bx^*\|\\&\qquad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\&\qquad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(Sz_n-x^*),\gamma f(x_n)-Ax^*\rangle\\&\quad\le\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\&\qquad+2(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n\|Sv_n-z_n\|\|BSv_n-Bx^*\|\\&\qquad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\&\qquad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(Sz_n-x^*),\gamma f(x_n)-Ax^*\rangle.\end{aligned}\tag{3.58}$$

From condition (C1), (3.45), and (3.53), we obtain

$$\lim_{n\to\infty}\|Sv_n - z_n\| = 0. \tag{3.59}$$

Consequently, from (3.50) we derive that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|z_n - x^*\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|v_n - x^*\|^2 + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&= (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\|T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n\Psi_2 u_n) - T_{s_n}^{(\Theta_2,\varphi)}(x^* - s_n\Psi_2 x^*)\right\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\|(u_n - s_n\Psi_2 u_n) - (x^* - s_n\Psi_2 x^*)\|^2 \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|u_n - x^*\|^2 + s_n(s_n - 2\beta)\|\Psi_2 u_n - \Psi_2 x^*\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\le (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})\left\{\|x_n - x^*\|^2 + s_n(s_n - 2\beta)\|\Psi_2 u_n - \Psi_2 x^*\|^2\right\} \\
&\quad + (1-\epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle \\
&\le \|x_n - x^*\|^2 + (1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})s_n(s_n - 2\beta)\|\Psi_2 u_n - \Psi_2 x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\epsilon_n\langle((1-\beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^*\rangle.
\end{aligned} \tag{3.60}$$

From (3.51) and (3.60), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\left\|y_n-x^*\right\|^2\\
&\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\\
&\quad\times\Big\{\|x_n-x^*\|^2+(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})s_n(s_n-2\beta)\|\Psi_2u_n-\Psi_2x^*\|^2\\
&\qquad+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&\qquad+2\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\Big\}\\
&=\|x_n-x^*\|^2+(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})s_n(s_n-2\beta)\|\Psi_2u_n-\Psi_2x^*\|^2\\
&\quad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&\quad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle.
\end{aligned}\tag{3.61}$$

So, we obtain

$$\begin{aligned}
&(1-j)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})c(2\beta-d)\|\Psi_2u_n-\Psi_2x^*\|^2\\
&\quad\le(1-\gamma_n)(1-\epsilon_n\overline{\gamma})(1-\beta_n-\epsilon_n\overline{\gamma})s_n(2\beta-s_n)\|\Psi_2u_n-\Psi_2x^*\|^2\\
&\quad\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2\\
&\qquad+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&\qquad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle\\
&\quad\le\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2\\
&\qquad+2(1-\gamma_n)\beta_n\epsilon_n\langle x_n-x^*,\gamma f(x_n)-Ax^*\rangle\\
&\qquad+2(1-\gamma_n)\epsilon_n\langle((1-\beta_n)I-\epsilon_nA)(z_n-x^*),\gamma f(x_n)-Ax^*\rangle.
\end{aligned}\tag{3.62}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\epsilon_n=0$, we obtain

$$\lim_{n\to\infty}\|\Psi_2u_n-\Psi_2x^*\|=0.\tag{3.63}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|\Psi_1x_n-\Psi_1x^*\|=0.\tag{3.64}$$

In addition, from the firmly nonexpansivity of $T_{r_n}^{(\Theta_1,\varphi)}$, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\| T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n) - T_{r_n}^{(\Theta_1,\varphi)}(x^* - r_n\Psi_1 x^*)\right\|^2 \\
&\le \langle (x_n - r_n\Psi_1 x_n) - (x^* - r_n\Psi_1 x^*), u_n - x^* \rangle \\
&= \frac{1}{2}\Big\{ \|(x_n - r_n\Psi_1 x_n) - (x^* - r_n\Psi_1 x^*)\|^2 + \|u_n - x^*\|^2 \\
&\quad - \|(x_n - r_n\Psi_1 x_n) - (x^* - r_n\Psi_1 x^*) - (u_n - x^*)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n - r_n(\Psi_1 x_n - \Psi_1 x^*)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, \Psi_1 x_n - \Psi_1 x^* \rangle - r_n^2\|\Psi_1 x_n - \Psi_1 x^*\|^2 \Big\}.
\end{aligned} \tag{3.65}$$

