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Research Article

Micropolar Fluids with Vanishing Viscosity

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A study of the convergence of weak solutions of the nonstationary micropolar fluids, in bounded domains of $\mathbb{R}^n$, when the viscosities tend to zero, is established. In the limit, a fluid governed by an Euler-like system is found.

1. Introduction

The aim of this work is to analyze the convergence of the evolution equations for the motion of incompressible micropolar fluids, when the viscosities related to the physical properties of the fluid tend to zero. The equations that describe the motion of a viscous incompressible micropolar fluid express the balance of mass, momentum, and angular momentum. In a bounded domain $\Omega \subset \mathbb{R}^3$ and in a time interval $(0, T]$, $0 < T < +\infty$, this model is given by the following system of differential equations:

$$(\mathbf{u}_\nu)_t - \nu_1 \Delta \mathbf{u}_\nu + \mathbf{u}_\nu \cdot \nabla \mathbf{u}_\nu + \nabla p_\nu = 2\mu_r \operatorname{rot} \mathbf{w}_\nu + \mathbf{f}, \quad \text{in } Q, \tag{1.1}$$

$$\operatorname{div} \mathbf{u}_\nu = 0, \quad \text{in } Q, \tag{1.2}$$

$$(\mathbf{w}_\nu)_t - \nu_2 \Delta \mathbf{w}_\nu - \nu_3 \nabla \operatorname{div} \mathbf{w}_\nu + \mathbf{u}_\nu \cdot \nabla \mathbf{w}_\nu + 4\mu_r \mathbf{w}_\nu = 2\mu_r \operatorname{rot} \mathbf{u}_\nu + \mathbf{g}, \quad \text{in } Q, \tag{1.3}$$

with $Q = \Omega \times (0, T]$, where the unknowns are $\mathbf{u}_\nu$, $\mathbf{w}_\nu$, and p_ν, which denote, respectively, the velocity of the fluid, the microrotational velocity, and the hydrostatic pressure of the fluid, at a point $(\mathbf{x}, t)$. ν_1, ν_2, and ν_3 are positive constants which satisfy $\nu_1 = \mu + \mu_r$, $\nu_2 = c_a + c_d$, $\nu_3 = c_0 + c_d - c_a$, with $c_0 + c_d > c_a$, where $\mu, \mu_r, c_0, c_a, c_d$ represent viscosity coefficients. In particular, μ is the usual Newtonian viscosity, μ_r is called the viscosity of microrotation and c_0, c_a, c_d are new viscosities related to the asymmetry of the stress tensor. The fields $\mathbf{f}$ and $\mathbf{g}$ are given and denote external sources of linear and angular momentum, respectively.

With (1.1)–(1.3) the following initial and boundary conditions are prescribed

$$\mathbf{u}_\nu(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{w}_\nu(\mathbf{x}, 0) = \mathbf{0}, \quad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}_\nu(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{w}_\nu(\mathbf{x}, t) = \mathbf{0}, \quad \text{on } \partial\Omega \times [0, T], \tag{1.5}$$

where, for the simplicity in this exposition, homogeneous boundary conditions have been taken. The initial data is also assumed to be equal to zero due to the nature of the solutions of the Euler-like system (1.6)–(1.10) below.

Theory of micropolar fluids was proposed by Eringen [1] and describes flows of fluids whose particles undergo translations androtations as well. In this sense, micropolar fluids permit to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations for viscous incompressible fluids. Indeed, if $\mu_r = 0$ in system (1.1)–(1.3), the equations are decoupled and (1.1) reduces to the incompressible Navier-Stokes equations (see [2]). For the derivation and physical discussion of system (1.1)–(1.3), see the references [1, 3, 4].

There is extensive literature related to the solutions of micropolar fluids. In a hilbertian context, in [4–6] and some references therein, results of existence, uniqueness and regularity of weak solutions were found. On the other hand, in [7, 8], by using semigroups approach, some recent results related to the initial value problem (1.1)–(1.5) with initial data in L^p-spaces, including the stability of strong steady solutions, were performed.

This work is concerned with the behavior of the micropolar fluids, in a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$ smooth enough, when the viscosities ν_1, ν_2, ν_3 tend to zero. We will prove that there is a subspace $\mathcal{F}_0$ of $(L^\infty(0, T; \mathbf{H}_0^3(\Omega)))^2$ such that, for external sources $(\mathbf{f}, \mathbf{g})$ in $\mathcal{F}_0$, the weak solutions of the micropolar fluid system (1.1)–(1.3) converge in $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, when the viscosities ν_1, ν_2, ν_3 in (1.1)–(1.3) tend to zero, to the solution $(\mathbf{u}, \mathbf{w})$ of the following Euler-like system:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega \times [0, T], \tag{1.6}$$

$$\operatorname{div} \mathbf{u} = 0, \quad \text{in } \Omega \times [0, T], \tag{1.7}$$

$$\mathbf{w}_t + \mathbf{u} \cdot \nabla \mathbf{w} = \mathbf{g}, \quad \text{in } \Omega \times [0, T], \tag{1.8}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{w}(\mathbf{x}, 0) = \mathbf{0}, \quad \text{in } \Omega, \tag{1.9}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{w}(\mathbf{x}, t) = \mathbf{0}, \quad \text{on } \partial\Omega \times [0, T]. \tag{1.10}$$

As far as it is known, the analysis of convergence of the evolution equations for the motion of incompressible micropolar fluids, when the viscosities tend to zero, in an open set $\Omega \times (0, T)$ with Ω being a bounded domain of $\mathbb{R}^3$, is still unknown. In [9] a nonhomogeneous, viscous incompressible asymmetric fluid in $\Omega = \mathbb{R}^3$ was considered, and the existence of a small time interval where the fluid variables converge uniformly as the viscosities tend to zero was proved. However, the results of [9] are not applicable in our case, that is, when Ω is a bounded domain of $\mathbb{R}^3$. Indeed, the analysis of our situation is still more difficult. The difficulties arise from the lack of smoothness of the weak solution. To overcome this difficulty a penalization argument is needed. This argument generalizes the penalization method given in [10], for the Navier-Stokes equations, to this case of micropolar fluids. In fact, if we take the viscosity of microrotation $\mu_r = 0$, our results imply the other ones in [10], where the analysis of the convergence in an appropriate sense, of solutions of Navier-Stokes equations to the solutions of the Euler equations on a small time interval, is given. It is worthwhile to remark that [10] has been the unique work where the convergence of nonstationary Navier-Stokes equations, with vanishing viscosity, to the Euler equations, in a bounded domain of $\mathbb{R}^3$, has been considered. In the whole space $\mathbb{R}^3$, the authors of [11–13] analyzed the convergence, as the viscosity tends to zero, of the Navier-Stokes equations to the solution of the Euler equations on a small time interval. The two-dimensional case is more usual in the literature. In fact, the book [14] presents a result where the fundamental argument involves the stream formulation for the Navier-Stokes equations, which is not applicable in the three-dimensional case.

This paper is organized as follows. In Section 2 the basic notation is stated and the main results are formulated. In Section 3, the analysis of convergence of solutions of the initial value problem (1.1)–(1.5), when the viscosities ν_1, ν_2, ν_3 tend to zero, is done. This analysis is based on the ideas of [10] for Navier-Stokes equations in bounded domains.

2. Statements and Notations

Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth enough boundary $\partial\Omega$. We consider the usual Sobolev spaces $H^m(\Omega) = \{f \in L^2(\Omega) : \|D^k f\|_{L^2} < \infty, |k| \leq m\}, m \geq 1$, with norm denoted by $\|\cdot\|_{H^m}$. $H_0^1(\Omega)$ is the closure $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^1}$. In order to distinguish the scalar-value functions to vector-value functions, bold characters will be used; for instance, $\mathbf{H}^m = (H^m(\Omega))^3$ and so on. The solenoidal functional spaces $\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega)/\operatorname{div}\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$ and $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega)/\operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}$, will be also used. Here the Helmholtz decomposition of the space $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{G}$, where $\mathbf{G} = \{\varphi : \varphi = \nabla p,\ p \in H^1(\Omega)\}$, is recalled. Throughout the paper, P denotes the orthogonal projection from $\mathbf{L}^2$ onto $\mathbf{H}$. The norm in the L^p-spaces will be denoted by $\|\cdot\|_p$. In particular, the norm in L^2 and its scalar product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Moreover $\langle\cdot,\cdot\rangle$ will denote some duality products. We remark that, in the rest of this paper, the letter C denotes inessential positive constants which may vary from line to line.

In order to study the behavior of system (1.1)–(1.5), when the viscosities ν_1, ν_2, ν_3 tend to zero, the initial value problem (1.6)–(1.10) is required to study. An immediate question related to the system (1.6)–(1.10) is to know about the existence of its solution. In the following lemma a partial result about the existence and uniqueness of solution of problem (1.6)–(1.10) is given. For that, let us consider the following functional space:

$$\mathcal{F}_0 = \left\{ \left(\Phi + t^2 P(\Phi\cdot\nabla\Phi), \Psi + t^2\Phi\cdot\nabla\Psi\right) : \Phi \in \mathbf{V}\cap\mathbf{H}^3, \Psi \in \mathbf{H}_0^1\cap\mathbf{H}^3 \right\} \subset \left(L^\infty\left(0,T;\mathbf{H}_0^3\right)\right)^2. \tag{2.1}$$

Thus we have the following lemma.

Lemma 2.1. *Let* $(\mathbf{f},\mathbf{g}) \in \mathcal{F}_0$*. Then there is a unique solution* $\mathbf{u} \in L^\infty(0,T;\mathbf{V} \cap \mathbf{H}^3)$, $\mathbf{w} \in L^\infty(0,T;\mathbf{H}_0^1 \cap \mathbf{H}^3)$, $p \in L^\infty(0,T;H^2/\mathbb{R})$ *of problem* (1.6)–(1.10).

