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Research Article

Convexities and Existence of the Farthest Point

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Five counterexamples are given, which show relations among the new convexities and some important convexities in Banach space. Under the assumption that Banach space X is nearly very convex, we give a sufficient condition that bounded, weakly closed subset of X has the farthest points. We also give a sufficient condition that the farthest point map is single valued in a residual subset of X when X is very convex.

1. Introduction

Let X be a Banach space, and let X^* be its dual space. Let us denote by $B(X)$ and $S(X)$ the closed unit ball and the unit sphere of X, respectively. Let $x \in S(X), \Sigma(x) = \{x^* \in S(X^*) : x^*(x) = 1\}$. For any sequence $\{x_n\} \subset X$, define $\operatorname{sep}(x_n) \equiv \inf\{\|x_n - x_m\| : n \neq m\}$. Let B be a bounded subset of X. We define a real-valued function $r_B : X \to R$ by

$$r_B(x) = \sup\{\|x - y\| : y \in B\} \tag{1.1}$$

and call $r_B(x)$ the farthest distance from x to B. The function r_B is convex and Lipschitz-continuous. In fact, $|r_B(x) - r_B(y)| \leq \|x - y\|$ for all $x, y \in X$.

A point $z \in B$ is called a farthest point of B if there exists an $x \in X$ such that $\|x - z\| = r_B(x)$.

The mapping $F_B : X \to 2^B$ defined by $F_B(x) = \{z \in B : \|x - z\| = r_B(x)\}$ is called the farthest point map of B.

The existence of a farthest point of B is equivalent to the fact that the set

$$D = \{x \in X : \|x - z\| = r_B(x) \text{ for some } z \in B\} \tag{1.2}$$

is nonempty. In [1–5], the existence of a farthest point of B is studied. Edelstein [2] showed that if X is uniformly convex space, then the set D defined above is dense in X. Asplund [1] showed that if X is both reflexive and locally uniformly rotund (LUR), then the set D is dense in X. Lau [4] proved that if B is weakly compact subset of X, then D contains a dense G_δ set of X, and if X is reflexive Banach space, then for every bounded, weakly closed subset B in X, D contains a dense G_δ subset of X.

We say that X is strongly convex (resp., very convex/nearly strongly convex/nearly very convex) if any $x \in S(X)$ and $\{x_n\} \subset B(X)$ with $x^*(x_n) \to 1$ as $n \to \infty$ for some $x^* \in \Sigma(x)$ imply $x_n \to x$ as $n \to \infty$ (resp., $x_n \xrightarrow{w} x$ as $n \to \infty/\{x_n\}$ is relatively compact/$\{x_n\}$ is weakly relatively compact).

The author [6] proved that strong convexity (resp., very convex/nearly strong convexity/nearly very convex) has important applications in approximation theory. Bandyopadhyay et al. [7] also proposed two generalizations of LUR and weakly locally uniform rotundity (WLUR), which were called almost locally uniform rotundity (ALUR) and weakly almost locally uniform rotundity (WALUR). A Banach space X is said to be ALUR (resp., WALUR) if for any $x \in S(X), \{x_n\} \subset B(X)$ and $\{x_m^*\} \subset B(X^*)$, the condition $\lim_m \lim_n x_m^*((x_n + x)/2) = 1$ implies $x_n \to x$ (reps., $x_n \xrightarrow{w} x$). Recently, we have proved that ALUR and strong convexity, WALUR and very convex are equivalent, respectively [8]. Sullivan [9] defined very rotund space. A Banach space X is known as very rotund if no $x^* \in S(X^*)$ is simultaneously a norming element for some $x \in S(X)$ and $x^{**} \in S(X^{**})$, where $x \neq x^{**}$. The author [10] proved that very rotund space coincides with very convex space. By [6–12], we know that the four new convexities mentioned above have a lot of good properties and applications.

It is known that LUR, WLUR, midpoint locally uniform rotundity (MLUR) and weakly midpoint locally uniform rotundity (WMLUR) are four important convexities in the geometric theory of Banach spaces. By [9–12], the relation of the convexities mentioned above is shown in Figure 1 below.

