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Research Article

Existence and Nonexistence Results for Classes of Singular Elliptic Problem

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The singular semilinear elliptic problem $-\Delta u + k(x)u^{-\gamma} = \lambda u^p$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, is considered, where Ω is a bounded domain with smooth boundary in R^N, $k \in C^{\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, and γ, p, λ are three positive constants. Some existence or nonexistence results are obtained for solutions of this problem by the sub-supersolution method.

1. Introduction and Main Results

In this paper, we study the existence or the nonexistence of solutions to the following singular semilinear elliptic problem

$$\begin{gathered} -\Delta u + k(x)u^{-\gamma} = \lambda u^p, \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with $C^{2+\alpha}$ boundary for some $\alpha \in (0,1)$, $k \in C^{\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, and $\gamma, p,$ and λ are three nonnegative constants. This problem arises in the study of non-Newtonian fluids, chemical heterogeneous catalysts, in the theory of heat conduction in electrically conducting materials (see [1–7] and their references).

Many authors have considered this problem. For examples, when $k(x) < 0$ in Ω, problem (1.1) was studied in [3, 8–11]; when $k(x) > 0$ in Ω, problem (1.1) was considered in [12–14]. Particularly, when $k(x) \equiv 1$, it has been established in Zhang [14] that there exists $\overline{\lambda} > 0$ such that problem (1.1) has at least one solution in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for all $\lambda > \overline{\lambda}$ and has no solution in $C^2(\Omega) \cap C(\overline{\Omega})$ if $\lambda < \overline{\lambda}$. After that Shi and Yao in [13] have also obtained the same results with $k \in C^{2,\alpha}(\overline{\Omega})$ and $k(x) > 0$ in $\overline{\Omega}$. Recently, Ghergu and Rădulescu in [12] considered more general sublinear singular elliptic problem with $k \in C^{\alpha}(\overline{\Omega})$.

In this paper, we consider the case that $k \in C^{\alpha}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, and k may have zeros in $\overline{\Omega}$. The following main results are obtained by the sub-supersolution method with restriction on the boundary in Cui [15].

Theorem 1.1. *Suppose that $k \in C^{\alpha}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}), k \geq 0$, and $k \not\equiv 0$. Assume that $0 < \gamma < 1$ and $0 < p < 1$, then there exists $\overline{\lambda} \in (0, \infty)$ such that problem* (1.1) *has at least one solution $u_\lambda \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ and $u_\lambda^{-\gamma} \in L^1(\Omega)$ for all $\lambda > \overline{\lambda}$, and problem* (1.1) *has no solution in $C^2(\Omega) \cap C(\overline{\Omega})$ if $\lambda < \overline{\lambda}$. Moreover, problem* (1.1) *has a maximal solution v_λ which is increasing with respect to λ for all $\lambda > \overline{\lambda}$.*

Remark 1.2. Theorem 1.1 generalizes Theorem 1.2 in [13] in coefficient $k(x)$ of the singular term. Consequently, it also generalizes Theorem 1 in [14]. Moreover, there are functions k satisfying our Theorem 1.1 and not satisfying Theorem 1.2 in [13]. For example, let

$$k(x) = \begin{cases} -\dfrac{1}{\ln(|x - x_0|/(2d))}, & x \in \overline{\Omega} \setminus \{x_0\}, \\ 0, & x = x_0, \end{cases} \tag{1.2}$$

where $x_0 \in \partial\Omega$, and

$$d = \operatorname{diam}(\Omega) \overset{\Delta}{=} \max\left\{ \left| x - y \right| \mid x, y \in \overline{\Omega} \right\}. \tag{1.3}$$

Certainly, this example does not satisfy Theorem 1.2 in [12] yet.

Theorem 1.3. *Suppose that $k \in C^{\alpha}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ and $k(x) > 0$ in $\overline{\Omega}$. If $\gamma \geq 1$, problem* (1.1) *has no solution in $C^2(\Omega) \cap C(\overline{\Omega})$ for all $\lambda > 0$ and $p > 0$.*

Remark 1.4. Obviously, Theorem 1.3 is a generalization of Theorem 2 in [14]. There are also functions $k(x)$ satisfying our Theorem 1.3 and not satisfying Theorem 2 in [14] and Theorem 1.1 in [12]. For example, let

$$k(x) = \begin{cases} -\dfrac{1}{\ln(|x - x_0|/(2d))} + \varepsilon, & x \in \overline{\Omega} \setminus \{x_0\}, \\ \varepsilon, & x = x_0, \end{cases} \tag{1.4}$$

where $x_0 \in \partial\Omega$, ε is any positive constant and $d = \operatorname{diam}(\Omega)$ is the diameter of Ω.