Proof. The proof follows by using the arguments of [10, Lemma 3.1]. Indeed, with $(\mathbf{f},\mathbf{g})$ being an element of $\mathcal{F}_0$, we consider $(\Phi,\Psi) \in \mathbf{V}\cap\mathbf{H}^3 \times \mathbf{H}_0^1 \cap \mathbf{H}^3$ and define

$$\mathbf{u}(\mathbf{x},t) = t\,\Phi(\mathbf{x}) \in L^\infty\left(0,T;\mathbf{V}\cap\mathbf{H}^3\right), \qquad \mathbf{w}(\mathbf{x},t) = t\Psi(\mathbf{x}) \in L^\infty\left(0,T;\mathbf{H}_0^1\cap\mathbf{H}^3\right). \tag{2.2}$$

Note that the pair $(\mathbf{u},\mathbf{w})$ satisfies conditions (1.4) and (1.5). Moreover, $\mathbf{u}\cdot\nabla\mathbf{u} \in L^\infty(0,T;\mathbf{L}^2)$ and thus, $\mathbf{u}\cdot\nabla\mathbf{u} = (I-P)(\mathbf{u}\cdot\nabla\mathbf{u}) + P(\mathbf{u}\cdot\nabla\mathbf{u})$. Then, $\mathbf{u}_t(\mathbf{x},t) = \Phi(\mathbf{x})$ and $\mathbf{w}_t(\mathbf{x},t) = \Psi(\mathbf{x})$. Hence

$$\begin{gathered}\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \Phi + P(\mathbf{u}\cdot\nabla\mathbf{u}) = \Phi + t^2 P(\Phi\cdot\nabla\Phi) = \mathbf{f},\\ \mathbf{w}_t + \mathbf{u}\cdot\nabla\mathbf{w} = \Psi + \mathbf{u}\cdot\nabla\mathbf{w} = \Psi + t^2\Phi\cdot\nabla\Psi = \mathbf{g},\end{gathered} \tag{2.3}$$

with $\nabla p = -(I-P)(\mathbf{u}\cdot\nabla\mathbf{u}) \in L^\infty(0,T;\mathbf{H}^1)$. Therefore the proof of the existence is finished.

In order to prove the uniqueness, we consider $(\mathbf{u}_1,\mathbf{w}_1,p_1)$ and $(\mathbf{u}_2,\mathbf{w}_2,p_2)$ two solutions of (1.6)–(1.10) and define $\tilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\tilde{\mathbf{w}} = \mathbf{w}_1 - \mathbf{w}_2$. Then, from (1.6) and (1.8), we have

$$\tilde{\mathbf{u}}_t + \mathbf{u}_1\cdot\nabla\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\mathbf{u}_2 + \nabla\left(p_1 - p_2\right) = \mathbf{0}, \tag{2.4}$$

$$\tilde{\mathbf{w}}_t + \tilde{\mathbf{u}}\cdot\nabla\mathbf{w}_1 + \mathbf{u}_2\cdot\nabla\tilde{\mathbf{w}} = \mathbf{0}. \tag{2.5}$$

Taking the inner product of (2.4) with the function $\tilde{\mathbf{u}}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}\|^2 = -(\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}_2,\tilde{\mathbf{u}}) \le C\|\tilde{\mathbf{u}}\|^2\|\nabla\mathbf{u}_2\|_\infty. \tag{2.6}$$

Since $\mathbf{u}_2 \in \mathbf{H}^3(\Omega)$ and $\mathbf{H}^2(\Omega) \subset \mathbf{L}^\infty(\Omega)$, we get

$$\frac{d}{dt}\|\tilde{\mathbf{u}}\|^2 - C_1\|\tilde{\mathbf{u}}\|^2 \le 0 \Longrightarrow \frac{d}{dt}\left(\exp^{-C_1 t}\|\tilde{\mathbf{u}}\|^2\right) \le 0. \tag{2.7}$$

Integrating the last inequality from 0 to t, $t \le T$, we have $\exp^{-C_1 t}\|\tilde{\mathbf{u}}\|^2 \le 0$, which implies $\|\tilde{\mathbf{u}}\| = 0$. Consequently $\mathbf{u}_1 = \mathbf{u}_2$.

Similarly, by taking the inner product of (2.5) with the function $\tilde{\mathbf{w}}$ we find

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{w}}\|^2 = -(\tilde{\mathbf{u}}\cdot\nabla\mathbf{w}_1,\tilde{\mathbf{w}}) = 0. \tag{2.8}$$

Then, by integrating the last equality from 0 to t, we have $\|\tilde{\mathbf{w}}\| = 0$ and thus $\mathbf{w}_1 = \mathbf{w}_2$. □

In the next theorem our main result is stated.

Theorem 2.2. *Let* $(\mathbf{f}, \mathbf{g})$ *be in* $\mathcal{F}_0$. *Then one has the following.*

(1) Existence

There is a weak solution $(\mathbf{u}_\nu, \mathbf{w}_\nu)$ *of problem* (1.1)–(1.5) *verifying*

$$\mathbf{u}_\nu \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}), \qquad \mathbf{w}_\nu \in L^\infty\left(0, T; \mathbf{L}^2\right) \cap L^2\left(0, T; \mathbf{H}_0^1\right), \tag{2.9}$$

where $\mathbf{u}_\nu$ *and* $\mathbf{w}_\nu$ *are dependent on* ν_1, ν_2, ν_3.

(2) Convergence

If $(\mathbf{u}, \mathbf{w})$ *is the unique solution of problem* (1.6)–(1.10) *given by Lemma 2.1, then*

$$\begin{aligned} \|\mathbf{u}_\nu - \mathbf{u}\|_{L^2(0,T;H)} &= O\left((\nu_1 + \nu_2 + \nu_3)^{1/2}\right), \\ \|\mathbf{w}_\nu - \mathbf{w}\|_{L^2(0,T;L^2)} &= O\left((\nu_1 + \nu_2 + \nu_3)^{1/2}\right). \end{aligned} \tag{2.10}$$

Moreover, if $\nu_3 < \nu_1 < \nu_2 < k\nu_1$ *for some constant* k*, as* $\nu_1, \nu_2, \nu_3 \to 0$ *one has*

$$\mathbf{u}_\nu \longrightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; \mathbf{V}), \qquad \mathbf{w}_\nu \longrightarrow \mathbf{w} \quad \text{weakly in } L^2\left(0, T; \mathbf{H}_0^1\right). \tag{2.11}$$

Remark 2.3. (1) Due to that we are interested in the convergence of system (1.1)–(1.5) when ν_1, ν_2, ν_3 go to zero, the assumptions in item (2) of Theorem 2.2 are verified. Moreover, since $\nu_1 = \mu + \mu_r$, if $\mu_r = 0$, system (1.1)–(1.5) decouples and therefore, if ν_1 tends to zero, the known results for the Navier-Stokes equations are recovered.

(2) Note that although in Theorem 2.2 the external sources $\mathbf{f}$ and $\mathbf{g}$ are assumed in the class $\mathcal{F}_0$, the case of constant external sources is covered.

3. Vanishing Viscosity: Proof of Theorem 2.2

The aim of this section is to prove Theorem 2.2. For this the following auxiliary result is needed.

Lemma 3.1. *Let* $\mathbf{u} \in \mathbf{H}_0^1$*, and for real constants* $\xi, \epsilon > 0$ *consider the operator* B_ξ *defined by* $B_\xi \mathbf{u} = (\xi + \epsilon\|\nabla \mathbf{u}\|^2)\nabla \mathbf{u}$. *Then for all* $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1$*, the following inequality holds*

$$\left(B_\xi \mathbf{u} - B_\xi \mathbf{v}, \nabla(\mathbf{u} - \mathbf{v})\right) \geq \xi \, \|\nabla(\mathbf{u} - \mathbf{v})\|^2 + \frac{\epsilon}{2}\|\nabla \mathbf{v}\|^2 \|\nabla(\mathbf{u} - \mathbf{v})\|^2. \tag{3.1}$$

Proof. Using the equality $2(\mathbf{u},\mathbf{v}-\mathbf{u}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2$ and the definition of $B_\xi\mathbf{u}$, we obtain

$$\begin{aligned}
\left(B_\xi\mathbf{u}-B_\xi\mathbf{v},\nabla(\mathbf{u}-\mathbf{v})\right) &= \left(\xi+\epsilon\|\nabla\mathbf{v}\|^2\right)\|\nabla(\mathbf{u}-\mathbf{v})\|^2+\epsilon\left(\|\nabla\mathbf{u}\|^2-\|\nabla\mathbf{v}\|^2\right)(\nabla\mathbf{u},\nabla(\mathbf{u}-\mathbf{v}))\\
&= \left(\xi+\epsilon\|\nabla\mathbf{v}\|^2\right)\|\nabla(\mathbf{u}-\mathbf{v})\|^2+\frac{\epsilon}{2}\left(\|\nabla\mathbf{u}\|^2-\|\nabla\mathbf{v}\|^2\right)^2\\
&\quad+\frac{\epsilon}{2}\left(\|\nabla\mathbf{u}\|^2-\|\nabla\mathbf{v}\|^2\right)\|\nabla(\mathbf{u}-\mathbf{v})\|^2\\
&\geq \xi\,\|\nabla(\mathbf{u}-\mathbf{v})\|^2+\frac{\epsilon}{2}\|\nabla\mathbf{v}\|^2\|\nabla(\mathbf{u}-\mathbf{v})\|^2.
\end{aligned}\tag{3.2}$$

Hence the proof of lemma is finished. □

The next theorem is crucial in the proof of our main result.

