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Research Article

On a Class of Nonautonomous Max-Type Difference Equations

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This paper addresses the max-type difference equation $x_n = \max\{f_n/x_{n-k}^{\alpha}, B/x_{n-m}^{\beta}\}$, $n \in \mathbb{N}_0$, where $k, m \in \mathbb{N}$, $B > 0$, and $(f_n)_{n\in\mathbb{N}_0}$ is a positive sequence with a finite limit. We prove that every positive solution to the equation converges to $\max\{(\lim_{n\to\infty} f_n)^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$ under some conditions. Explicit positive solutions to two particular cases are also presented.

1. Introduction

The study of difference equations, which usually depicts the evolution of certain phenomena over the course of time, has a long history. Many experts recently pay some attention to so-called *max-type* difference equations which stem from certain models in control theory, see, for example, [1–23] and the references therein.

The study of the following family of max-type difference equations

$$x_n = \max\left\{B_n^{(0)}, B_n^{(1)}\frac{x_{n-p_1}^{r_1}}{x_{n-q_1}^{s_1}}, B_n^{(2)}\frac{x_{n-p_2}^{r_2}}{x_{n-q_2}^{s_2}}, \dots, B_n^{(k)}\frac{x_{n-p_k}^{r_k}}{x_{n-q_k}^{s_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $p_i, q_i \in \mathbb{N}$ such that $1 \le p_1 < \cdots < p_k$, $1 \le q_1 < \cdots < q_k$, $r_i, s_i \in \mathbb{R}_+$, $k \in \mathbb{N}$ and $B_n^{(i)}$ $(i = 0, 1, \dots, k)$ are real sequences, was proposed by S. Stević at numerous conferences, for example, [10, 11]. For some results in this direction, see [1, 2, 4, 12–23].

In the beginning of the investigation the following equation was studied:

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}}, \frac{A_n^{(2)}}{x_{n-2}}, \ldots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in \mathbb{N}$, $(A_n^{(i)})_{n\in\mathbb{N}_0}$, $i = 1, \ldots, k$ are real sequences and the initial values are nonzero (see, e.g., [3, 5, 6, 9] and the related references therein).

In [22], Sun studied the second-order difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

with $\alpha, \beta \in (0,1), A, B > 0$, and proved that each positive solution to (1.3) converges to the equilibrium point $\max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$, by considering several subcases. However, the method used there is a bit complicated and difficult for extending. Hence in [14] Stević extended this, as well as the main result in [13], by presenting a more concise and elegant proof of the next theorem.

Theorem 1.1 (see [14, Theorem 1]). *Every positive solution to the difference equation*

$$x_n = \max\left\{\frac{A_1}{x_{n-p_1}^{\alpha_1}}, \frac{A_2}{x_{n-p_2}^{\alpha_2}}, \ldots, \frac{A_k}{x_{n-p_k}^{\alpha_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $p_i, i = 1, \ldots, k$ are natural numbers such that $1 \le p_1 < \cdots < p_k$, $k \in \mathbb{N}$ and $A_i > 0$, $\alpha_i \in (-1, 1)$, $i = 1, \ldots, k$, converges to $\max_{1\le i\le k}\{A_i^{1/(\alpha_i+1)}\}$.

Definition 1.2. Let $F : \mathbb{N}_0 \times \mathbb{R}^k \to \mathbb{R}$ be a function of $k+1$ variables, then the difference equation

$$x_n = F(n, x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{1.5}$$

is called *nonautonomous* or time variant.

Note that the following nonautonomous difference equation

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}^{\alpha_1}}, \frac{A_n^{(2)}}{x_{n-2}^{\alpha_2}}, \ldots, \frac{A_n^{(l)}}{x_{n-l}^{\alpha_l}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $l \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, and $(A_n^{(i)})_{n\in\mathbb{N}_0}, i = 1, \ldots, l$ are real sequences (not all constant), is a natural generalization of (1.2), (1.3), and (1.4). It is a special case of (1.1) of particular interest.

The aforementioned works are mainly devoted to the study of (1.6) with constant or periodic numerators.

This paper is devoted to the study of the following nonautonomous max-type difference equation with two delays:

$$x_n = \max\left\{\frac{f_n}{x_{n-k}^{\alpha}}, \frac{B}{x_{n-m}^{\beta}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.7}$$

where $k, m \in \mathbb{N}$, α, $\beta \in \mathbb{R}$ are fixed and $(f_n)_{n\in\mathbb{N}_0}$ is a positive sequence with a finite limit. Inspired by the methods and proofs of the above-mentioned papers, here we try to find some sufficient conditions such that every positive solution to (1.7) converges to $\max\{(\lim_{n\to\infty} f_n)^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

This paper proceeds as follows. Several useful lemmas are given in Section 2. In Section 3 we establish three main results about the global attractivity of (1.7) under some conditions. Finally motivated by a recent theorem in [21], explicit solutions to two particular cases of (1.7) are presented in Section 4.