2. Proof of Theorems

Consider the more general semilinear elliptic problem

$$\begin{aligned} -\Delta u &= f(x,u), \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where the function $f(x,s)$ is locally Hölder continuous in $\Omega \times (0,\infty)$ and continuously differentiable with respect to the variable s. A function $\underline{u}$ is called to be a subsolution of problem (2.1) if $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, and

$$\begin{aligned} -\Delta \underline{u} &\le f(x,\underline{u}), \quad \text{in } \Omega, \\ \underline{u} &> 0, \quad \text{in } \Omega, \\ \underline{u} &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

A function $\overline{u}$ is called to be a supersolution of problem (2.1) if $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, and

$$\begin{aligned} -\Delta \overline{u} &\ge f(x,\overline{u}), \quad \text{in } \Omega, \\ \overline{u} &> 0, \quad \text{in } \Omega, \\ \overline{u} &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

According to Lemma 3 in the study of Cui [15], we can easily have the following basic existence of classical solution to problem (2.1).

Lemma 2.1. *Let $f \in C^{\alpha}_{\text{loc}}(\Omega \times (0,\infty))$ be continuously differentiable with respect to the variable s. Suppose that problem* (2.1) *has a supersolution $\overline{u}$ and a subsolution $\underline{u}$ such that*

$$\underline{u}(x) \le \overline{u}(x), \quad \text{in } \Omega, \tag{2.4}$$

then problem (2.1) *has at least one solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ satisfying*

$$\underline{u}(x) \le u(x) \le \overline{u}(x), \quad \text{in } \overline{\Omega}. \tag{2.5}$$

Let λ_1 be the first eigenvalue of the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.6}$$

and $\varphi_1 > 0$ in Ω the corresponding eigenfunction. Then $\varphi_1 \in C^{2+\alpha}(\overline{\Omega})$. Moreover one has the following lemma.

Lemma 2.2 (see [10]). *One has*

$$\int_{\Omega} \varphi_1^r \, dx < \infty \tag{2.7}$$

if and only if $r > -1$.

Now we give the proof of our theorems.

Proof of Theorem 1.1. Let $p \in (0,1)$, and let u^* denote the unique solution of

$$\begin{aligned} -\Delta u &= u^p, \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

where u^* belongs to $C^2(\overline{\Omega})$ (see [16]). Then $u = \lambda^{1/(1-p)} u^*$ is a solution of

$$\begin{aligned} -\Delta u &= \lambda u^p, \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.9}$$

where $0 < p < 1$ and $\lambda > 0$. Then fix $\lambda > 0$ and set

$$\overline{u} = \lambda^{1/(1-p)} u^*, \tag{2.10}$$

thus we can easily obtain that $\overline{u}$ is a supersolution of problem (1.1).

Now, we want to find a subsolution of problem (1.1). Let

$$\underline{u} = M \varphi_1^{2/(1+\gamma)}, \tag{2.11}$$

where M is a positive constant; now we will prove that $\underline{u}$ is a subsolution of problem (1.1). By Hopf's maximum principle in [17], there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned} |\nabla \varphi_1| &\geq \delta, \quad \text{on } \Omega \setminus \Omega', \\ \varphi_1 &\geq \delta, \quad \text{on } \Omega', \end{aligned} \tag{2.12}$$

where $\Omega' = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon_0\}$. On Ω', we choose $M \geq M_1 \triangleq ((\|k\|_\infty (1 + \gamma))/\lambda_1 \delta^2)^{1/(1+\gamma)}$, then we have

$$\frac{k(x)}{M^\gamma \varphi_1^{2\gamma/(1+\gamma)}} \leq \frac{\lambda_1 M}{1+\gamma} \varphi_1^{2/(1+\gamma)}, \tag{2.13}$$

where $\|k\|_\infty = \max\{|k(x)| \mid x \in \overline{\Omega}\}$ for $k \in C(\overline{\Omega})$. On $\Omega \setminus \Omega'$, we choose $M \geq M_2 \triangleq (\|k\|_\infty (1+\gamma)^2 / 2(1-\gamma)\delta^2)^{1/(1+\gamma)}$, then one obtains

$$\frac{k(x)}{M^\gamma \varphi_1^{2\gamma/(1+\gamma)}} \leq \frac{2(1-\gamma)M|\nabla\varphi_1|^2}{(1+\gamma)^2 \varphi_1^{2\gamma/(1+\gamma)}}. \tag{2.14}$$