Theorem 3.2. *Let* $(\mathbf{f},\mathbf{g})$ *be in* $\mathcal{F}_0$ *and* $\nu=\min\{\nu_1,\nu_2,\nu_3\}$. *Then, for each* ϵ *with* $0<\epsilon<\nu$ *there is a unique solution* $(\mathbf{u}_{\nu\epsilon},\mathbf{w}_{\nu\epsilon})\in L^4(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H})\times L^4(0,T;\mathbf{H}_0^1)\cap L^\infty(0,T;\mathbf{L}^2)$ *of the problem*

$$(\mathbf{u}_{\nu\epsilon})_t-\left(\nu_1+\epsilon\|\nabla\mathbf{u}_{\nu\epsilon}\|^2\right)\Delta\mathbf{u}_{\nu\epsilon}+\mathbf{u}_{\nu\epsilon}\cdot\nabla\mathbf{u}_{\nu\epsilon}+\nabla p_{\nu\epsilon}=2\mu_r\operatorname{rot}\mathbf{w}_{\nu\epsilon}+\mathbf{f},\quad \text{in } Q,\tag{3.3}$$

$$\operatorname{div}\mathbf{u}_{\nu\epsilon}=0,\quad \text{in } Q,\tag{3.4}$$

$$(\mathbf{w}_{\nu\epsilon})_t-\left(\nu_2+\epsilon\|\nabla\mathbf{w}_{\nu\epsilon}\|^2\right)\Delta\mathbf{w}_{\nu\epsilon}-\nu_3\nabla\operatorname{div}\mathbf{w}_{\nu\epsilon}+\mathbf{u}_{\nu\epsilon}\cdot\nabla\mathbf{w}_{\nu\epsilon}+4\mu_r\mathbf{w}_{\nu\epsilon}=2\mu_r\operatorname{rot}\mathbf{u}_{\nu\epsilon}+\mathbf{g},\quad \text{in } Q,\tag{3.5}$$

$$\mathbf{u}_{\nu\epsilon}(\mathbf{x},0)=\mathbf{w}_{\nu\epsilon}(\mathbf{x},0)=0,\quad \text{in } \Omega,\tag{3.6}$$

$$\mathbf{u}_{\nu\epsilon}(\mathbf{x},t)=\mathbf{w}_{\nu\epsilon}(\mathbf{x},t)=0,\quad \text{on } \partial\Omega\times[0,T].\tag{3.7}$$

Proof. In order to prove the existence of solutions of system (3.3)–(3.7), the Galerkin method is used. Let $\mathbf{V}_k$ the subspace of $\mathbf{V}$ spanned by $\{\Phi^1(\mathbf{x}),\ldots,\Phi^k(\mathbf{x})\}$, and $\mathbf{H}_k$ be the subspace of $\mathbf{H}_0^1$ spanned by $\{\Psi^1(x),\ldots,\Psi^k(\mathbf{x})\}$. For each $k\geq 1$, the following approximations $\mathbf{u}_{\nu\epsilon}^k$ and $\mathbf{w}_{\nu\epsilon}^k$, of $\mathbf{u}_{\nu\epsilon}$ and $\mathbf{w}_{\nu\epsilon}$, are defined:

$$\mathbf{u}_{\nu\epsilon}^k(\mathbf{x},t)=\sum_{i=1}^k c_{ik}(t)\Phi^i(\mathbf{x}),\qquad \mathbf{w}_{\nu\epsilon}^k(\mathbf{x},t)=\sum_{i=1}^k d_{ik}(t)\Psi^i(\mathbf{x}),\tag{3.8}$$

for $t \in (0,T)$, where the coefficients $c_{ik}(t)$ and $d_{ik}(t)$ are calculated such that $\mathbf{u}^k_{\nu\epsilon}$ and $\mathbf{w}^k_{\nu\epsilon}$ solve the following system:

$$\begin{gathered}\left(\left(\mathbf{u}^k_{\nu\epsilon}\right)_t, \Phi^i\right) + \left(\nu_1 + \epsilon\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^2\right)\left(\nabla \mathbf{u}^k_{\nu\epsilon}, \nabla \Phi^i\right) + \left(\mathbf{u}^k_{\nu\epsilon} \cdot \nabla \mathbf{u}^k_{\nu\epsilon}, \Phi^i\right) \\ = 2\mu_r\left(\operatorname{rot} \mathbf{w}^k_{\nu\epsilon}, \Phi^i\right) + \left(\mathbf{f}, \Phi^i\right), \\ \left(\left(\mathbf{w}^k_{\nu\epsilon}\right)_t, \Psi^i\right) + \left(\nu_2 + \epsilon\left\|\nabla \mathbf{w}^k_{\nu\epsilon}\right\|^2\right)\left(\nabla \mathbf{w}^k_{\nu\epsilon}, \nabla \Psi^i\right) + \nu_3\left(\operatorname{div} \mathbf{w}^k_{\nu\epsilon}, \operatorname{div} \Psi^i\right) \\ + \left(\mathbf{u}^k_{\nu\epsilon} \cdot \nabla \mathbf{w}^k_{\nu\epsilon}, \Psi^i\right) + 4\mu_r\left(\mathbf{w}^k_{\nu\epsilon}, \Psi^i\right) = 2\mu_r\left(\operatorname{rot} \mathbf{u}^k_{\nu\epsilon}, \Psi^i\right) + \left(\mathbf{g}, \Psi^i\right),\end{gathered} \tag{3.9}$$

for all $\Phi^i \in \mathbf{V}_k$ and $\Psi^i \in \mathbf{H}_k$.

Then, by multiplying (3.9) by c_{ik} and d_{ik}, respectively, summing over i from 0 to k and taking into account (3.8), we have

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}^k_{\nu\epsilon}\right\|^2 + \nu_1\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^2 + \epsilon\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^4 = 2\mu_r\left(\operatorname{rot} \mathbf{w}^k_{\nu\epsilon}, \mathbf{u}^k_{\nu\epsilon}\right) + \left(\mathbf{f}, \mathbf{u}^k_{\nu\epsilon}\right), \\ \frac{1}{2}\frac{d}{dt}\left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2 + \nu_2\left\|\nabla \mathbf{w}^k_{\nu\epsilon}\right\|^2 + \epsilon\left\|\nabla \mathbf{w}^k_{\nu\epsilon}\right\|^4 + \nu_3\left\|\operatorname{div} \mathbf{w}^k_{\nu\epsilon}\right\|^2 + 4\mu_r\left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2 \\ = 2\mu_r\left(\operatorname{rot} \mathbf{u}^k_{\nu\epsilon}, \mathbf{w}^k_{\nu\epsilon}\right) + \left(\mathbf{g}, \mathbf{w}^k_{\nu\epsilon}\right).\end{gathered} \tag{3.10}$$

Now, by applying Hölder's and Young's inequalities we get

$$\begin{gathered}2\mu_r\left(\left(\operatorname{rot} \mathbf{w}^k_{\nu\epsilon}, \mathbf{u}^k_{\nu\epsilon}\right) + \left(\operatorname{rot} \mathbf{u}^k_{\nu\epsilon}, \mathbf{w}^k_{\nu\epsilon}\right)\right) \leq \frac{\nu_1}{2}\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^2 + C\left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2, \\ \left(\mathbf{f}, \mathbf{u}^k_{\nu\epsilon}\right) + \left(\mathbf{g}, \mathbf{w}^k_{\nu\epsilon}\right) \leq C\left(\|\mathbf{f}\|^2 + \|\mathbf{g}\|^2 + \left\|\mathbf{u}^k_{\nu\epsilon}\right\|^2 + \left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2\right).\end{gathered} \tag{3.11}$$

Then, summing (3.10), with the help of last inequalities, we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\left\|\mathbf{u}^k_{\nu\epsilon}\right\|^2 + \left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2\right) + \frac{\nu_1}{2}\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^2 + \nu_2\left\|\nabla \mathbf{w}^k_{\nu\epsilon}\right\|^2 + \epsilon\left(\left\|\nabla \mathbf{u}^k_{\nu\epsilon}\right\|^4 + \left\|\nabla \mathbf{w}^k_{\nu\epsilon}\right\|^4\right) \\ \leq C\left(\left\|\mathbf{u}^k_{\nu\epsilon}\right\|^2 + \left\|\mathbf{w}^k_{\nu\epsilon}\right\|^2\right) + C\left(\|\mathbf{f}\|^2 + \|\mathbf{g}\|^2\right),\end{gathered} \tag{3.12}$$

and hence, by integrating (3.12) from 0 to t, $t \in (0,T]$, we find

$$\begin{aligned}
&\left\|\mathbf{u}_{\nu\epsilon}^{k}(t)\right\|^{2}+\left\|\mathbf{w}_{\nu\epsilon}^{k}(t)\right\|^{2}+\int_{0}^{t}\left(\nu_{1}\left\|\nabla \mathbf{u}_{\nu\epsilon}^{k}\right\|^{2}+2\nu_{2}\left\|\nabla \mathbf{w}_{\nu\epsilon}^{k}\right\|^{2}\right)ds \\
&\quad+2\epsilon\int_{0}^{t}\left(\left\|\nabla \mathbf{u}_{\nu\epsilon}^{k}\right\|^{4}+\left\|\nabla \mathbf{w}_{\nu\epsilon}^{k}\right\|^{4}\right)ds \\
&\quad\leq C\int_{0}^{t}\left(\left\|\mathbf{u}_{\nu\epsilon}^{k}\right\|^{2}+\left\|\mathbf{w}_{\nu\epsilon}^{k}\right\|^{2}\right)ds+C\int_{0}^{t}\left(\|\mathbf{f}\|^{2}+\|\mathbf{g}\|^{2}\right)ds.
\end{aligned} \tag{3.13}$$