The structure of this paper is as follows. In Section 2, We will give five counterexamples, which show the relations among the four new convexities mentioned above and LUR, WLUR, MLUR, WMLUR, and rotund (R).

In Section 3, we prove that if X is nearly very convex space and if for every $x \in X$, there exists $x_0^* \in \partial r_B(x)$ which attains its norm, then for every bounded, weakly closed subset B of X, the set D defined above contains a G_δ subset of X, which improves the results of Edelstain [2], Asplund [1], and Lau [4]. Finally, we also prove a sufficient condition that the farthest point map F_B is single valued in a residual subset of X when X is very convex.

2. Some Counterexamples about Convexities

Lemma 2.1 (see [13]). *Let $\{x_\alpha : \alpha \in D\}$ be a net in X if for every $\varepsilon > 0$, there exists $\alpha_s \in D$ such that the tail $\{x_\alpha : \alpha \geq \alpha_s\}$ has a finite ε-net, then $\{x_\alpha\}$ is a relatively compact subset in X.*

Lemma 2.2. *If X is nearly strongly convex space, then X has Kadec property, that is, if whenever $x \in S(X)$ and $\{x_\alpha\}$ is a net in $S(X)$ such that $x_\alpha \xrightarrow{w} x$, then $x_\alpha \to x$. Particularly, X has property H, that is, if whenever $x \in S(X)$ and $\{x_n\}$ is a sequence in $S(X)$ such that $x_n \xrightarrow{w} x$, then $x_n \to x$.*

Proof. Suppose that net $\{x_\alpha : \alpha \in D\} \subset S(X)$, $x \in S(X)$ such that $x_\alpha \xrightarrow{w} x$, we will prove that $x_\alpha \to x$.

$$\begin{tikzcd}
\text{LUR} \arrow[r] \arrow[dddd] & \text{Strong convexity} \arrow[rr] \arrow[dr] \arrow[dddd] & & \text{MLUR} \arrow[dd] \arrow[dddd, bend left] \\
& & \text{Nearly strong convexity} \arrow[dd] & \\
& & & \text{R} \\
& & \text{Nearly very convex} & \\
\text{WLUR} \arrow[r] & \text{Very convex} \arrow[rr] \arrow[ur] & & \text{WMLUR} \arrow[uu]
\end{tikzcd}$$

Figure 1: The relationship between the convexities.

Case 1. If for every $\varepsilon > 0$, there exists $\alpha_s \in D$ such that $\{x_\alpha : \alpha \geq \alpha_s\}$ has a finite ε-net, by Lemma 2.1 and $x_\alpha \xrightarrow{w} x$, we may obtain that $x_\alpha \to x$.

Case 2. If for every $\varepsilon_0 > 0$ such that all tails of net $\{x_\alpha : \alpha \in D\}$ have no finite ε_0-net, we take $f \in S(X^*)$ with f in $\Sigma(x)$. Since $x_\alpha \xrightarrow{w} x$, $f(x_\alpha) \to f(x) = 1$. Choosing $\{\beta_n\} \subset D$, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq \cdots$, we know that

$$f(x_\alpha) > 1 - \frac{1}{n}, \tag{2.1}$$

for any $\alpha \in D$ with $\alpha \geq \beta_n$. Take $\alpha_1 \in D$ such that $\alpha_1 \geq \beta_1$. For x_{α_1} and β_2, we can choose $\alpha_2 \in D$, $\alpha_2 \geq \beta_2$ such that $\|x_{\alpha_2} - x_{\alpha_1}\| \geq \varepsilon_0$. Otherwise, $\{x_\alpha : \alpha \geq \beta_2\} \subset \{x \in X : \|x - x_{\alpha_1}\| < \varepsilon_0\}$, that is, the tail $\{x_\alpha : \alpha \geq \beta_2\}$ has finite ε_0-net. This is a contradiction with the assumption. For x_{α_1}, x_{α_2}, and β_3, there exists $\alpha_3 \in D$, $\alpha_3 \geq \beta_3$ such that $\|x_{\alpha_3} - x_{\alpha_1}\| \geq \varepsilon_0$ and $\|x_{\alpha_3} - x_{\alpha_2}\| \geq \varepsilon_0$. Otherwise, $\{x_\alpha : \alpha \geq \beta_3\} \subset \{x \in X : \|x - x_{\alpha_1}\| < \varepsilon_0\} \cup \{x \in X : \|x - x_{\alpha_2}\| < \varepsilon_0\}$, that is, the tail $\{x_\alpha : \alpha \geq \beta_3\}$ has finite ε_0-net. This is a contradiction with the assumption. According to the same method, we may choose a sequence $\{x_{\alpha_n}\} \subset \{x_\alpha : \alpha \in D\}$ such that