Thus, we choose $M \geq \max\{M_1, M_2\}$, then fixing M, let $\lambda > \lambda' \triangleq (3\lambda_1 M^{1-p})/(1+\gamma)\ \|\varphi_1\|_\infty^{2(1-p)/(1+\gamma)}$, it follows from (2.13) and (2.14) that

$$\begin{aligned}
-\Delta\underline{u} + k(x)\underline{u}_\lambda^{-\gamma} &= -M\Delta\varphi_1^{2/(1+\gamma)} + \frac{k(x)}{M^\gamma \varphi_1^{2\gamma/(1+\gamma)}} \\
&= -M\left(\frac{2(1-\gamma)}{(1+\gamma)^2}|\nabla\varphi_1|^2 \varphi_1^{-2\gamma/(1+\gamma)} + \frac{2}{1+\gamma}\varphi_1^{(1-\gamma)/(1+\gamma)}\Delta\varphi_1\right) + \frac{k(x)}{M^\gamma \varphi_1^{2\gamma/(1+\gamma)}} \\
&= \frac{2\lambda_1 M}{1+\gamma}\varphi_1^{2/(1+\gamma)} + \frac{k(x)}{M^\gamma \varphi_1^{2\gamma/(1+\gamma)}} - \frac{2(1-\gamma)M|\nabla\varphi_1|^2}{(1+\gamma)^2 \varphi_1^{2\gamma/(1+\gamma)}} \\
&\leq \frac{3\lambda_1 M}{1+\gamma}\varphi_1^{2/(1+\gamma)} \\
&\leq \lambda\left(M\varphi_1^{2/(1+\gamma)}\right)^p \\
&= \lambda\underline{u}_\lambda^p.
\end{aligned} \tag{2.15}$$

Thus we proved that $\underline{u} = M\varphi_1^{2/(1+\gamma)}$ is a subsolution of problem (1.1) for all $\lambda > \lambda'$. According to Lemma 4 in [14], there exists a positive constant C such that

$$\varphi_1(x) \leq Cu^*(x), \quad \text{in } \overline{\Omega}. \tag{2.16}$$

Set $\lambda \geq \lambda'' \triangleq (MC\|\varphi_1\|_\infty^{(1-\gamma)/(1+\gamma)})^{1-p}$, then we have

$$\overline{u} = \lambda^{1/(1-p)}u^* \geq \underline{u} = M\varphi_1^{2/(1+\gamma)}, \quad \text{in } \Omega. \tag{2.17}$$

Thus we choose $\lambda^* = \max\{\lambda', \lambda''\}$; via Lemma 2.1, problem (1.1) has at least one solution $u_\lambda \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ and satisfying

$$\underline{u}(x) \leq u_\lambda(x) \leq \overline{u}(x), \quad \text{in } \overline{\Omega}, \tag{2.18}$$

for all $\lambda \geq \lambda^*$.

Since $u_\lambda \geq M\varphi_1^{2/(1+\gamma)}$ in $\overline{\Omega}$ for all $\lambda \geq \lambda^*$ and $-2\gamma/(1+\gamma) > -1$, according to Lemma 2.2 one has

$$\int_\Omega u_\lambda^{-\gamma}(x)dx \leq \frac{1}{M^\gamma}\int_\Omega \varphi_1^{-2\gamma/(1+\gamma)}(x)dx < +\infty. \tag{2.19}$$

So we obtain $u_\lambda^{-\gamma} \in L^1(\Omega)$.