2. Auxiliary Results

To establish the main results in Section 3, here we present several lemmas. First we extend Lemma 2.4 in [21] by proving the following result.

Lemma 2.1. *Consider the nonautonomous difference equation*

$$z_n = \min\{C_1(n) - \alpha_1(n)z_{n-1}, \ldots, C_k(n) - \alpha_k(n)z_{n-k}\}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $k \in \mathbb{N}$ and $\alpha_i(n), C_i(n)$, $i = 1, 2, \ldots, k$ are sequences. If $C_i(n)$s are nonnegative sequences and there always exists $i_0 \in \{1, 2, \ldots, k\}$ such that $C_{i_0}(n) = 0$ for each fixed $n \in \mathbb{N}_0$, then

$$|z_n| \le \max\{|\alpha_1(n)||z_{n-1}| - C_1(n), \ldots, |\alpha_k(n)||z_{n-k}| - C_k(n)\}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Proof. Suppose that $n \in \mathbb{N}_0$ is fixed, and denote by $S \subseteq \{1, \ldots, k\}$ the set of all indices for which the terms in (2.1) are negative.

If $S = \emptyset$, which means all terms in the right-hand side of (2.1) are nonnegative, then apparently

$$0 \le z_n \le -\alpha_{i_0}(n)z_{n-i_0} \tag{2.3}$$

which implies

$$|z_n| \le |\alpha_{i_0}(n)||z_{n-i_0}| - C_{i_0}(n). \tag{2.4}$$

Otherwise, $S \neq \emptyset$, which means that there exist indices such that the corresponding terms in (2.1) are negative, then we derive

$$z_n = \min_{j\in S}\{C_j(n) - \alpha_j(n)z_{n-j}\} < 0. \tag{2.5}$$

Since $\alpha_j(n)z_{n-j}$ must be positive for $j \in S$, it follows from (2.5) that

$$|z_n| = \max_{j\in S}\{\alpha_j(n)z_{n-j} - C_j(n)\} = \max_{j\in S}\{|\alpha_j(n)||z_{n-j}| - C_j(n)\}. \tag{2.6}$$

Inequality (2.2) follows easily from (2.4) and (2.6). □

The following lemma is widely used in the literature.

Lemma 2.2 (see [24]). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers which satisfies the inequality*

$$a_{n+k} \le q\max\{a_{n+k-1}, a_{n+k-2}, \ldots, a_n\}, \quad \text{for } n \in \mathbb{N}, \tag{2.7}$$

where $q > 0$ and $k \in \mathbb{N}$ are fixed. Then there exists an $M \ge 0$ such that

$$a_n \le M\left(\sqrt[k]{q}\right)^n, \quad n \in \mathbb{N}, \tag{2.8}$$

which implies $a_n \to 0$ as $n \to \infty$ if $0 < q < 1$.

Lemma 2.3. *Assume that $(x_n)_{n\ge -k}$ is a sequence of nonnegative numbers satisfying the difference inequality*

$$x_n \le \max\{\gamma_1 x_{n-1} - d_1(n), \ldots, \gamma_k x_{n-k} - d_k(n)\}, \quad n \in \mathbb{N}_0, \tag{2.9}$$

where $k \in \mathbb{N}, \gamma_i \in [0,1)$, and $d_i(n),\ i = 1,\ldots,k$ are nonnegative sequences. If there exists at least one positive γ_i, then the sequence x_n converges to zero as $n \to \infty$.

Proof. This lemma follows directly from Lemma 2.2 since

$$\begin{aligned} x_n &\le \max\{\gamma_1 x_{n-1} - d_1(n), \ldots, \gamma_k x_{n-k} - d_k(n)\} \\ &\le \max\{\gamma_1 x_{n-1}, \ldots, \gamma_k x_{n-k}\} \le \gamma\max\{x_{n-1}, \ldots, x_{n-k}\}, \end{aligned} \tag{2.10}$$

where $0 < \gamma = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} < 1$. □

Remark 2.4. If in Lemma 2.3, we assume $\gamma_i = 0$, $i = 1,\ldots,k$, then the statement also holds, since in this case, if such a sequence exists, then the solution must be trivial, that is, $x_n = 0$, $n \in \mathbb{N}_0$ (for some results on the existence of nontrivial solutions, see, e.g., [25–27] and the references therein).

Through some simple calculations, we have the following result.