$$\operatorname{sep}(x_{\alpha_n}) = \inf\{\|x_{\alpha_n} - x_{\alpha_m}\| : n \neq m\} \geq \varepsilon_0, \quad \alpha_n \geq \beta_n, \; n = 1, 2, \ldots. \tag{2.2}$$

Hence, we know that $\{x_{\alpha_n}\}$ is not relatively compact.

On the other hand, by (2.1), we have that

$$1 \geq f(x_{\alpha_n}) > 1 - \frac{1}{n}. \tag{2.3}$$

This shows that $f(x_{\alpha_n}) \to 1$. Since X is nearly strongly convex, $\{x_{\alpha_n}\}$ is relatively compact which is a contradiction. □

Example 2.3. There exists an MLUR space which is not a nearly strongly convex space.

Recall the equivalent norm defined on c_0 by Smith [14]. For $k \in \mathbb{N}$, define a mapping $V_k : c_0 \to \mathbb{R}^1, V_k(x) = V_k(\xi_1, \xi_2, \ldots) = \sup\{|\xi_1 - \xi_i| : i \geq k\}$. Let $\{\alpha_n\}$ be a sequence of positive real numbers, and $\sum_{n=2}^{\infty} \alpha_n^2 = 1$. Define two mappings V and T from $c_0 \to l_2$ as follows:

$$V(x) = (\alpha_2 V_2(x), \alpha_3 V_3(x), \ldots), \qquad T(x) = (\xi_1, \alpha_2 \xi_2, \alpha_3 \xi_3, \ldots). \tag{2.4}$$

Since $\{V_k(x)\}$ and $\{\xi_k\}$ are both bounded sequences, and $\sum_{n=2}^{\infty} \alpha_n^2 = 1$, then we know that $\{V_k(x)\}$ and $T(x)$ are in l_2. For x in c_0, let

$$\|x\|_G = \left(\|x\|_\infty^2 + \|V(x)\|_2^2 + \|T(x)\|_2^2\right)^{1/2}. \tag{2.5}$$

Since T is one to one continuous, linear, and $V(x+y) \leq V(x) + V(y)$ for all x, y in c_0, the $\|\cdot\|_G$ is a norm, and $\|\cdot\|_\infty \leq \|\cdot\|_G \leq \sqrt{6}\|\cdot\|_\infty$. This shows that $\|\cdot\|_G$ is an equivalent norm on c_0.

Smith shows that $(c_0, \|\cdot\|_G)$ is MLUR. We say that $(c_0, \|\cdot\|_G)$ has no property H. Indeed, let $x = e_1$, $x_n = e_1 + e_n$, then $\|x_n\|_G \to \|x\|_G = \sqrt{3}$ and $x_n \xrightarrow{w} x$, but $x_n \nrightarrow x$. By Lemma 2.2, we know that $(c_0, \|\cdot\|_G)$ is not nearly strongly convex.

Example 2.4. There exists a very convex space which is not nearly strongly convex space.