Let $\Omega_j = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > r/2j\}$, $j = 1, 2, 3, \ldots$, and let u_j be the unique solution of

$$\begin{gathered} -\Delta u + k(x)u_{j-1}^{-\gamma} = \lambda u_{j-1}^{p}, \quad \text{in } \Omega_j, \\ u = u_{j-1}, \quad \text{on } \overline{\Omega} \setminus \Omega_j, \end{gathered} \tag{2.20}$$

for $j = 1, 2, 3, \ldots$, and with $u_0 = \overline{u} = \lambda^{1/(1-p)}u^*$, where

$$r = \max_{x\in\Omega}\min_{y\in\partial\Omega}\left|x - y\right|. \tag{2.21}$$

We claim that u_j is nonincreasing with respect to j in $\overline{\Omega}$ for all $j \in N$. Indeed, since $\overline{u}$ is a supersolution of problem (1.1) for all $\lambda > 0$, then we have

$$\begin{aligned} -\Delta(u_0 - u_1) &= -\Delta u_0 + \Delta u_1 \\ &= -\Delta u_0 + k(x)u_0^{-\gamma} - \lambda u_0^{p} \\ &= -\Delta \overline{u} + k(x)\overline{u}_\lambda^{-\gamma} - \lambda \overline{u}_\lambda^{p} \\ &> 0, \end{aligned} \tag{2.22}$$

for all $x \in \Omega_1$. Since $u_1 = u_0$ in $\overline{\Omega} \setminus \Omega_1$, so by the maximum principle, one has $u_0 \geq u_1$ in $\overline{\Omega}$. So when $j = 0$ our claim is true. We assume that our claim is true when $j = n$; that is, $u_n \leq u_{n-1}$ in $\overline{\Omega}$. Then we obtain

$$\begin{aligned} -\Delta(u_n - u_{n+1}) &= -\Delta u_n + \Delta u_{n+1} \\ &= \lambda\left(u_{n-1}^{p} - u_n^{p}\right) + k(x)\left(u_n^{-\gamma} - u_{n-1}^{-\gamma}\right) \\ &> 0, \end{aligned} \tag{2.23}$$

for all $x \in \Omega_{n+1}$. Since $u_n = u_{n+1}$ in $\overline{\Omega} \setminus \Omega_{n+1}$, so by the maximum principle, one has $u_n \geq u_{n+1}$ in $\overline{\Omega}$. Thus by the induction, one obtains

$$u_{j+1} \leq u_j, \quad \text{in } \overline{\Omega}, \tag{2.24}$$

for all $j \in N$. Then by the monotonicity of u_j, we have

$$\begin{aligned} -\Delta u_j &= \lambda u_{j-1}^{p} - k(x) u_{j-1}^{-\gamma} \\ &\geq \lambda u_j^{p} - k(x) u_j^{-\gamma}, \end{aligned} \tag{2.25}$$

for all $x \in \Omega_j$ and $j \in N^+$. According to the definitions of u_j and u_0, we obtain that u_j is a supersolution of problem (1.1) for all $j \in N^+$. Let u_λ be a classical solution of problem (1.1), thus one has

$$u_\lambda(x) \leq u_{j+1}(x) \leq u_j(x) \leq u_0(x), \quad \text{in } \overline{\Omega}. \tag{2.26}$$

Assume that $v_\lambda(x) = \lim_{j\to\infty} u_j(x)$ for all $x \in \overline{\Omega}$, then by standard elliptic arguments (see [17]) it follows that v_λ is a solution of problem (1.1), and $v_\lambda \geq u_\lambda$ in Ω for any u_λ. Therefore, v_λ is the maximal solution of problem (1.1). According to the above arguments, problem (1.1) has a maximal solution for $\lambda \geq \lambda^*$.

To complete the proof of Theorem 1.1, setting

$$\begin{aligned} \sigma &= \{\lambda > 0 \mid \text{problem (1.1) has at least one solution } u_\lambda\}, \\ \overline{\lambda} &= \inf \sigma, \end{aligned} \tag{2.27}$$

then $[\lambda^*, +\infty) \subset \sigma$, $\overline{\lambda} \leq \lambda^*$. It suffices to prove that if $\lambda_0 \in \sigma$, then $[\lambda_0, +\infty) \subset \sigma$; that is, assume that $\lambda > \lambda_0$, then problem (1.1) has at least one solution. Let u_{λ_0} be a solution of problem (1.1) corresponding to λ_0, then u_{λ_0} is a subsolution of problem (1.1) with every fixed $\lambda > \lambda_0$. Since $\overline{u} = \lambda^{1/(1-p)} u^*$ is a supersolution of problem (1.1) for any $\lambda > 0$, then one has

$$\lambda^{1/(1-p)} u^* \geq \lambda_0^{1/(1-p)} u^* \geq u_{\lambda_0}, \quad \text{in } \Omega, \tag{2.28}$$

for all $\lambda > \lambda_0$. According to Lemma 2.1, problem (1.1) has at least one solution $u_\lambda \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for all $\lambda > \lambda_0$. Moreover,

$$u_{\lambda_0}(x) \leq u_\lambda(x) \leq \overline{u}(x), \quad \text{in } \Omega. \tag{2.29}$$