Applying Gronwall's inequality in (3.13) we get

$$\begin{aligned}
&\left\|\mathbf{u}_{\nu\epsilon}^{k}(t)\right\|^{2}+\left\|\mathbf{w}_{\nu\epsilon}^{k}(t)\right\|^{2}+\int_{0}^{t}\left(\nu_{1}\left\|\nabla \mathbf{u}_{\nu\epsilon}^{k}\right\|^{2}+2\nu_{2}\left\|\nabla \mathbf{w}_{\nu\epsilon}^{k}\right\|^{2}\right)ds \\
&\quad+2\epsilon\int_{0}^{t}\left(\left\|\nabla \mathbf{u}_{\nu\epsilon}^{k}\right\|^{4}+\left\|\nabla \mathbf{w}_{\nu\epsilon}^{k}\right\|^{4}\right)ds \leq CTe^{CT}\leq C.
\end{aligned} \tag{3.14}$$

Thus, from (3.14) we conclude that there is $\mathbf{u}_{\nu\epsilon}, \mathbf{w}_{\nu\epsilon}$ such that as $k \to \infty$

$$\begin{aligned}
&\mathbf{u}_{\nu\epsilon}^{k} \longrightarrow \mathbf{u}_{\nu\epsilon} \quad \text{weakly-* in } L^{\infty}(0,T;\mathbf{H}) \text{ and weakly in } L^{2}(0,T;\mathbf{V}), \\
&\mathbf{w}_{\nu\epsilon}^{k} \longrightarrow \mathbf{w}_{\nu\epsilon} \quad \text{weakly-* in } L^{\infty}\left(0,T;\mathbf{L}^{2}\right) \text{ and weakly in } L^{2}\left(0,T;\mathbf{H}_{0}^{1}\right).
\end{aligned} \tag{3.15}$$

Now, since $\mathbf{V} \subset \mathbf{L}^{6}$ and $\mathbf{H}_{0}^{1} \subset \mathbf{L}^{6}$, from (3.14) we have that $\mathbf{u}_{\nu\epsilon}^{k}, \mathbf{w}_{\nu\epsilon}^{k} \in L^{4}(0,T;\mathbf{L}^{6})$; consequently $(\mathbf{u}_{\nu\epsilon}^{k})_{t} \in L^{4/3}(0,T;\mathbf{V}^{*})$ and $(\mathbf{w}_{\nu\epsilon}^{k})_{t} \in L^{4/3}(0,T;\mathbf{H}^{-1})$, with $\mathbf{V}^{*}$ and $\mathbf{H}^{-1}$ being the topological duals of $\mathbf{V}$ and $\mathbf{H}_{0}^{1}$, respectively. Therefore

$$\begin{aligned}
&\left(\mathbf{u}_{\nu\epsilon}^{k}\right)_{t} \longrightarrow (\mathbf{u}_{\nu\epsilon})_{t} \quad \text{weakly in } L^{4/3}(0,T;\mathbf{V}^{*}), \\
&\left(\mathbf{w}_{\nu\epsilon}^{k}\right)_{t} \longrightarrow (\mathbf{w}_{\nu\epsilon})_{t} \quad \text{weakly in } L^{4/3}\left(0,T;\mathbf{H}^{-1}\right).
\end{aligned} \tag{3.16}$$

Since $\mathbf{V} \subset \mathbf{H}$ is compact and $\mathbf{H} \subset \mathbf{V}^{*}$ is continuous, as well as $\mathbf{V} \subset \mathbf{L}^{2}$ is compact and $\mathbf{L}^{2} \subset \mathbf{H}^{-1}$ is continuous, then as $k \to \infty$, we obtain

$$\mathbf{u}_{\nu\epsilon}^{k} \longrightarrow \mathbf{u}_{\nu\epsilon} \quad \text{in } L^{2}(0,T;\mathbf{H}), \qquad \mathbf{w}_{\nu\epsilon}^{k} \longrightarrow \mathbf{w}_{\nu\epsilon} \quad \text{in } L^{2}\left(0,T;\mathbf{L}^{2}\right). \tag{3.17}$$

In order to pass to the limit in (3.9) we take into account (3.15) and (3.17). Indeed, the convergence in the linear terms follows directly. Moreover, as in [2, page 289], we can prove that as $k \to \infty$

$$\int_0^T \left| \left(\mathbf{u}_{\nu\epsilon}^k \cdot \nabla \mathbf{u}_{\nu\epsilon}^k - \mathbf{u}_{\nu\epsilon} \cdot \nabla \mathbf{u}_{\nu\epsilon}, \Phi^i \right) \phi(t) \right| dt \longrightarrow 0,$$
$$\int_0^T \left| \left(\mathbf{u}_{\nu\epsilon}^k \cdot \nabla \mathbf{w}_{\nu\epsilon}^k - \mathbf{u}_{\nu\epsilon} \cdot \nabla \mathbf{w}_{\nu\epsilon}, \Psi^i \right) \phi(t) \right| dt \longrightarrow 0, \tag{3.18}$$

for all $\Phi^i \in \mathbf{V}_k, \Psi^i \in \mathbf{H}_k$ and all $\phi \in C_0^\infty(0,T)$. Finally, from (3.15) we have

$$\left(\nabla \mathbf{u}_{\nu\epsilon}^k, \nabla \Phi^i \right) \longrightarrow \left(\nabla \mathbf{u}_{\nu\epsilon}, \nabla \Phi^i \right), \quad \left(\nabla \mathbf{w}_{\nu\epsilon}^k, \nabla \Psi^i \right) \longrightarrow \left(\nabla \mathbf{w}_{\nu\epsilon}, \nabla \Psi^i \right), \quad \text{as } k \longrightarrow \infty, \tag{3.19}$$

and hence, by taking $\nabla\Phi^i = \nabla \mathbf{u}_{\nu\epsilon}^k$, $\nabla \Psi^i = \nabla \mathbf{w}_{\nu\epsilon}^k$ in (3.19), and then, $\nabla\Phi^i = \nabla \mathbf{u}_{\nu\epsilon}$, $\nabla\Psi^i = \nabla \mathbf{w}_{\nu\epsilon}$, as $k \to \infty$ we get

$$\left\| \nabla \mathbf{u}_{\nu\epsilon}^k \right\|^2 \longrightarrow \left(\nabla \mathbf{u}_{\nu\epsilon}, \nabla \mathbf{u}_{\nu\epsilon}^k \right) \longrightarrow \| \nabla \mathbf{u}_{\nu\epsilon} \|^2, \qquad \left\| \nabla \mathbf{w}_{\nu\epsilon}^k \right\|^2 \longrightarrow \left(\nabla \mathbf{w}_{\nu\epsilon}, \nabla \mathbf{w}_{\nu\epsilon}^k \right) \longrightarrow \| \nabla \mathbf{w}_{\nu\epsilon} \|^2. \tag{3.20}$$

Moreover, as

$$\left(\left\| \nabla \mathbf{u}_{\nu\epsilon}^k \right\|^2 \nabla \mathbf{u}_{\nu\epsilon}^k - \| \nabla \mathbf{u}_{\nu\epsilon} \|^2 \nabla \mathbf{u}_{\nu\epsilon}, \nabla \Phi^i \right) = \left(\left\| \nabla \mathbf{u}_{\nu\epsilon}^k \right\|^2 - \| \nabla \mathbf{u}_{\nu\epsilon} \|^2 \right) \left(\nabla \mathbf{u}_{\nu\epsilon}^k, \nabla \Phi^i \right) + \| \nabla \mathbf{u}_{\nu\epsilon} \|^2 \left(\nabla \mathbf{u}_{\nu\epsilon}^k - \nabla \mathbf{u}_{\nu\epsilon}, \nabla \Phi^i \right),$$
$$\left(\left\| \nabla \mathbf{w}_{\nu\epsilon}^k \right\|^2 \nabla \mathbf{w}_{\nu\epsilon}^k - \| \nabla \mathbf{w}_{\nu\epsilon} \|^2 \nabla \mathbf{w}_{\nu\epsilon}, \nabla \Psi^i \right) = \left(\left\| \nabla \mathbf{w}_{\nu\epsilon}^k \right\|^2 - \| \nabla \mathbf{w}_{\nu\epsilon} \|^2 \right) \left(\nabla \mathbf{w}_{\nu\epsilon}^k, \nabla \Psi^i \right) + \| \nabla \mathbf{w}_{\nu\epsilon} \|^2 \left(\nabla \mathbf{w}_{\nu\epsilon}^k - \nabla \mathbf{w}_{\nu\epsilon}, \nabla \Psi^i \right), \tag{3.21}$$

we conclude that

$$\int_0^T \left(\left\| \nabla \mathbf{u}_{\nu\epsilon}^k \right\|^2 \nabla \mathbf{u}_{\nu\epsilon}^k - \| \nabla \mathbf{u}_{\nu\epsilon} \|^2 \nabla \mathbf{u}_{\nu\epsilon}, \nabla \Phi^i \right) \phi(t) dt \longrightarrow 0,$$
$$\int_0^T \left(\left\| \nabla \mathbf{w}_{\nu\epsilon}^k \right\|^2 \nabla \mathbf{w}_{\nu\epsilon}^k - \| \nabla \mathbf{w}_{\nu\epsilon} \|^2 \nabla \mathbf{w}_{\nu\epsilon}, \nabla \Psi^i \right) \phi(t) dt \longrightarrow 0. \tag{3.22}$$