Lemma 2.5. *Every positive solution $(x_n)_{n\ge -1}$ to the first-order difference equation*

$$x_n = A^{1-1/\omega}(x_{n-1})^{1/\omega}, \quad n \in \mathbb{N}_0, \tag{2.11}$$

with $A > 0,\ \omega > 0,\ x_{-1} > 0$, *has the form*

$$x_n = A^{1-1/\omega^{n+1}}(x_{-1})^{1/\omega^{n+1}}, \quad n \in \mathbb{N}_0. \tag{2.12}$$

Note that Lemma 2.5 leads to the following corollary.

Corollary 2.6. *Each positive solution* $(Z_n)_{n\geq -k}$ *to the k th-order difference equation*

$$Z_n = A^{1-1/\omega}(Z_{n-k})^{1/\omega}, \quad n \in \mathbb{N}_0, \tag{2.13}$$

where $A > 0,\ \omega > 0,\ k \in \mathbb{N}$ *and the initial values* $Z_{-1}, \ldots, Z_{-k}$ *are positive, has the following form:*

$$Z_n = A^{1-(1/\omega)^{[n/k]+1}}\left(Z_{\rho(n,k)-k}\right)^{(1/\omega)^{[n/k]+1}}, \quad n \geq 0, \tag{2.14}$$

where $[\cdot]$ *represents the integer part function and* $\rho(n,k) = n - k \cdot [n/k]$.

Remark 2.7. By Corollary 2.6 we have that for any positive solution $(Z_n)_{n\geq -k}$ to (2.13) the following three statements hold true if $\omega > 1$:

(1) $\lim_{n\to\infty} Z_n = A$;

(2) if $Z_i < A$ for every $i \in \{-k, \ldots, -1\}$, then the subsequences $(Z_{jk+i})_{j\geq 0}$ are all strictly increasing;

(3) if $Z_i > A$ for every $i \in \{-k, \ldots, -1\}$, then the subsequences $(Z_{jk+i})_{j\geq 0}$ are all strictly decreasing.

3. Main Results

In this section, we prove the main results of this paper, which concern the global attractivity of positive solutions to (1.7) under some conditions. In the sequel, we assume that there is a finite limit of the positive sequence $(f_n)_{n\in\mathbb{N}_0}$ in (1.7).

Theorem 3.1. *Consider* (1.7), *where* $(f_n)_{n\in\mathbb{N}_0}$ *is a positive monotone sequence with finite limit* $A > 0$. *If* $|\alpha| < 1,\ |\beta| < 1,\ B > A^{(\beta+1)/(\alpha+1)}$, *then every positive solution to* (1.7) *converges to* $B^{1/(\beta+1)}$.

Proof. By the change $x_n = y_n B^{1/(\beta+1)}$, (1.7) is transformed into

$$y_n = \max\left\{\frac{C_n}{y_{n-k}^{\alpha}}, \frac{1}{y_{n-m}^{\beta}}\right\}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

with $C_n = f_n/B^{(\alpha+1)/(\beta+1)},\ n \in \mathbb{N}_0$. Note that the sequence $(C_n)_{n\in\mathbb{N}_0}$ is also monotone and $\lim_{n\to\infty} C_n = A/B^{(\alpha+1)/(\beta+1)} < 1$.

According to the assumption the sequence $(f_n)_{n\in\mathbb{N}_0}$ is nondecreasing or nonincreasing. If $(f_n)_{n\in\mathbb{N}_0}$ is nonincreasing, then for some fixed $\varepsilon \in (0, B^{(\alpha+1)/(\beta+1)} - A)$, there exists a natural number N such that for every $n \geq N$ we have $f_n - A < \varepsilon$, which implies

$$0 < C_n < 1, \quad n \geq N. \tag{3.2}$$

On the other hand, if $(f_n)_{n\in\mathbb{N}_0}$ is nondecreasing then obviously $C_n < 1$ for each $n \in \mathbb{N}_0$, hence (3.2) also holds for this case.

Let $D \in (0,1)$ be fixed. Employing the transformation $y_n = D^{z_n}$, (3.1) becomes

$$D^{z_n} = \max\left\{ \frac{C_n}{D^{\alpha z_{n-k}}}, \frac{1}{D^{\beta z_{n-m}}} \right\}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

which implies

$$z_n = \min\left\{\log_D C_n - \alpha z_{n-k}, -\beta z_{n-m}\right\}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

Note that $\log_D C_n > 0$ for all $n \geq N$. From this and by Lemma 2.1 we get

$$|z_n| \leq \max\left\{|\alpha||z_{n-k}| - \log_D C_n, |\beta||z_{n-m}|\right\}, \quad n \geq N. \tag{3.5}$$