Consequently, the maximal solution v_λ of problem (1.1) is increasing with respect to λ for all $\lambda > \overline{\lambda}$. So the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.3. Suppose to the contrary that there exists $\lambda > 0$ such that problem (1.1) has one solution $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$. Let e be the unique solution of

$$\begin{aligned} -\Delta u &= 1, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.30}$$

$e \in C^{2+\alpha}(\overline{\Omega})$. By the maximum principle, $e > 0$ in Ω. We claim that for any solution u_λ of problem (1.1), there exists a constant $M = M(\lambda) > 0$ such that

$$Me(x) > u_\lambda(x), \quad \text{in } \Omega. \tag{2.31}$$

Indeed, let $M = \lambda \|u_\lambda\|_\infty^p + 1$, then one obtains

$$\begin{aligned} -\Delta(Me - u_\lambda) &= -M\Delta e + \Delta u_\lambda \\ &= \lambda \|u_\lambda\|_\infty^p + 1 - \lambda u_\lambda^p(x) + k(x) u_\lambda^{-\gamma} \\ &> 0, \end{aligned} \tag{2.32}$$

for all $x \in \Omega$. Since $(Me - u_\lambda)|_{\partial\Omega} = 0$, by the maximum principle we have

$$Me(x) > u_\lambda(x), \quad \text{in } \Omega. \tag{2.33}$$

According to Lemma 4 in [14], there exists a positive constant C such that

$$e(x) \le C\varphi_1(x), \quad \text{in } \Omega. \tag{2.34}$$

Since $\gamma \ge 1$, from Lemma 2.2, it follows that

$$\int_\Omega u_\lambda^{-\gamma}(x)dx \ge \frac{1}{(CM)^\gamma} \int_\Omega \varphi_1^{-\gamma}(x)dx = +\infty. \tag{2.35}$$

Thus we obtain

$$\int_\Omega u_\lambda^{-\gamma} dx = +\infty. \tag{2.36}$$

Set

$$\Omega_i = \left\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{r}{2i},\ i \in N^+ \right\}, \tag{2.37}$$

and $\Omega = \bigcup_{i=1}^\infty \Omega_i$, then $\Omega_i \subset \Omega$ and $u_\lambda \in C^2(\overline{\Omega}_i)$, satisfying

$$-\Delta u_\lambda + k(x) u_\lambda^{-\gamma} = \lambda u_\lambda^p, \tag{2.38}$$

for all $x \in \overline{\Omega}_i$ and $i \in N^+$. Consequently, integrating (2.38) we have

$$-\int_{\Omega_i} \Delta u_\lambda dx + \int_{\Omega_i} k(x) u_\lambda^{-\gamma} dx = \lambda \int_{\Omega_i} u_\lambda^p dx \le \lambda \int_\Omega u_\lambda^p dx, \tag{2.39}$$

noting that

$$\int_{\Omega_i} \Delta u_\lambda dx = \int_{\partial\Omega_i} \frac{\partial u_\lambda}{\partial n} ds, \tag{2.40}$$

where n denotes the outward normal to $\partial\Omega_i$. From (2.39) and (2.40), letting $i \to \infty$, one has

$$\int_{\Omega} k(x) u_\lambda^{-\gamma} dx - \int_{\partial\Omega} \frac{\partial u_\lambda}{\partial n} ds \leq \lambda \|u_\lambda\|_\infty^p |\Omega|, \tag{2.41}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. According to (2.36) and $k(x) > 0$ in $\overline{\Omega}$, one obtains

$$\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial n} ds = +\infty. \tag{2.42}$$

But this is impossible, by Hopf's maximum principle, we have

$$\frac{\partial u_\lambda}{\partial n} < 0, \tag{2.43}$$

for all $x \in \partial\Omega$, where n denotes the outward normal to $\partial\Omega$ at x. Therefore Theorem 1.3 is true. □

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