So, we obtain

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|\Psi_1 x_n - \Psi_1 x^*\|. \tag{3.66}$$

Similarly, we can prove that

$$\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - v_n\|^2 + 2s_n\|u_n - v_n\|\|\Psi_2 u_n - \Psi_2 x^*\|. \tag{3.67}$$

Substituting (3.66) into (3.50), we have

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|z_n - x^*\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^* \rangle \\
&\quad + 2\epsilon_n\langle ((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^* \rangle \\
&\le (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\|u_n - x^*\|^2 + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 \\
&\quad + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^* \rangle \\
&\quad + 2\epsilon_n\langle ((1 - \beta_n)I - \epsilon_n A)(z_n - x^*), \gamma f(x_n) - Ax^* \rangle \\
&\le (1 - \epsilon_n\overline{\gamma})(1 - \beta_n - \epsilon_n\overline{\gamma})\Big\{ \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|\Psi_1 x_n - \Psi_1 x^*\| \Big\} \\
&\quad + (1 - \epsilon_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \epsilon_n^2\left\|\gamma f(x_n) - Ax^*\right\|^2 + 2\beta_n\epsilon_n\langle x_n - x^*, \gamma f(x_n) - Ax^* \rangle
\end{aligned}$$

$$
\begin{aligned}
&+2\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle\\
&=\left(1-\epsilon_n\overline{\gamma}\right)^2\|x_n-x^*\|^2-\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)\|x_n-u_n\|^2\\
&\quad+\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)2r_n\|x_n-u_n\|\|\Psi_1x_n-\Psi_1x^*\|\\
&\quad+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\left\langle x_n-x^*,\gamma f(x_n)-Ax^*\right\rangle\\
&\quad+2\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle\\
&\le\|x_n-x^*\|^2-\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)\|x_n-u_n\|^2\\
&\quad+\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)2r_n\|x_n-u_n\|\|\Psi_1x_n-\Psi_1x^*\|\\
&\quad+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\left\langle x_n-x^*,\gamma f(x_n)-Ax^*\right\rangle\\
&\quad+2\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle.
\end{aligned}
\tag{3.68}
$$

Using (3.51) and (3.68), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2&\le\gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\left\|y_n-x^*\right\|^2\\
&\le\gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\\
&\quad\times\Big\{\|x_n-x^*\|^2-\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)\|x_n-u_n\|^2\\
&\qquad+\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)2r_n\|x_n-u_n\|\|\Psi_1x_n-\Psi_1x^*\|\\
&\qquad+\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2\beta_n\epsilon_n\left\langle x_n-x^*,\gamma f(x_n)-Ax^*\right\rangle\\
&\qquad+2\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle\Big\}\\
&=\|x_n-x^*\|^2-(1-\gamma_n)\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)\|x_n-u_n\|^2\\
&\quad+2(1-\gamma_n)\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)r_n\|x_n-u_n\|\|\Psi_1x_n-\Psi_1x^*\|\\
&\quad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\left\langle x_n-x^*,\gamma f(x_n)-Ax^*\right\rangle\\
&\quad+2(1-\gamma_n)\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle.
\end{aligned}
\tag{3.69}
$$

It follows that

$$
\begin{aligned}
&(1-\gamma_n)\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)\|x_n-u_n\|^2\\
&\quad\le\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\
&\qquad+2(1-\gamma_n)\left(1-\epsilon_n\overline{\gamma}\right)\left(1-\beta_n-\epsilon_n\overline{\gamma}\right)r_n\|x_n-u_n\|\|\Psi_1x_n-\Psi_1x^*\|\\
&\qquad+(1-\gamma_n)\epsilon_n^2\left\|\gamma f(x_n)-Ax^*\right\|^2+2(1-\gamma_n)\beta_n\epsilon_n\left\langle x_n-x^*,\gamma f(x_n)-Ax^*\right\rangle\\
&\qquad+2(1-\gamma_n)\epsilon_n\left\langle\left((1-\beta_n)I-\epsilon_nA\right)(z_n-x^*),\gamma f(x_n)-Ax^*\right\rangle.
\end{aligned}
\tag{3.70}
$$