Recall the equivalent norm on Hilbert space by Troyanski in Isratescu [15]. Let X be a Hilbert space, and let $\{e_i\}_{i=0}^{\infty}$ be an orthogonal basis. For any $x \in X$, $x = \lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n + \cdots$, let

$$\|x\|_1^2 = \max\left\{|\lambda_0|^2 + |\lambda_2|^2 + \cdots + |\lambda_{2n}|^2 + \cdots, |\lambda_1|^2 + |\lambda_3|^2 + \cdots + |\lambda_{2n+1}|^2 + \cdots\right\}. \tag{2.6}$$

It is obvious that $\|\cdot\|_1$ is an equivalent norm on X. Further, we set

$$\|x\| = \left(\|x\|_1^2 + \sum_{i=1}^{\infty} \frac{1}{2^i}|\lambda_i|^2\right)^{1/2}. \tag{2.7}$$

Clearly, this is again a norm on X which is an equivalent original one. Troyanski shows that $(X, \|\cdot\|)$ is R and reflexive [15], but it has no property H. Hence, $(X, \|\cdot\|)$ is very convex, but is not nearly strongly convex by Lemma 2.2.

Example 2.5. There exists a nearly very convex space which is neither very convex space nor nearly strongly convex space.

For any $x \in l_2$, let

$$|||x||| = \max\left\{\frac{1}{\sqrt{2}}\|x\|_2, \|x\|_\infty\right\}. \tag{2.8}$$

Since $(1/\sqrt{2})\|x\|_2 \leq |||x||| \leq \|x\|_2$, $|||x|||$ is an equivalent norm on l_2. Since $(l_2, |||\cdot|||)$ is reflexive, we know that $(l_2, |||\cdot|||)$ is nearly very convex space. We say that $(l_2, |||\cdot|||)$ has no property H. Indeed, let $x = e_1$, $x_n = e_1 + e_n$, then $|||x||| = |||x_n||| = 1$, but $|||x_n - x_m||| = 1$. Hence, B $(l_2, |||\cdot|||)$ is not nearly strongly convex by Lemma 2.2. We say that $(l_2, |||\cdot|||)$ is not very convex. In fact, let $e_{1,2} = (1,1,0,\ldots)$, $e_{1,3} = (1,0,1,0,\ldots)$, then $|||e_{1,2} + e_{1,3}||| = 2$. This shows that $(l_2, |||\cdot|||)$ is not R, and therefore, $(l_2, |||\cdot|||)$ is not very convex.

Example 2.6. There exists a strongly convex space which is not WLUR space.

Let $E = (l_2, \|\cdot\|)$, where $x = (a_1, a_2, \ldots) \in l_2$,

$$\|x\|^2 = \left\{|a_1| + \left(a_2^2 + a_3^2 + \cdots\right)^{1/2}\right\}^2 + \left\{\left(\frac{a_2}{2}\right)^2 + \cdots + \left(\frac{a_n}{n}\right)^2 + \cdots\right\}. \tag{2.9}$$

Let $X = (\Sigma \oplus E)_{l_2}$. In [16], it is proved that X is 2R, but is not KUR. Since 2R implies R, reflexive and property H, we get that X is strongly convex space.

X is not WLUR space. Indeed, let $\{e_i\}_{i=1}^{\infty}$ be the natural basis, and $x = (e_1, e_1, 0, \ldots)$, $x_n = (1/2(e_1 + e_n), e_1, 0, \ldots)$, then $\lim_{n\to\infty}\|x_n\| = \sqrt{2} = \|x\|$ and $\lim_{n\to\infty}\|x_n + x\| = \sqrt{8}$. However,

$$d(x_n, \operatorname{span}\{x\}) = \inf\{\|x_n - y\| : y \in \operatorname{span}\{x\}\} \geq \frac{1}{2}. \tag{2.10}$$

Choose $f \in S(X^*)$ such that $f(x_n) \geq 1/2$, $f(y) = 0$, for all $y \in \operatorname{span}\{x\}$. It is easily proved that

$$\lim_{n\to\infty}\left\|\frac{x}{\|x\|} + \frac{x_n}{\|x_n\|}\right\| = \frac{1}{\sqrt{2}}\lim_{n\to\infty}\|x + x_n\| = 2, \tag{2.11}$$

but $f((x/\|x\|) + (x_n/\|x_n\|)) \geq |f(x_n/\|x_n\|)| = 1/(2\sqrt{2})$. This shows that X is not WLUR space.

Example 2.7. There exists a nearly strongly convex space which is not a strongly convex space.