Now the uniqueness of solution will be analyzed. Let $(\mathbf{u}^1_{\nu\epsilon}, \mathbf{w}^1_{\nu\epsilon}, p^1_{\nu\epsilon})$ and $(\mathbf{u}^2_{\nu\epsilon}, \mathbf{w}^2_{\nu\epsilon}, p^2_{\nu\epsilon})$ be two solutions of (3.3)–(3.7). We denote $\tilde{\mathbf{u}}_{\nu\epsilon} = \mathbf{u}^1_{\nu\epsilon} - \mathbf{u}^2_{\nu\epsilon}$, $\tilde{\mathbf{w}}_{\nu\epsilon} = \mathbf{w}^1_{\nu\epsilon} - \mathbf{w}^2_{\nu\epsilon}$, and $\tilde{p}_{\nu\epsilon} = p^1_{\nu\epsilon} - p^2_{\nu\epsilon}$. Then we have

$$\begin{aligned} &(\tilde{\mathbf{u}}_{\nu\epsilon})_t - \left(\nu_1 + \epsilon\left\|\nabla \mathbf{u}^1_{\nu\epsilon}\right\|^2\right)\Delta \mathbf{u}^1_{\nu\epsilon} + \left(\nu_1 + \epsilon\left\|\nabla \mathbf{u}^2_{\nu\epsilon}\right\|^2\right)\Delta \mathbf{u}^2_{\nu\epsilon} + \tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla \mathbf{u}^1_{\nu\epsilon} \\ &\quad + \nabla \tilde{p}_{\nu\epsilon} + \mathbf{u}^2_{\nu\epsilon}\cdot\nabla\tilde{\mathbf{u}}_{\nu\epsilon} = 2\mu_r \operatorname{rot}\tilde{\mathbf{w}}_{\nu\epsilon}, \end{aligned} \tag{3.23}$$

$$\operatorname{div}\tilde{\mathbf{u}}_{\nu\epsilon} = 0, \tag{3.24}$$

$$\begin{aligned} &(\tilde{\mathbf{w}}_{\nu\epsilon})_t - \left(\nu_2 + \epsilon\left\|\nabla \mathbf{w}^1_{\nu\epsilon}\right\|^2\right)\Delta \mathbf{w}^1_{\nu\epsilon} + \left(\nu_2 + \epsilon\left\|\nabla \mathbf{w}^2_{\nu\epsilon}\right\|^2\right)\Delta \mathbf{w}^2_{\nu\epsilon} - \nu_3\nabla\operatorname{div}\tilde{\mathbf{w}}_{\nu\epsilon} \\ &\quad + \tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla \mathbf{w}^1_{\nu\epsilon} + \mathbf{u}^2_{\nu\epsilon}\cdot\nabla\tilde{\mathbf{w}}_{\nu\epsilon} + 4\mu_r\tilde{\mathbf{w}}_{\nu\epsilon} = 2\mu_r \operatorname{rot}\tilde{\mathbf{u}}_{\nu\epsilon}. \end{aligned} \tag{3.25}$$

Taking the inner product of (3.23) with $\tilde{\mathbf{u}}_{\nu\epsilon}$, of (3.25) with $\tilde{\mathbf{w}}_{\nu\epsilon}$, by using (3.24), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 + \left(B_{\nu_1}\mathbf{u}^1_{\nu\epsilon}, \nabla\tilde{\mathbf{u}}_{\nu\epsilon}\right) - \left(B_{\nu_1}\mathbf{u}^2_{\nu\epsilon}, \nabla\tilde{\mathbf{u}}_{\nu\epsilon}\right) &= -\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{u}^1_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon}\right) \\ &\quad + 2\mu_r(\operatorname{rot}\tilde{\mathbf{w}}_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon}), \\ \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 + \left(B_{\nu_2}\mathbf{w}^1_{\nu\epsilon}, \nabla\tilde{\mathbf{w}}_{\nu\epsilon}\right) - \left(B_{\nu_2}\mathbf{w}^2_{\nu\epsilon}, \nabla\tilde{\mathbf{w}}_{\nu\epsilon}\right) &\le -\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{w}^1_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon}\right) \\ &\quad + 2\mu_r|(\operatorname{rot}\tilde{\mathbf{u}}_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon})|. \end{aligned} \tag{3.26}$$

Hence, by using Lemma 3.1 we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 + \nu_1\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 + \frac{\epsilon}{2}\left\|\nabla\mathbf{u}^2_{\nu\epsilon}\right\|^2\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 &\le \left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{u}^1_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon}\right)\right| \\ &\quad + 2\mu_r|(\operatorname{rot}\tilde{\mathbf{w}}_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon})|, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 + \nu_2\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 + \frac{\epsilon}{2}\left\|\nabla\mathbf{w}^2_{\nu\epsilon}\right\|^2\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 &\le \left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{w}^1_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon}\right)\right| \\ &\quad + 2\mu_r|(\operatorname{rot}\tilde{\mathbf{u}}_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon})|. \end{aligned} \tag{3.27}$$

Consequently

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 + \nu_1\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 &\le \left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{u}^1_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon}\right)\right| + 2\mu_r|(\operatorname{rot}\tilde{\mathbf{w}}_{\nu\epsilon}, \tilde{\mathbf{u}}_{\nu\epsilon})|, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 + \nu_2\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 &\le \left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{w}^1_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon}\right)\right| + 2\mu_r|(\operatorname{rot}\tilde{\mathbf{u}}_{\nu\epsilon}, \tilde{\mathbf{w}}_{\nu\epsilon})|. \end{aligned} \tag{3.28}$$

Now, by using Hölder's and Young's inequalities we have

$$
\begin{aligned}
\left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{u}^1_{\nu\epsilon},\tilde{\mathbf{u}}_{\nu\epsilon}\right)\right| &\le \|\tilde{\mathbf{u}}_{\nu\epsilon}\|_3\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|\left\|\mathbf{u}^1_{\nu\epsilon}\right\|_6 \\
&\le C\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^{1/2}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^{3/2}\left\|\nabla\mathbf{u}^1_{\nu\epsilon}\right\| \\
&\le C\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2\left\|\nabla\mathbf{u}^1_{\nu\epsilon}\right\|^4+\frac{\nu_1}{4}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2, \\
\left|\left(\tilde{\mathbf{u}}_{\nu\epsilon}\cdot\nabla\mathbf{w}^1_{\nu\epsilon},\tilde{\mathbf{w}}_{\nu\epsilon}\right)\right| &\le C\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^{1/2}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^{1/2}\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|\left\|\nabla\mathbf{w}^1_{\nu\epsilon}\right\| \\
&\le C\|\tilde{\mathbf{u}}_{\nu\epsilon}\|\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|\left\|\nabla\mathbf{w}^1_{\nu\epsilon}\right\|^2+\nu_2\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|^2 \\
&\le C\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2\left\|\nabla\mathbf{w}^1_{\nu\epsilon}\right\|^4+\frac{\nu_1}{4}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2+\nu_2\|\nabla\tilde{\mathbf{w}}_{\nu\epsilon}\|^2, \\
2\mu_r|(\operatorname{rot}\tilde{\mathbf{w}}_{\nu\epsilon},\tilde{\mathbf{u}}_{\nu\epsilon})| &\le C\|\tilde{\mathbf{w}}_{\nu\epsilon}\|\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\| \le C\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2+\frac{\nu_1}{4}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2, \\
2\mu_r|(\operatorname{rot}\tilde{\mathbf{u}}_{\nu\epsilon},\tilde{\mathbf{w}}_{\nu\epsilon})| &\le C\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2+\frac{\nu_1}{4}\|\nabla\tilde{\mathbf{u}}_{\nu\epsilon}\|^2.
\end{aligned}
\tag{3.29}
$$

Then, by summing (3.28), and taking into account the last inequalities, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2+\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2\right) &\le C\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2+C\left(\left\|\nabla\mathbf{u}^1_{\nu\epsilon}\right\|^4+\left\|\nabla\mathbf{w}^1_{\nu\epsilon}\right\|^4\right)\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2 \\
&\le CM(t)\left(\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2+\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2\right),
\end{aligned}
\tag{3.30}
$$

where $M(t)=1+\|\nabla\mathbf{u}^2_{\nu\epsilon}(t)\|^4+\|\nabla\mathbf{w}^2_{\nu\epsilon}(t)\|^4$.

Since $M(t)\in L^1(0,T)$, by integrating (3.30) from 0 to t, and then applying Gronwall's inequality, we conclude that

$$
\|\tilde{\mathbf{u}}_{\nu\epsilon}\|^2+\|\tilde{\mathbf{w}}_{\nu\epsilon}\|^2\le 0,
\tag{3.31}
$$

which implies $\tilde{\mathbf{u}}_{\nu\epsilon}=0$ and $\tilde{\mathbf{w}}_{\nu\epsilon}=0$. Consequently the uniqueness of solution is proved. □

Proposition 3.3. *Under the assumptions of Lemma 2.1 and Theorem 3.2, if* $(\mathbf{u}_{\nu\epsilon},\mathbf{w}_{\nu\epsilon})$ *and* $(\mathbf{u},\mathbf{w})$ *are the solutions of problems* (3.3)–(3.7) *and* (1.6)–(1.10), *respectively, then*

$$
\begin{aligned}
\|\mathbf{u}_{\nu\epsilon}-\mathbf{u}\|_{L^\infty(0,T;H)} &= O\left((\nu_1+\epsilon+\nu_2+\nu_3)^{1/2}\right), \\
\nu_1^{1/2}\|\nabla(\mathbf{u}_{\nu\epsilon}-\mathbf{u})\|_{L^2(0,T;L^2)} &= O\left((\nu_1+\epsilon+\nu_2+\nu_3)^{1/2}\right), \\
\|\mathbf{w}_{\nu\epsilon}-\mathbf{w}\|_{L^\infty(0,T;L^2)} &= O\left((\nu_1+\epsilon+\nu_2+\nu_3)^{1/2}\right), \\
\nu_2^{1/2}\|\nabla(\mathbf{w}_{\nu\epsilon}-\mathbf{w})\|_{L^2(0,T;L^2)} &= O\left((\nu_1+\epsilon+\nu_2+\nu_3)^{1/2}\right).
\end{aligned}
\tag{3.32}
$$