Since $\lim_{n\to\infty}\epsilon_n = 0$, $\lim_{n\to\infty}\|\Psi_1 x_{\mathrm{n}} - \Psi_1 x^*\| = 0$, and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.71}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.72}$$

Furthermore, by the triangular inequality, we also have

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\|. \tag{3.73}$$

Applying (3.71) and (3.72), we have

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.74}$$

Since

$$\|z_n - v_n\| \le \|z_n - x_n\| + \|x_n - v_n\|, \tag{3.75}$$

so we get

$$\lim_{n\to\infty}\|z_n - v_n\| = 0. \tag{3.76}$$

Also, observe that

$$\begin{aligned}\|Sz_n - z_n\| &\le \|Sz_n - Sv_n\| + \|Sv_n - z_n\|\\ &\le \|z_n - v_n\| + \|Sv_n - z_n\|.\end{aligned} \tag{3.77}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0. \tag{3.78}$$

Step 4. We prove that the mapping $P_{\mathfrak{F}}(\gamma f + (I - A))$ has a unique fixed point.

Since f is a contraction of C into itself with coefficient $\eta \in (0,1)$, then, we have

$$\begin{aligned}\left\|P_{\mathfrak{F}}(\gamma f + (I - A))(x) - P_{\mathfrak{F}}(\gamma f + (I - A))(y)\right\| &\le \left\|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\right\|\\ &\le \gamma\left\|f(x) - f(y)\right\| + \|I - A\|\left\|x - y\right\|\\ &\le \gamma\eta\left\|x - y\right\| + (1 - \overline{\gamma})\left\|x - y\right\|\\ &= (1 - (\overline{\gamma} - \eta\gamma))\left\|x - y\right\|, \quad \forall x, y \in C.\end{aligned} \tag{3.79}$$

Since $0 < 1-(\overline{\gamma}-\eta\gamma) < 1$, it follows that $P_{\mathcal{F}}(\gamma f+(I-A))$ is a contraction of C into itself. Therefore by the Banach Contraction Mapping Principle, it has a unique fixed point, say $z \in C$, that is,

$$z = P_{\mathcal{F}}(\gamma f + (I - A))(z). \tag{3.80}$$

Step 5. We claim that $\limsup_{n\to\infty}\langle \gamma f z - Az, x_n - z\rangle \leq 0$, where z is the unique solution of the variational inequality $\langle \gamma f(z) - Az, x - z\rangle \leq 0$, for all $x \in \mathcal{F}$.

Since $z = P_{\mathcal{F}}(\gamma f + (I - A))(z)$ is a unique solution of the variational inequality (3.14), to show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f z - Az, x_n - z\rangle = \lim_{i\to\infty}\langle \gamma f z - Az, x_{n_i} - z\rangle. \tag{3.81}$$

Correspondingly, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f z - Az, z_n - z\rangle = \lim_{i\to\infty}\langle \gamma f z - Az, z_{n_i} - z\rangle. \tag{3.82}$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $z_{n_i} \rightharpoonup w$. From $\|Sz_n - z_n\| \to 0$, we obtain $Sz_{n_i} \rightharpoonup w$.

Next, we show that $w \in F(S) \cap \mathrm{GMEP}(\Theta_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\Theta_2, \varphi, \Psi_2) \cap \mathrm{VI}(C, B)$.

First, we show that $w \in F(S)$.

Assume $w \notin F(S)$. Since $z_{n_i} \rightharpoonup w$ and $w \neq Sw$, it follows by Opial's condition (Lemma 2.4) that

$$\begin{aligned}
\liminf_{i\to\infty}\|z_{n_i} - w\| &< \liminf_{i\to\infty}\|z_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty}\{\|z_{n_i} - Sz_{n_i}\| + \|Sz_{n_i} - Sw\|\} \\
&= \liminf_{i\to\infty}\|Sz_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty}\|z_{n_i} - w\|.
\end{aligned} \tag{3.83}$$

This is a contradiction. Thus, we have $w \in F(S)$.

Next, we prove that $w \in \mathrm{GMEP}(\Theta_1, \varphi, \Psi_1)$.