For $x = (x_1, x_2)$ in $\mathbb{R}^2$, let $\|x\| = |x_1| + |x_2|$, then $(\mathbb{R}^2, \|\cdot\|)$ is reflexive and has property H. Hence, it is nearly strongly convex. It is easy to prove that $(\mathbb{R}^2, \|\cdot\|)$ is not R space. Therefore $(\mathbb{R}^2, \|\cdot\|)$ is not strongly convex.

Remark 2.8 (Smith [17]). gave three examples $(l_2, \|\cdot\|_W)$, $(l_2, \|\cdot\|_A)$, $(l_1, \|\cdot\|_H)$. He shows that $(l_2, \|\cdot\|_W)$ is WLUR not MLUR, and $(l_2, \|\cdot\|_A)$ is MLUR not WLUR, and $(l_1, \|\cdot\|_H)$ is R not MLUR. It is easily proved that $(l_1, \|\cdot\|_H)$ is not WMLUR either.

By the above five counterexamples and Remark 2.8, we know that, except for very convex implied WMLUR, none of the above converse implied relations in the diagram is generally true.

3. Convexities and Existence of the Farthest Point

Before proceeding to this part, let's recall that the subdifferential of convex function f on Banach space X is defined by

$$\partial f(x) = \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x), \forall y \in X\}. \tag{3.1}$$

$x \to \partial f(x)$ is called subdifferential mapping.

Remark 3.1. It was shown in [4] that if B is a bounded closed subset in Banach space X, then for any $x \in X$ and $x^* \in \partial r_B(x)$, we have that $\|x^*\| \leq 1$ and thus

$$\sup\{x^*(x-z) : z \in B\} \leq \|x^*\| \cdot r_B(x) \leq r_B(x). \tag{3.2}$$

Hence, for any $x \in X$ and $x^* \in \partial r_B(x)$, we have that

$$\inf_{z \in B} x^*(z-x) \geq -r_B(x). \tag{3.3}$$

Lemma 3.2 (Lau [4]). *Let X be a Banach space and B a bounded subset in X, then the set*

$$F = \left\{ x \in X : \inf_{z \in B} x^*(z-x) > -r_B(x) \text{ for some } x^* \in \partial r_B(x) \right\} \tag{3.4}$$

is a first category in X.

Theorem 3.3. *Let X be a nearly very convex Banach space and B a bounded, weakly closed subset of X. Further, for any $x \in X$, if there exists an $x_0^* \in \partial r_B(x)$ which attains its norm, then*

$$D = \{x \in X : \|x-z\| = r_B(x) \text{ for some } z \in B\} \tag{3.5}$$

contains a dense G_δ set of X. In particular, the set of farthest points of B is nonempty.

Proof. Define F as in Lemma 3.2 and

$$F_n = \left\{ x \in B : \inf_{z \in B} x^*(z-x) \geq -r_B(x) + \frac{1}{n} \text{ for some } x^* \in \partial r_B(x) \right\}. \tag{3.6}$$

Let $Q = X \setminus F$, then

$$Q = X \setminus \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (X \setminus F_n), \tag{3.7}$$

where each $X \setminus F_n$ is an open, dense subset in B. Hence, Q is a dense G_δ set in X.

Now, we prove $Q \subset D$ as follows. For any $x \in Q$, take $x_0^* \in \partial r_B(x)$ such that x_0^* attains its norm. Since $\inf_{z \in B} x_0^*(z-x) \geq -r_B(x)$, by the definition of Q, we have that $\inf_{z \in B} x_0^*(z-x) \leq -r_B(x)$. Take sequence $\{z_n\} \subset B$ such that

$$\lim_{n \to \infty} x_0^*(z_n - x) = -r_B(x). \tag{3.8}$$

Clearly, $(z_n - x)/(-r_B(x)) \in B(X)$. Given that X is nearly very convex and that B is weakly closed, there exist $z_0 \in B$ and subsequence $\{z_{n_k}\} \subset \{z_n\}$ such that $(z_{n_k} - x)/(-r_B(x)) \xrightarrow{w}$