Proof. Considering the differences between (1.6) and (3.3), as well as between (1.8) and (3.5) and then by taking the inner product with $\mathbf{v} = \mathbf{u}_{\nu\epsilon} - \mathbf{u}$ and $\mathbf{z} = \mathbf{w}_{\nu\epsilon} - \mathbf{w}$, respectively, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \left(\nu_1 + \epsilon\|\nabla\mathbf{u}_{\nu\epsilon}\|^2\right)(\nabla\mathbf{u}_{\nu\epsilon}, \nabla\mathbf{v}) + (\mathbf{v}\cdot\nabla\mathbf{u}, \mathbf{v}) = 2\mu_r(\operatorname{rot}\mathbf{w}_{\nu\epsilon}, \mathbf{v}),\\
&\frac{1}{2}\frac{d}{dt}\|\mathbf{z}\|^2 + \left(\nu_2 + \epsilon\|\nabla\mathbf{w}_{\nu\epsilon}\|^2\right)(\nabla\mathbf{w}_{\nu\epsilon}, \nabla\mathbf{z}) + \nu_3(\operatorname{div}\mathbf{w}_{\nu\epsilon}, \operatorname{div}\mathbf{z}) + (\mathbf{v}\cdot\nabla\mathbf{w}, \mathbf{z})\\
&\qquad + 4\mu_r(\mathbf{w}_{\nu\epsilon}, \mathbf{z}) = 2\mu_r(\operatorname{rot}\mathbf{u}_{\nu\epsilon}, \mathbf{z}).
\end{aligned} \tag{3.33}$$

Recalling the notation $B_\xi\widetilde{\varphi} = (\xi + \epsilon\|\nabla\widetilde{\varphi}\|^2)\nabla\widetilde{\varphi}$ we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + (B_{\nu_1}\mathbf{u}_{\nu\epsilon} - B_{\nu_1}\mathbf{u}, \nabla\mathbf{v})\\
&\quad = -(B_{\nu_1}\mathbf{u}, \nabla\mathbf{v}) - (\mathbf{v}\cdot\nabla\mathbf{u}, \mathbf{v}) + 2\mu_r(\operatorname{rot}\mathbf{z}, \mathbf{v}) + 2\mu_r(\operatorname{rot}\mathbf{w}, \mathbf{v}),
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{z}\|^2 + (B_{\nu_2}\mathbf{w}_{\nu\epsilon} - B_{\nu_2}\mathbf{w}, \nabla\mathbf{z}) + \nu_3\|\operatorname{div}\mathbf{z}\|^2 + 4\mu_r\|\mathbf{z}\|^2\\
&\quad = -(B_{\nu_2}\mathbf{w}, \nabla\mathbf{z}) - \nu_3(\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{z}) - 4\mu_r(\mathbf{w}, \mathbf{z}) - (\mathbf{v}\cdot\nabla\mathbf{w}, \mathbf{z})\\
&\qquad + 2\mu_r(\operatorname{rot}\mathbf{v}, \mathbf{z}) + 2\mu_r(\operatorname{rot}\mathbf{u}, \mathbf{z}).
\end{aligned} \tag{3.35}$$

Using Hölder's and Young's inequalities we bound the right hand of (3.34) and (3.35) as follows:

$$(B_{\nu_1}\mathbf{u}, \nabla\mathbf{v}) \le \frac{\nu_1 + \epsilon\|\nabla\mathbf{u}\|^2}{2}\left(\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{v}\|^2\right), \tag{3.36}$$

$$(\mathbf{v}\cdot\nabla\mathbf{u}, \mathbf{v}) \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{v}\|^2 \le C\|\mathbf{v}\|^2, \tag{3.37}$$

$$2\mu_r(\operatorname{rot}\mathbf{z}, \mathbf{v}) \le C\nu_1\|\nabla\mathbf{v}\|\,\|\mathbf{z}\| \le \frac{\nu_1^2}{8}\|\nabla\mathbf{v}\|^2 + C\|\mathbf{z}\|^2, \tag{3.38}$$

$$2\mu_r(\operatorname{rot}\mathbf{w}, \mathbf{v}) \le \nu_1^2\|\nabla\mathbf{w}\|^2 + C\|\mathbf{v}\|^2, \tag{3.39}$$

$$(B_{\nu_2}\mathbf{w}, \nabla\mathbf{z}) \le \frac{\nu_2 + \epsilon\|\nabla\mathbf{w}\|^2}{2}\left(\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right), \tag{3.40}$$

$$\nu_3(\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{z}) \le C\nu_3\|\nabla\mathbf{w}\|^2 + \nu_3\|\operatorname{div}\mathbf{z}\|^2, \tag{3.41}$$

$$4\mu_r(\mathbf{w}, \mathbf{z}) \le \nu_1^2\|\mathbf{w}\|^2 + C\|\mathbf{z}\|^2, \tag{3.42}$$

$$(\mathbf{v}\cdot\nabla\mathbf{w}, \mathbf{z}) \le C\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right), \tag{3.43}$$

$$2\mu_r(\operatorname{rot}\mathbf{v}, \mathbf{z}) \le \frac{\nu_1^2}{8}\|\nabla\mathbf{v}\|^2 + C\|\mathbf{z}\|^2, \tag{3.44}$$

$$2\mu_r(\operatorname{rot}\mathbf{u}, \mathbf{z}) \le \nu_1^2\|\nabla\mathbf{u}\|^2 + C\|\mathbf{z}\|^2. \tag{3.45}$$

Carrying (3.36)–(3.39) in (3.34) and (3.40)–(3.45) in (3.35), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + (B_{\nu_1}\mathbf{u}_{\nu\epsilon} - B_{\nu_1}\mathbf{u}, \nabla\mathbf{v}) &\le \frac{\nu_1 + \epsilon\|\nabla\mathbf{u}\|^2}{2}\left(\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{v}\|^2\right) + C\|\mathbf{v}\|^2 \\
&\quad + \frac{\nu_1}{8}\|\nabla\mathbf{v}\|^2 + C\|\mathbf{z}\|^2 + \nu_1\|\nabla\mathbf{w}\|^2, \\
\frac{1}{2}\frac{d}{dt}\|\mathbf{z}\|^2 + (B_{\nu_2}\mathbf{w}_{\nu\epsilon} - B_{\nu_2}\mathbf{w}, \nabla\mathbf{z}) &\le \frac{\nu_2 + \epsilon\|\nabla\mathbf{w}\|^2}{2}\left(\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2\right) + C\nu_3\|\nabla\mathbf{w}\|^2 \\
&\quad + \nu_1\|\mathbf{w}\|^2 + C\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right) + \frac{\nu_1}{8}\|\nabla\mathbf{v}\|^2 + C\nu_1\|\nabla\mathbf{u}\|^2.
\end{aligned} \tag{3.46}$$

Now, by using the equality $2(\mathbf{u}, \mathbf{v} - \mathbf{u}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2$, the definition of $B_\xi\varphi$ and Lemma 3.1, from (3.46) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right) + \frac{\nu_1}{4}\|\nabla\mathbf{v}\|^2 + \frac{\nu_2}{2}\|\nabla\mathbf{z}\|^2 \\
&\quad \le \frac{\nu_1 + \epsilon\|\nabla\mathbf{u}\|^2}{2}\|\nabla\mathbf{u}\|^2 + \frac{\nu_2 + \epsilon\|\nabla\mathbf{w}\|^2}{2}\|\nabla\mathbf{w}\|^2 + C\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right) + C(\nu_1 + \nu_3),
\end{aligned} \tag{3.47}$$

which implies

$$\frac{d}{dt}\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right) + \nu_1\|\nabla\mathbf{v}\|^2 + \nu_2\|\nabla\mathbf{z}\|^2 \le C(\epsilon + \nu_1 + \nu_2 + \nu_3) + C\left(\|\mathbf{v}\|^2 + \|\mathbf{z}\|^2\right). \tag{3.48}$$

Since $\mathbf{v}(0) = 0$ and $\mathbf{z}(0) = 0$, by integrating (3.48) from 0 to t, $t \in [0, T]$, and then applying Gronwall's inequality, we obtain

$$\|\mathbf{v}(t)\|^2 + \|\mathbf{z}(t)\|^2 + \int_0^t \left(\nu_1\|\nabla\mathbf{v}(s)\|^2 + \nu_2\|\nabla\mathbf{z}(s)\|^2\right)ds \le C(\epsilon + \nu_1 + \nu_2 + \nu_3)T\exp(CT), \tag{3.49}$$

and hence the proof of estimates (3.32) is concluded. □

Proposition 3.4. *Under the assumptions of Theorem 3.2 and considering $\nu_3 < \nu_2$, then as $\epsilon \to 0$ the solution $(\mathbf{u}_{\nu\epsilon}, \mathbf{w}_{\nu\epsilon})$ of* (3.3)–(3.7) *verifies the following convergences:*

$$\mathbf{u}_{\nu\epsilon} \longrightarrow \mathbf{u}_\nu \quad in\ L^2(0, T; \mathbf{H}), \qquad \mathbf{w}_{\nu\epsilon} \longrightarrow \mathbf{w}_\nu \quad in\ L^2\left(0, T; \mathbf{L}^2\right), \tag{3.50}$$

where $(\mathbf{u}_\nu, \mathbf{w}_\nu)$ is a solution of problem (1.1)–(1.3).

Proof. Let $\mathbf{u}_{\nu\epsilon}, \mathbf{w}_{\nu\epsilon}$ be as in Theorem 3.2. Then from (3.14) we have

$$\begin{gathered}
\|\mathbf{u}_{\nu\epsilon}\|^2_{L^\infty(0,T;H)} + \nu_1\|\mathbf{u}_{\nu\epsilon}\|^2_{L^2(0,T;V)} + \epsilon\|\mathbf{u}_{\nu\epsilon}\|^4_{L^4(0,T;V)} \le C, \\
\|\mathbf{w}_{\nu\epsilon}\|^2_{L^\infty(0,T;L^2)} + \nu_2\|\mathbf{w}_{\nu\epsilon}\|^2_{L^2(0,T;H_0^1)} + \epsilon\|\mathbf{w}_{\nu\epsilon}\|^4_{L^4(0,T;H_0^1)} \le C,
\end{gathered} \tag{3.51}$$

where C is a constant which does not depend on ν_1, ν_2, ν_3, and ϵ.