For any $y \in C$, we have

$$\begin{aligned}
0 &\leq \Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \Psi_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\
&= \Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle r_n\Psi_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\
&= \Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n + r_n\Psi_1 x_n\rangle,
\end{aligned} \tag{3.84}$$

which yield that

$$\Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - r_n\Psi_1 x_n)\rangle \geq 0, \quad \forall y \in C. \tag{3.85}$$

It follows from Lemma 2.9 that $u_n = T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)$ for all $n \geq 1$.

Thus, we conclude that $u_n = T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n\Psi_1 x_n)$ is equivalent to

$$\Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \Psi_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.86}$$

From (H2), we also have

$$\varphi(y) - \varphi(u_n) + \langle \Psi_1 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -\Theta_1(u_n, y) \geq \Theta_1(y, u_n). \tag{3.87}$$

Replacing n by n_i, we obtain

$$\varphi(y) - \varphi(u_{n_i}) + \langle \Psi_1 x_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq \Theta_1(y, u_{n_i}). \tag{3.88}$$

Let $u_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Since $y \in C$ and $w \in C$, we obtain $u_t \in C$. So, from (3.88) we have

$$\begin{aligned}
\langle u_t - u_{n_i}, \Psi_1 u_t\rangle &\geq \langle u_t - u_{n_i}, \Psi_1 u_t\rangle - \varphi(u_t) + \varphi(u_{n_i}) - \langle \Psi_1 x_{n_i}, u_t - u_{n_i}\rangle \\
&\quad - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \Theta_1(u_t, u_{n_i}) \\
&\geq \langle u_t - u_{n_i}, \Psi_1 u_t - \Psi_1 u_{n_i}\rangle + \langle u_t - u_{n_i}, \Psi_1 u_{n_i} - \Psi_1 x_{n_i}\rangle - \varphi(u_t) \\
&\quad + \varphi(u_{n_i}) - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \Theta_1(u_t, u_{n_i}).
\end{aligned} \tag{3.89}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|\Psi_1 u_{n_i} - \Psi_1 x_{n_i}\| \to 0$. Further, from the inverse strong monotonicity of Ψ_1, we have

$$\langle u_t - u_{n_i}, \Psi_1 u_t - \Psi_1 u_{n_i}\rangle \geq 0. \tag{3.90}$$

So, from (H4), (H5), and the weak lower semicontinuity of φ, $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$, and $u_{n_i} \rightharpoonup w$, we have

$$\langle u_t - w, \Psi_1 u_t\rangle \geq -\varphi(u_t) + \varphi(w) + \Theta_1(u_t, w), \quad \text{as } i \to \infty. \tag{3.91}$$

From (H1), (H4), and (3.91), we also get

$$\begin{aligned}0 &= \Theta_1(u_t,u_t)+\varphi(u_t)-\varphi(u_t)\\ &\le t\Theta_1(u_t,y)+(1-t)\Theta_1(u_t,w)+t\varphi(y)+(1-t)\varphi(w)-\varphi(u_t)\\ &= t\left[\Theta_1(u_t,y)+\varphi(y)-\varphi(u_t)\right]+(1-t)\left[\Theta_1(u_t,w)+\varphi(w)-\varphi(u_t)\right]\\ &\le t\left[\Theta_1(u_t,y)+\varphi(y)-\varphi(u_t)\right]+(1-t)\langle u_t-w,\Psi_1 u_t\rangle\\ &= t\left[\Theta_1(u_t,y)+\varphi(y)-\varphi(u_t)\right]+(1-t)t\langle y-w,\Psi_1 u_t\rangle.\end{aligned} \tag{3.92}$$

Dividing by t, we get

$$\Theta_1(u_t,y)+\varphi(y)-\varphi(u_t)+(1-t)\langle y-w,\Psi_1 u_t\rangle\ge 0. \tag{3.93}$$

Letting $t\longrightarrow 0$ in the above inequality, we arrive that

$$\Theta_1(w,y)+\varphi(y)-\varphi(w)+\langle y-w,\Psi_1 w\rangle\ge 0. \tag{3.94}$$

Thus, $w\in \mathrm{GMEP}(\Theta_1,\varphi,\Psi_1)$. Similarly, we can prove that $w\in \mathrm{GMEP}(\Theta_2,\varphi,\Psi_2)$.