$(z_0 - x)/(-r_B(x))$ as $k \to \infty$, that is, $z_{n_k} \xrightarrow{w} z_0$ as $k \to \infty$. Hence, $x_0^*(z_0 - x) = -r_B(x)$. It follows that

$$r_B(x) \geq \|x - z_0\| \geq \left|x_0^*(z_0 - x)\right| = r_B(x). \tag{3.9}$$

Thus, $\|z_0 - x\| = r_B(x)$. This shows that $Q \subset D$. □

Corollary 3.4 (Lau [4]). *If X is a reflexive Banach space. Then for every bounded, weakly closed subset B in X, the set*

$$D = \{x \in X : \|x - z\| = r_B(x) \text{ for some } z \in B\} \tag{3.10}$$

contains a dense G_δ set of X, and hence, the set of farthest points of B is nonempty.

Corollary 3.5 (Asplund [1]). *If X is a reflexive LUR Banach space, then Corollary 3.4 holds for every bounded closed subset B in X.*

Theorem 3.6. *Let X be a very convex Banach space and B a bounded, weakly closed subset of X. For any $x \in X$, if there exists an $x_0^* \in \partial r_B(x)$ which attains its norm, then the farthest point map F_B is single valued in a residual subset of X.*

Proof. By Theorem 3.3,

$$Q = X \setminus F = \left\{x \in X : \sup_{z \in B} x^*(x - z) = r_B(x) \text{ for any } x^* \in \partial r_B(x)\right\} \tag{3.11}$$

is a dense G_δ subset of X, where F is defined as in Lemma 3.2.

Now we prove that $F_B(x)$ is single valued for all $x \in Q$.

If F_B is not single valued on Q, then there are $x_0 \in Q$ and $z_1, z_2 \in B$ with $z_1 \neq z_2$ such that $\|x_0 - z_i\| = r_B(x_0)$, $i = 1, 2$. By Hahn-Banach theorem, we have $x_i^* \in S(X^*)$, $i = 1, 2$ such that

$$x_i^*(x_0 - z_i) = \|x_0 - z_i\|, \quad i = 1, 2. \tag{3.12}$$

For any $z' \in B, y \in X$, we have that

$$\begin{aligned} x_i^*(y - x_0) &= x_i^*(y - z') - x_i^*(x_0 - z') \\ &\leq \sup_{z \in B} \sup_{x^* \in B(X^*)} x^*(y - z) - x_i^*(x_0 - z') \\ &= r_B(y) - x_i^*(x_0 - z'). \end{aligned} \tag{3.13}$$

Thus,

$$\sup_{z \in B} x_i^*(x_0 - z) \leq r_B(y) - x_i^*(y - x_0). \tag{3.14}$$

Since $x_0 \in Q$, $r_B(x_0) \leq r_B(y) - x_i^*(y - x_0)$. This shows that $x_i^* \in \partial r_B(x_0)$. Let $x_0^* = (1/2)x_1^* + (1/2)x_2^*$, then $x_0^* \in \partial r_B(x_0)$ due to the convexity of $\partial r_B(x_0)$, and $\sup_{z\in B} x_0^*(x_0 - z) = r_B(x_0)$. Take a sequence $\{z_n\} \subset B$ such that

$$\lim_{n\to\infty} x_0^*(x_0 - z_n) = r_B(x_0). \tag{3.15}$$

It follows that $\lim_{n\to\infty} x_i^*(x_0 - z_n) = r_B(x_0)$, $i = 1, 2$. By (3.12), $x_i^* \in \Sigma((x_0 - z_i)/(r_B(x_0))), i = 1, 2$. Because X is very convex, $(x_0 - z_n)/(r_B(x_0)) \xrightarrow{w} (x_0 - z_i)/(r_B(x_0)), i = 1, 2$ as $n \to \infty$. According to uniqueness of weak limit point, we have that $z_1 = z_2$, which is a contradiction. □

Remark 3.7. For closed-convex subset B and bounded closed, relatively weakly compact K in X, Ni and Li [18] proved that the set of all points in B such that the farthest problem $\max\{x, K\}$ is well posed is a dense G_δ subset in B provided that B is both strictly convex and Kadec with respect to K. This shows that the farthest point map F_B is single valued in a residual subset of X. By Example 2.4 in this paper, there exists a Banach space where assumptions in Theorem 3.6 are satisfied, but its unit ball $B(X)$ is not Kadec. Let $B(X) = B = K$, then we know that conditions of Theorem 3.6 are different from conditions of the result by Ni and Li. Hence, the result by Ni and Li does not imply Theorem 3.6.