Then, since $\mathbf{V} \subset \mathbf{L}^6$ and $\mathbf{H}_0^1 \subset \mathbf{L}^6$, we have that $\mathbf{u}_{\nu\epsilon}, \mathbf{w}_{\nu\epsilon} \in L^4(0,T;\mathbf{L}^6)$. Consequently $(\mathbf{u}_{\nu\epsilon})_t \in L^{4/3}(0,T;\mathbf{V}^*)$ and $(\mathbf{w}_{\nu\epsilon})_t \in L^{4/3}(0,T;\mathbf{H}^{-1})$, with $\mathbf{V}^*$ and $\mathbf{H}^{-1}$ being the topological dual of $\mathbf{V}$ and $\mathbf{H}_0^1$, respectively.

Now, since $|(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{z})| \le \|\mathbf{u}\|_3\|\nabla\mathbf{w}\|\|\mathbf{z}\|_6 \le C\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_V^{1/2}\|\nabla\mathbf{w}\|\|\nabla\mathbf{z}\|$, from (3.3) and (3.5) we obtain

$$\begin{aligned}
\|(\mathbf{u}_{\nu\epsilon})_t\|_{V^*} &\le C\Big(\nu_1\|\mathbf{u}_{\nu\epsilon}\|_V + \epsilon\|\mathbf{u}_{\nu\epsilon}\|_V^3 + \|\mathbf{u}_{\nu\epsilon}\|^{1/2}\|\mathbf{u}_{\nu\epsilon}\|_V^{3/2} + \nu_1\|\mathbf{w}_{\nu\epsilon}\| + \|\mathbf{f}\|\Big), \\
\|(\mathbf{w}_{\nu\epsilon})_t\|_{H^{-1}} &\le C\Big(\nu_2\|\mathbf{w}_{\nu\epsilon}\|_{H_0^1} + \epsilon\|\mathbf{w}_{\nu\epsilon}\|^3_{H_0^1} + \nu_3\|\mathbf{w}_{\nu\epsilon}\|_{H_0^1} + \|\mathbf{u}_{\nu\epsilon}\|^{3/2}\|\mathbf{u}_{\nu\epsilon}\|_V^{3/2}\Big) \\
&\quad + C\left(\|\mathbf{w}_{\nu\epsilon}\|^{3/2}_{H_0^1} + \nu_1\|\mathbf{w}_{\nu\epsilon}\| + \nu_1\|\mathbf{u}_{\nu\epsilon}\| + \|\mathbf{g}\|\right).
\end{aligned} \tag{3.52}$$

Then, by using $(\sum_{i=1}^n |a_i|)^{4/3} \le C\sum_{i=1}^n |a_i|^{4/3}$ and (3.51), from the last inequalities we get

$$\begin{aligned}
\|(\mathbf{u}_{\nu\epsilon})_t\|^{4/3}_{V^*} &\le C\Big(\nu_1^{4/3}\|\mathbf{u}_{\nu\epsilon}\|_V^{4/3} + \epsilon^{4/3}\|\mathbf{u}_{\nu\epsilon}\|_V^4 + \|\mathbf{u}_{\nu\epsilon}\|_V^2 + \nu_1^{4/3} + \|\mathbf{f}\|^{4/3}\Big), \\
\|(\mathbf{w}_{\nu\epsilon})_t\|^{4/3}_{H^{-1}} &\le C\left(\nu_2^{4/3}\|\mathbf{w}_{\nu\epsilon}\|^{4/3}_{H_0^1} + \epsilon^{4/3}\|\mathbf{w}_{\nu\epsilon}\|^4_{H_0^1} + \nu_3^{4/3}\|\mathbf{w}_{\nu\epsilon}\|^{4/3}_{H_0^1}\right) \\
&\quad + C\Big(\|\mathbf{u}_{\nu\epsilon}\|_V^2 + \|\mathbf{w}_{\nu\epsilon}\|^2_{H_0^1} + \nu_1^{4/3} + \|\mathbf{g}\|^{4/3}\Big).
\end{aligned} \tag{3.53}$$

Integrating the last inequalities from 0 to T we conclude

$$\begin{aligned}
\int_0^T \|(\mathbf{u}_{\nu\epsilon})_t\|^{4/3}_{V^*}dt &\le C\left(\nu_1^{4/3}\int_0^T \|\mathbf{u}_{\nu\epsilon}\|_V^{4/3}dt + \nu_1^{4/3}T + \int_0^T \|\mathbf{f}\|^{4/3}dt\right) \\
&\quad + C\Big(\epsilon^{4/3}\|\mathbf{u}_{\nu\epsilon}\|^4_{L^4(0,T;V)} + \|\mathbf{u}_{\nu\epsilon}\|^2_{L^2(0,T;V)}\Big), \\
\int_0^T \|(\mathbf{w}_{\nu\epsilon})_t\|^{4/3}_{H^{-1}}dt &\le C\left(\left(\nu_2^{4/3} + \nu_3^{4/3}\right)\int_0^T \|\mathbf{w}_{\nu\epsilon}\|^{4/3}_{H_0^1}dt + \nu_1^{4/3}T + \int_0^T \|\mathbf{g}\|^{4/3}dt\right) \\
&\quad + C\Big(\epsilon^{4/3}\|\mathbf{w}_{\nu\epsilon}\|^4_{L^4(0,T;H_0^1)} + \|\mathbf{u}_{\nu\epsilon}\|^2_{L^2(0,T;V)} + \|\mathbf{w}_{\nu\epsilon}\|^2_{L^2(0,T;H_0^1)}\Big).
\end{aligned} \tag{3.54}$$

Using Hölder's inequality and (3.51), since $\nu_3 < \nu_2$ we obtain

$$\begin{aligned}
&\nu_1^{4/3}\int_0^T \|\mathbf{u}_{\nu\epsilon}\|_V^{4/3}dt \le C\nu_1^{4/3}\|\mathbf{u}_{\nu\epsilon}\|_{L^2(0,T;V)}^{4/3} \le C\nu_1^{2/3},\\
&\left(\nu_2^{4/3}+\nu_3^{4/3}\right)\int_0^T \|\mathbf{w}_{\nu\epsilon}\|_{H_0^1}^{4/3}dt \le \ \nu_2^{4/3}C\|\mathbf{w}_{\nu\epsilon}\|_{L^2(0,T;H_0^1)}^{4/3} \le C\nu_2^{2/3}.
\end{aligned} \tag{3.55}$$

Hence, from (3.54)–(3.55) and (3.51) we get

$$\begin{aligned}
&\|(\mathbf{u}_{\nu\epsilon})_t\|_{L^{4/3}(0,T;V^*)}^{4/3} \le C\left(\nu_1^{2/3}+\nu_1^{4/3}+1+\epsilon^{1/3}+\nu_1^{-1}\right),\\
&\|(\mathbf{w}_{\nu\epsilon})_t\|_{L^{4/3}(0,T;H^{-1})}^{4/3} \le C\left(\nu_2^{2/3}+\nu_1^{4/3}+1+\epsilon^{1/3}+\nu_1^{-1}+\nu_2^{-1}\right).
\end{aligned} \tag{3.56}$$

Thus, since $\epsilon < \nu_1$, from last two inequalities we have

$$\begin{aligned}
&\|(\mathbf{u}_{\nu\epsilon})_t\|_{L^{4/3}(0,T;V^*)} \le C\left(\nu_1^{1/2}+\nu_1+\nu_1^{1/4}+\nu_1^{-3/4}+1\right),\\
&\|(\mathbf{w}_{\nu\epsilon})_t\|_{L^{4/3}(0,T;H^{-1})} \le C\left(\nu_2^{1/2}+\nu_1+\nu_1^{1/4}+\nu_1^{-3/4}+\nu_2^{-3/4}+1\right),
\end{aligned} \tag{3.57}$$

where the constant C is independent of ν_1, ν_2, ν_3, and ϵ.

From (3.51) and (3.57), taking subsequences if necessary, we deduce that, as $\epsilon \to 0$

$$\begin{aligned}
&\mathbf{u}_{\nu\epsilon} \rightharpoonup \mathbf{u}_\nu \quad \text{weak-}^* \text{ in } L^\infty(0,T;\mathbf{H}),\\
&\nu_1^{1/2}\mathbf{u}_{\nu\epsilon} \rightharpoonup \nu_1^{1/2}\mathbf{u}_\nu \quad \text{weakly in } L^2(0,T;\mathbf{V}),\\
&\epsilon^{1/4}\mathbf{u}_{\nu\epsilon} \rightharpoonup 0 \quad \text{weakly in } L^4(0,T;\mathbf{V}),\\
&(\mathbf{u}_{\nu\epsilon})_t \rightharpoonup (\mathbf{u}_\nu)_t \quad \text{weakly in } L^{4/3}(0,T;\mathbf{V}^*),\\
&\mathbf{u}_{\nu\epsilon}(0) \rightharpoonup \mathbf{u}_\nu(0) \quad \text{weakly in } \mathbf{V}^*.
\end{aligned} \tag{3.58}$$

Similarly, as $\epsilon \to 0$

$$\begin{aligned}
&\mathbf{w}_{\nu\epsilon} \rightharpoonup \mathbf{w}_\nu \quad \text{weak-}^* \text{ in } L^\infty\left(0,T;\mathbf{L}^2\right),\\
&\nu_2^{1/2}\mathbf{w}_{\nu\epsilon} \rightharpoonup \nu_2^{1/2}\mathbf{w}_\nu \quad \text{weakly in } L^2\left(0,T;\mathbf{H}_0^1\right),\\
&\epsilon^{1/4}\mathbf{w}_{\nu\epsilon} \rightharpoonup 0 \quad \text{weakly in } L^4\left(0,T;\mathbf{H}_0^1\right),\\
&(\mathbf{w}_{\nu\epsilon})_t \rightharpoonup (\mathbf{w}_\nu)_t \quad \text{weakly in } L^{4/3}\left(0,T;\mathbf{H}^{-1}\right),\\
&\mathbf{w}_{\nu\epsilon}(0) \rightharpoonup \mathbf{w}_\nu(0) \quad \text{weakly in } \mathbf{H}^{-1}.
\end{aligned} \tag{3.59}$$