Finally, now we prove that $w\in \mathrm{VI}(C,B)$.

We define the maximal monotone operator:

$$Qw_1=\begin{cases}Bw_1+N_C w_1, & w_1\in C,\\ \emptyset, & w_1\notin C.\end{cases} \tag{3.95}$$

Since B is relaxed (u,v)-cocoercive and condition (C5), we have

$$\langle Bx-By,x-y\rangle\ge(-u)\|Bx-By\|^2+v\|x-y\|^2\ge\left(v-u\omega^2\right)\|x-y\|^2\ge 0, \tag{3.96}$$

which yields that B is monotone. Thus, Q is maximal monotone. Let $(w_1,w_2)\in G(Q)$. Since $w_2-Bw_1\in N_C w_1$ and $z_n\in C$, we have

$$\langle w_1-z_n,w_2-Bw_1\rangle\ge 0. \tag{3.97}$$

On the other hand, from $z_n=P_C(Sv_n-\alpha_n BSv_n)$, we have

$$\langle w_1-z_n,z_n-(Sv_n-\alpha_n BSv_n)\rangle\ge 0, \tag{3.98}$$

and hence

$$\left\langle w_1-z_n,\frac{z_n-Sv_n}{\alpha_n}+BSv_n\right\rangle\ge 0. \tag{3.99}$$

It follows that

$$\begin{aligned}
\langle w_1 - z_{n_i}, w_2 \rangle &\geq \langle w_1 - z_{n_i}, Bw_1 \rangle \\
&\geq \langle w_1 - z_{n_i}, Bw_1 \rangle - \left\langle w_1 - z_{n_i}, \frac{z_{n_i} - Sv_{n_i}}{\alpha_{n_i}} + BSv_{n_i} \right\rangle \\
&= \left\langle w_1 - z_{n_i}, Bw_1 - BSv_{n_i} - \frac{z_{n_i} - Sv_{n_i}}{\alpha_{n_i}} \right\rangle \\
&= \langle w_1 - z_{n_i}, Bw_1 - Bz_{n_i} \rangle + \langle w_1 - z_{n_i}, Bz_{n_i} - BSv_{n_i} \rangle \\
&\quad - \left\langle w_1 - z_{n_i}, \frac{z_{n_i} - Sv_{n_i}}{\alpha_{n_i}} \right\rangle \\
&\geq \langle w_1 - z_{n_i}, Bz_{n_i} - BSv_{n_i} \rangle - \left\langle w_1 - z_{n_i}, \frac{z_{n_i} - Sv_{n_i}}{\alpha_{n_i}} \right\rangle,
\end{aligned} \tag{3.100}$$

which implies that

$$\langle w_1 - w, w_2 \rangle \geq 0. \tag{3.101}$$

Since Q is maximal monotone, we obtain that $w \in Q^{-1}0$. From Lemma 2.3, we get that $w \in \mathrm{VI}(C, B)$. That is, $w \in \mathcal{F}$. Since $z = P_{\mathcal{F}}(\gamma f + (I - A))(z)$, we have

$$\begin{aligned}
\limsup_{n \to \infty} \langle \gamma f z - Az, x_n - z \rangle &= \limsup_{n \to \infty} \langle \gamma f z - Az, x_n - z \rangle \\
&= \lim_{i \to \infty} \langle \gamma f z - Az, x_{n_i} - z \rangle \\
&= \langle \gamma f z - Az, w - z \rangle \leq 0.
\end{aligned} \tag{3.102}$$

On the other hand, we have

$$\begin{aligned}
\langle \gamma f z - Az, y_n - z \rangle &= \langle \gamma f z - Az, y_n - x_n \rangle + \langle \gamma f z - Az, x_n - z \rangle \\
&\leq \| \gamma f z - Az \| \| y_n - x_n \| + \langle \gamma f z - Az, x_n - z \rangle.
\end{aligned} \tag{3.103}$$

From (3.43) and (3.102), we obtain that

$$\limsup_{n \to \infty} \langle \gamma f z - Az, y_n - z \rangle \leq 0. \tag{3.104}$$