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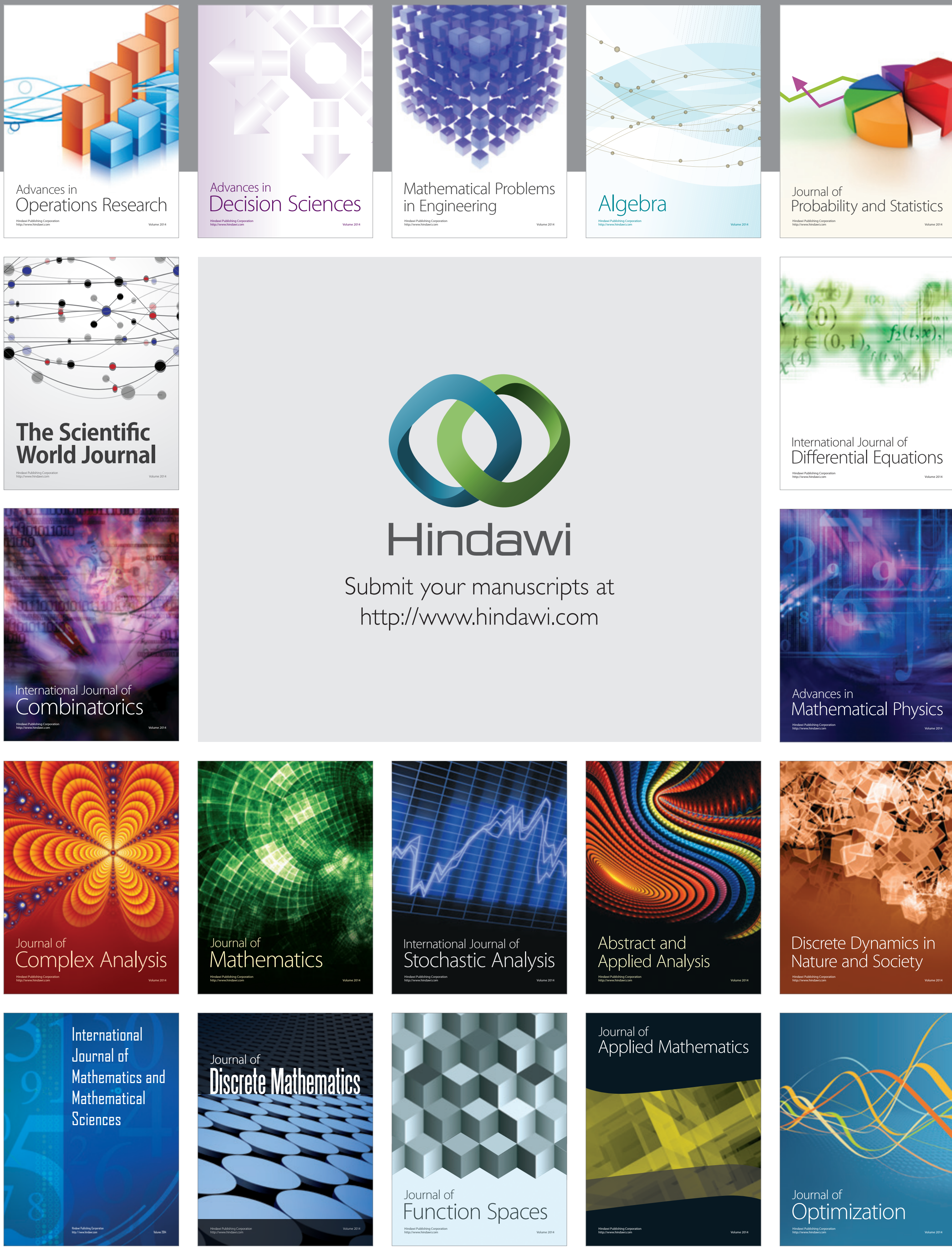
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