Since $\mathbf{V} \subset \mathbf{H}$ is compact and $\mathbf{H} \subset \mathbf{V}^*$ is continuous, as well as $\mathbf{V} \subset \mathbf{L}^2$ is compact and $\mathbf{L}^2 \subset \mathbf{H}^{-1}$ is continuous, then as $\epsilon \to 0$ we have

$$\mathbf{u}_{\nu\epsilon} \longrightarrow \mathbf{u}_\nu \quad \text{in } L^2(0,T;\mathbf{H}), \qquad \mathbf{w}_{\nu\epsilon} \longrightarrow \mathbf{w}_\nu \quad \text{in } L^2\left(0,T;\mathbf{L}^2\right). \tag{3.60}$$

We can verify that $(\mathbf{u}_\nu, \mathbf{w}_\nu)$ is a weak solution of (1.1)–(1.3). Indeed, we need to verify that $(\mathbf{u}_\nu, \mathbf{w}_\nu)$ satisfies the following variational system:

$$\begin{aligned}
&\frac{d}{dt}\left(\mathbf{u}_\nu, \widetilde{\Phi}\right) + \nu_1\left(\nabla \mathbf{u}_\nu, \nabla\widetilde{\Phi}\right) + \left(\mathbf{u}_\nu \cdot \nabla \mathbf{u}_\nu, \widetilde{\Phi}\right) = 2\mu_r\left(\operatorname{rot} \mathbf{w}_\nu, \widetilde{\Phi}\right) + \left(\mathbf{f}, \widetilde{\Phi}\right),\\
&\frac{d}{dt}\left(\mathbf{w}_\nu, \widetilde{\Psi}\right) + \nu_2\left(\nabla \mathbf{w}_\nu, \nabla\widetilde{\Psi}\right) + \nu_3\left(\operatorname{div} \mathbf{w}_\nu, \operatorname{div}\widetilde{\Psi}\right) + \left(\mathbf{u}_\nu \cdot \nabla \mathbf{w}_\nu, \widetilde{\Psi}\right) + 4\mu_r\left(\mathbf{w}_\nu, \widetilde{\Psi}\right)\\
&\quad = 2\mu_r\left(\operatorname{rot} \mathbf{u}_\nu, \widetilde{\Psi}\right) + \left(\mathbf{g}, \widetilde{\Psi}\right),
\end{aligned} \tag{3.61}$$

for all $\widetilde{\Phi} \in \mathbf{V}, \widetilde{\Psi} \in \mathbf{H}_0^1$.

Note that the before convergence results enable us to pass to the limit in the linear terms of (3.3)–(3.7), obtaining the linear term in (3.61). Furthermore, through standard arguments one can obtain

$$\left(\mathbf{u}_{\nu\epsilon} \cdot \nabla \mathbf{u}_{\nu\epsilon}, \widetilde{\Phi}\right) \longrightarrow \left(\mathbf{u}_\nu \cdot \nabla \mathbf{u}_\nu, \widetilde{\Phi}\right), \qquad \left(\mathbf{u}_{\nu\epsilon} \cdot \nabla \mathbf{w}_{\nu\epsilon}, \widetilde{\Psi}\right) \longrightarrow \left(\mathbf{w}_\nu \cdot \nabla \mathbf{w}_\nu, \widetilde{\Psi}\right). \tag{3.62}$$

Moreover it is not difficult to check that

$$\left(\mathbf{u}_\nu(\mathbf{x},0), \widetilde{\Phi}\right) = 0, \quad \forall\widetilde{\Phi} \in \mathbf{V}, \qquad \left(\mathbf{w}_\nu(\mathbf{x},0), \widetilde{\Psi}\right) = 0, \quad \forall\widetilde{\Psi} \in \mathbf{H}_0^1. \tag{3.63}$$

Finally, it is clear that for all $\widetilde{\Phi} \in \mathbf{V}, \widetilde{\Psi} \in \mathbf{H}_0^1$, as $\epsilon \to 0$ it holds

$$\left(\epsilon\|\nabla \mathbf{u}_{\nu\epsilon}\|^2\right)\left(\nabla \mathbf{u}_{\nu\epsilon}, \nabla\widetilde{\Phi}\right) \longrightarrow 0, \qquad \left(\epsilon\|\nabla \mathbf{w}_{\nu\epsilon}\|^2\right)\left(\nabla \mathbf{w}_{\nu\epsilon}, \nabla\widetilde{\Psi}\right) \longrightarrow 0. \tag{3.64}$$

□

Proof of the Theorem 2.2. The existence of a solution of (1.1)–(1.5) is given by using Proposition 3.4 as the limit $(\mathbf{u}_\nu, \mathbf{w}_\nu)$ of the sequence $(\mathbf{u}_{\nu\epsilon}, \mathbf{w}_{\nu\epsilon})$.

Now the second part of the Theorem 2.2 will be proved. Let $(\mathbf{u}, \mathbf{w})$ be solution of problem (1.6)–(1.10). From (3.58)-(3.59) we have

$$\begin{aligned}
\nabla(\mathbf{u}_{\nu\epsilon} - \mathbf{u}) \rightharpoonup \nabla(\mathbf{u}_\nu - \mathbf{u}) \quad &\text{weakly in } L^2\left(0,T;\mathbf{L}^2\right),\\
\nabla(\mathbf{w}_{\nu\epsilon} - \mathbf{w}) \rightharpoonup \nabla(\mathbf{w}_\nu - \mathbf{w}) \quad &\text{weakly in } L^2\left(0,T;\mathbf{L}^2\right).
\end{aligned} \tag{3.65}$$

Consequently

$$\begin{aligned}
\nu_1^{1/2}\|\nabla(\mathbf{u}_\nu-\mathbf{u})\|_{L^2(0,T;L^2)} &\le \liminf_{\epsilon\to 0}\nu_1^{1/2}\|\nabla(\mathbf{u}_{\nu\epsilon}-\mathbf{u})\|_{L^2(0,T;L^2)},\\
\nu_2^{1/2}\|\nabla(\mathbf{w}_\nu-\mathbf{w})\|_{L^2(0,T;L^2)} &\le \liminf_{\epsilon\to 0}\nu_2^{1/2}\|\nabla(\mathbf{w}_{\nu\epsilon}-\mathbf{w})\|_{L^2(0,T;L^2)}.
\end{aligned}\tag{3.66}$$

Hence, from the last inequalities and taking into account (3.32) we conclude that

$$\begin{aligned}
\nu_1^{1/2}\|\nabla(\mathbf{u}_\nu-\mathbf{u})\|_{L^2(0,T;L^2)} &\le C(\nu_1+\nu_2+\nu_3)^{1/2},\\
\nu_2^{1/2}\|\nabla(\mathbf{w}_\nu-\mathbf{w})\|_{L^2(0,T;L^2)} &\le C(\nu_1+\nu_2+\nu_3)^{1/2}.
\end{aligned}\tag{3.67}$$

Therefore, since $\max\{\nu_1,\nu_2,\nu_3\}<1$, with the additional condition $\nu_3<\nu_1<\nu_2<k\nu_1$ for some positive constant k, from (3.67) we get

$$\|\nabla(\mathbf{u}_\nu-\mathbf{u})\|_{L^2(0,T;L^2)}\le M_1,\qquad \|\nabla(\mathbf{w}_\nu-\mathbf{w})\|_{L^2(0,T;L^2)}\le M_2,\tag{3.68}$$

with M_1, M_2 positive constants independent of ν_1,ν_2,ν_3. Moreover, by using (3.32) we obtain

$$\begin{aligned}
\|\mathbf{u}_\nu-\mathbf{u}\|_{L^2(0,T;H)} &\le \|\mathbf{u}_{\nu\epsilon}-\mathbf{u}_\nu\|_{L^2(0,T;H)}+\|\mathbf{u}_{\nu\epsilon}-\mathbf{u}\|_{L^2(0,T;H)}\\
&\le \|\mathbf{u}_{\nu\epsilon}-\mathbf{u}_\nu\|_{L^2(0,T;H)}+C(\nu_1+\nu_2+\nu_3+\epsilon)^{1/2},\\
\|\mathbf{w}_\nu-\mathbf{w}\|_{L^2(0,T;L^2)} &\le \|\mathbf{w}_{\nu\epsilon}-\mathbf{w}_\nu\|_{L^2(0,T;L^2)}+\|\mathbf{w}_{\nu\epsilon}-\mathbf{w}\|_{L^2(0,T;L^2)}\\
&\le \|\mathbf{w}_{\nu\epsilon}-\mathbf{w}_\nu\|_{L^2(0,T;L^2)}+C(\nu_1+\nu_2+\nu_3+\epsilon)^{1/2}.
\end{aligned}\tag{3.69}$$

Thus, by taking into account (3.60), as $\epsilon\to 0$ we find

$$\begin{aligned}
\|\mathbf{u}_\nu-\mathbf{u}\|_{L^2(0,T;H)} &\le C(\nu_1+\nu_2+\nu_3)^{1/2},\\
\|\mathbf{w}_\nu-\mathbf{w}\|_{L^2(0,T;L^2)} &\le C(\nu_1+\nu_2+\nu_3)^{1/2},
\end{aligned}\tag{3.70}$$

with the constant C independent of ν_1,ν_2,ν_3. Hence, the proof of theorem is finished. □

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