Step 6. Finally, we show that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(\gamma f + (I - A))(z)$. Indeed, by (3.13) and using Lemmas 2.6 and 2.12(2), we observe that

$$\begin{aligned}
\|y_n - z\|^2 &= \|\epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)z_n - z\|^2 \\
&= \|((1-\beta_n)I - \epsilon_n A)(z_n - z) + \beta_n(x_n - z) + \epsilon_n(\gamma f(x_n) - Az)\|^2 \\
&\le \|((1-\beta_n)I - \epsilon_n A)(z_n - z) + \beta_n(x_n - z)\|^2 \\
&\quad + 2\epsilon_n\langle \gamma f(x_n) - Az, ((1-\beta_n)I - \epsilon_n A)(z_n - z) + \beta_n(x_n - z) + \epsilon_n(\gamma f(x_n) - Az)\rangle \\
&= \|((1-\beta_n)I - \epsilon_n A)(z_n - z) + \beta_n(x_n - z)\|^2 \\
&\quad + 2\epsilon_n\langle \gamma f(x_n) - Az, \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)z_n - z\rangle \\
&= \left\|(1-\beta_n)\frac{((1-\beta_n)I - \epsilon_n A)}{(1-\beta_n)}(z_n - z) + \beta_n(x_n - z)\right\|^2 + 2\epsilon_n\langle \gamma f(x_n) - Az, y_n - z\rangle \\
&\le (1-\beta_n)\left\|\frac{((1-\beta_n)I - \epsilon_n A)}{1-\beta_n}(z_n - z)\right\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\epsilon_n\gamma\langle f(x_n) - f(z), y_n - z\rangle + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\le (1-\beta_n)\left\|\frac{((1-\beta_n)I - \epsilon_n A)}{1-\beta_n}(z_n - z)\right\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\epsilon_n\gamma\eta\|x_n - z\|\|y_n - z\| + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\le \frac{\|(1-\beta_n)I - \epsilon_n A\|^2}{1-\beta_n}\|z_n - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \epsilon_n\gamma\eta\left(\|x_n - z\|^2 + \|y_n - z\|^2\right) + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&\le \frac{\|(1-\beta_n)I - \epsilon_n A\|^2}{1-\beta_n}\|x_n - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + \epsilon_n\gamma\eta\left(\|x_n - z\|^2 + \|y_n - z\|^2\right) + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&= \left(\frac{((1-\beta_n) - \overline{\gamma}\epsilon_n)^2}{1-\beta_n} + \beta_n + \epsilon_n\gamma\eta\right)\|x_n - z\|^2 \\
&\quad + \epsilon_n\gamma\eta\|y_n - z\|^2 + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle \\
&= \left(1 - (2\overline{\gamma} - \eta\gamma)\epsilon_n + \frac{\overline{\gamma}^2\epsilon_n^2}{1-\beta_n}\right)\|x_n - z\|^2 \\
&\quad + \epsilon_n\gamma\eta\|y_n - z\|^2 + 2\epsilon_n\langle \gamma f(z) - Az, y_n - z\rangle,
\end{aligned} \tag{3.105}$$

which implies that

$$\begin{aligned} \|y_n - z\|^2 \leq & \left(1 - \frac{2(\overline{\gamma} - \eta\gamma)\epsilon_n}{1 - \eta\gamma\epsilon_n}\right)\|x_n - z\|^2 \\ & + \frac{\epsilon_n}{1 - \eta\gamma\epsilon_n}\left[\frac{\overline{\gamma}^2\epsilon_n}{1 - \beta_n}\|x_n - z\|^2 + 2\langle \gamma f(z) - Az, y_n - z\rangle\right]. \end{aligned} \tag{3.106}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\gamma_n(x_n - z) + (1 - \gamma_n)(y_n - z)\|^2 \\ &\leq \gamma_n\|x_n - z\|^2 + (1 - \gamma_n)\|y_n - z\ \|^2. \end{aligned} \tag{3.107}$$

Substituting (3.106) into (3.107) yields that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \gamma_n\|x_n - z\|^2 + (1 - \gamma_n)\|y_n - z\|^2 \\ &\leq \gamma_n\|x_n - z\|^2 + (1 - \gamma_n) \\ &\quad \times \left\{\left(1 - \frac{2(\overline{\gamma} - \eta\gamma)\epsilon_n}{1 - \eta\gamma\epsilon_n}\right)\|x_n - z\|^2 \right. \\ &\quad \left. + \frac{\epsilon_n}{1 - \eta\gamma\epsilon_n}\left[\frac{\overline{\gamma}^2\epsilon_n}{1 - \beta_n}\|x_n - z\|^2 + 2\langle \gamma f(z) - Az, y_n - z\rangle\right]\right\} \\ &= \left(1 - \frac{2(1 - \gamma_n)(\overline{\gamma} - \eta\gamma)\epsilon_n}{1 - \eta\gamma\epsilon_n}\right)\|x_n - z\|^2 \\ &\quad + \frac{\epsilon_n(1 - \gamma_n)}{1 - \eta\gamma\epsilon_n}\left[\frac{\overline{\gamma}^2\epsilon_n}{1 - \beta_n}\|x_n - z\|^2 + 2\langle \gamma f(z) - Az, y_n - z\rangle\right]. \end{aligned} \tag{3.108}$$

Taking

$$\begin{aligned} \varrho_n &= \frac{2(1 - \gamma_n)(\overline{\gamma} - \eta\gamma)\epsilon_n}{1 - \eta\gamma\epsilon_n}, \\ \sigma_n &= \frac{\epsilon_n(1 - \gamma_n)}{1 - \eta\gamma\epsilon_n}\left\{\frac{\overline{\gamma}^2\epsilon_n}{1 - \beta_n}\|x_n - z\|^2 + 2\langle \gamma f(z) - Az, y_n - z\rangle\right\}, \end{aligned} \tag{3.109}$$

then, we can rewrite (3.108) as

$$\|x_{n+1} - z\|^2 \leq (1 - \varrho_n)\|x_n - z\|^2 + \sigma_n, \tag{3.110}$$

It follows from condition (C1) and (3.104) that

$$\lim_{n\to\infty} \varrho_n = 0, \qquad \sum_{n=1}^{\infty} \varrho_n = \infty. \tag{3.111}$$

Since

$$\limsup_{n\to\infty} \langle \gamma f z - Az, y_n - z \rangle \leq 0. \tag{3.112}$$

and $\{x_n - z\}$ is bounded, we have

$$\limsup_{n\to\infty} \sigma_n = \limsup_{n\to\infty} \frac{\epsilon_n (1-\gamma_n)}{1-\eta\gamma\epsilon_n} \left\{ \frac{\overline{\gamma}^2 \epsilon_n}{1-\beta_n} \|x_n - z\|^2 + 2\langle \gamma f(z) - Az, y_n - z \rangle \right\} \leq 0. \tag{3.113}$$

Applying Lemma 2.8 to (3.110), we conclude that $\{x_n\}$ converges strongly to z in norm. This completes the proof. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ_1 and Θ_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (H1)–(H5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let*

(i) *$\Psi_1 : C \to H$ be a ξ-inverse-strongly monotone mapping,*

(ii) *$\Psi_2 : C \to H$ be a β-inverse-strongly monotone mapping,*

(iii) *$f : C \to C$ be a contraction mapping with coefficient $\eta \in (0,1)$.*

Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$.

Assume that

$$\digamma := F(S) \cap GMEP(\Theta_1, \varphi, \Psi_1) \cap GMEP(\Theta_2, \varphi, \Psi_2) \neq \emptyset. \tag{3.114}$$

Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, and $\{u_n\}$ be the sequences generated by

$$\begin{aligned} u_n &= T_{r_n}^{(\Theta_1,\varphi)} (x_n - r_n \Psi_1 x_n), \\ v_n &= T_{s_n}^{(\Theta_2,\varphi)} (u_n - s_n \Psi_2 u_n), \\ y_n &= \epsilon_n f(x_n) + \beta_n x_n + \delta_n S v_n, \\ x_{n+1} &= \gamma_n x_n + (1-\gamma_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{3.115}$$

where $\{r_n\} \subset [a,b] \subset [0,2\xi]$, $\{s_n\} \subset [c,d] \subset [0,2\beta]$, $\{\gamma_n\} \subset [h,j] \subset (0,1)$, and $\{\gamma_n\}$, $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are three sequences in $(0,1)$ satisfying the following conditions:

(C1) *$\epsilon_n + \beta_n + \delta_n = 1$,*

(C2) *$\lim_{n\to\infty} \epsilon_n = 0$ and $\sum_{n=1}^{\infty} \epsilon_n = \infty$,* (C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(C4) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\xi$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C5) $0 < \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n < 2\beta$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} f(z)$, which is the unique solution of the variational inequality:

$$\langle f(z) - z, x - z\rangle \le 0, \quad \forall x \in \mathcal{F}. \tag{3.116}$$

Proof. In Theorem 3.3, put $A = I$, $\gamma = 1$, and $\delta_n = 1 - \epsilon_n - \beta_n$. Let $B = 0$ in Theorem 3.3; then we have $\mathrm{VI}(C, B) = C$ and

$$z_n = P_C(Sv_n - \alpha_n BSv_n) = Sv_n, \tag{3.117}$$

and we can obtain the desired conclusion from Theorem 3.3 immediately. □

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ_1 and Θ_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (H1)–(H5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with assumption (B1) or (B2). Let $f : C \to C$ be a contraction mapping with coefficient $\eta \in (0, 1)$ and let $S : C \to C$ be a nonexpansive mapping with $F(S) \ne \emptyset$.*

Assume that

$$\mathcal{F} := F(S) \cap \mathrm{MEP}(\Theta_1, \varphi) \cap \mathrm{MEP}(\Theta_2, \varphi) \ne \emptyset. \tag{3.118}$$

Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, and $\{u_n\}$ be the sequences generated by

$$\begin{gathered} \Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ \Theta_2(v_n, y) + \varphi(y) - \varphi(v_n) + \frac{1}{s_n}\langle y - v_n, v_n - u_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \epsilon_n f(x_n) + \beta_n x_n + \delta_n Sv_n, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, \quad \forall n \ge 1, \end{gathered} \tag{3.119}$$

where $\{r_n\} \subset [a, b] \subset (0, \infty)$, $\{s_n\} \subset [c, d] \subset (0, \infty)$, $\{\gamma_n\} \subset [h, j] \subset (0, 1)$, and $\{\gamma_n\}$, $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are three sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\epsilon_n + \beta_n + \delta_n = 1$,

(C2) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(C4) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < \infty$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C5) $0 < \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n < \infty$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$.

Then, $\{x_n\}$ *converges strongly to* $z = P_{\mathcal{F}} f(z)$*, which is the unique solution of the variational inequality:*

$$\langle f(z) - z, x - z \rangle \leq 0, \quad \forall x \in \mathcal{F}. \tag{3.120}$$

Proof. In Theorem 3.3, put $u_n = T_{r_n}^{(\Theta_1,\varphi)}(x_n - r_n \Psi_1 x_n)$ to be equivalent to

$$\Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \Psi_1 x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \tag{3.121}$$

and put $v_n = T_{s_n}^{(\Theta_2,\varphi)}(u_n - s_n \Psi_2 u_n)$ to be equivalent to

$$\Theta_2(v_n, y) + \varphi(y) - \varphi(v_n) + \langle \Psi_2 x_n, y - v_n \rangle + \frac{1}{s_n}\langle y - v_n, v_n - u_n \rangle \geq 0, \quad \forall y \in C. \tag{3.122}$$

Now, put $\Psi_1 = \Psi_2 = 0$. Then, it follows that

$$\begin{aligned} &\Theta_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ &\Theta_2(v_n, y) + \varphi(y) - \varphi(v_n) + \frac{1}{s_n}\langle y - v_n, v_n - u_n \rangle \geq 0, \quad \forall y \in C. \end{aligned} \tag{3.123}$$

Observe that for all $\xi > 0$, we see that

$$\langle \Psi_1 x - \Psi_2 y, x - y \rangle \geq \xi \|\Psi_1 x - \Psi_2 y\|^2, \quad \forall x, y \in C. \tag{3.124}$$

Thus, let $\{r_n\}$ be a sequence satisfying the restriction: $a \leq r_n \leq b$, where $a, b \in (0, \infty)$ with $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \infty$. Similarly, we obtain $c \leq s_n \leq d$, where $c, d \in (0, \infty)$ with $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < \infty$, and we obtain the desired result by Corollary 3.4. □

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