When both α and β are zero, it is clear that z_n is always zero for $n \geq N$. Otherwise, it follows from Lemma 2.3 that $\lim_{n\to\infty}|z_n| = 0$, which implies

$$\lim_{n\to\infty} z_n = 0. \tag{3.6}$$

Finally, from the above two transformations we get

$$\lim_{n\to\infty} x_n = B^{1/(\beta+1)} \lim_{n\to\infty} y_n = B^{1/(\beta+1)} D^{\lim_{n\to\infty} z_n} = B^{1/(\beta+1)}. \tag{3.7}$$

The proof is complete. □

Theorem 3.2. *Consider* (1.7). *Let* $(f_n)_{n\geq -k}$ *be a positive solution to* (2.13) *such that* $f_i < A$ *(or* $f_i > A$*),* $i = -k, \ldots, -1$*, and denote*

$$\Gamma = \sup_{n\geq m}\left\{ \frac{\ln f_{n-m} - \ln A}{\ln f_n - \ln A} \right\}. \tag{3.8}$$

If $\omega > 1$, $0 < \alpha\omega < 1$, $|\beta|\Gamma < 1$, $B < A^{(\beta+1)/(\alpha+1)}$, *then every positive solution to* (1.7) *converges to* $A^{1/(\alpha+1)}$.

Proof. Employing the transformation $x_n = y_n A^{1/(\alpha+1)}$, (1.7) becomes

$$y_n = \max\left\{ \frac{C_n}{y_{n-k}^{\alpha}}, \frac{\lambda}{y_{n-m}^{\beta}} \right\}, \quad n \in \mathbb{N}_0, \tag{3.9}$$

where $C_n = f_n/A$, $n \in \mathbb{N}_0$ and $\lambda = B/A^{(\beta+1)/(\alpha+1)}$.

Then by the change $y_n = C_n^{z_n}$, (3.9) is transformed into

$$C_n^{z_n} = \max\left\{ \frac{C_n}{C_{n-k}^{\alpha z_{n-k}}}, \frac{\lambda}{C_{n-m}^{\beta z_{n-m}}} \right\}, \quad n \geq \max\{k, m\}. \tag{3.10}$$

In the sequel, we proceed by considering two cases.

Case 1. Let $f_i < A, \ i = -k, \dots, -1$.
By Remark 2.7, we have $0 < C_n < 1, \ n \in \mathbb{N}_0$. From (3.10) we get

$$\begin{aligned} z_n &= \min\left\{1 - \alpha\left(\log_{C_n} C_{n-k}\right) z_{n-k}, \log_{C_n}\lambda - \beta\left(\log_{C_n} C_{n-m}\right) z_{n-m}\right\} \\ &= \min\left\{1 - \alpha\omega z_{n-k}, \log_{C_n}\lambda - \beta\left(\log_{C_n} C_{n-m}\right) z_{n-m}\right\}, \end{aligned} \tag{3.11}$$

for $n \geq \max\{k, m\}$. By the change $z_n = g_n + 1/(\alpha\omega + 1)$, (3.11) becomes

$$g_n = \min\left\{-\alpha\omega g_{n-k}, T_n - \beta\left(\log_{C_n} C_{n-m}\right) g_{n-m}\right\}, \quad n \geq \max\{k, m\}, \tag{3.12}$$

where

$$T_n = \log_{C_n}\lambda - \frac{\beta\log_{C_n} C_{n-m} + 1}{\alpha\omega + 1}. \tag{3.13}$$

Claim 1. There exists an integer $M > 0$ such that $T_n > 0$ for every $n \geq M$.

Proof. Since $f_n = AC_n$, we easily have that

$$|\beta| \log_{C_n} C_{n-m} \leq |\beta| \sup_{j \geq m}\left\{\log_{C_j} C_{j-m}\right\} = |\beta|\Gamma < 1, \quad n \geq m. \tag{3.14}$$

Hence

$$0 < \frac{\beta\log_{C_n} C_{n-m} + 1}{\alpha\omega + 1} < 2, \quad n \geq m. \tag{3.15}$$

On the other hand, for $\varepsilon = A(1 - \sqrt{\lambda})$, there exists an $M > 0$ such that for each $n \geq M$ we have $f_n > A - \varepsilon = A\sqrt{\lambda}$, which along with the fact $C_n \in (0, 1), \ n \in \mathbb{N}_0$, implies that

$$\log_{C_n}\lambda > 2, \quad n \geq M. \tag{3.16}$$

The claim follows directly from (3.15) and (3.16), as desired. □

Next, from Lemma 2.1 and (3.12) it follows that

$$\begin{aligned}|g_n| &\le \max\left\{\alpha\omega|g_{n-k}|, \beta\left(\log_{C_n} C_{n-m}\right)|g_{n-m}| - T_n\right\}\\ &\le \max\{\alpha\omega|g_{n-k}|, \beta\Gamma|g_{n-m}| - T_n\}, \quad n \ge \max\{k, m, M\}.\end{aligned} \tag{3.17}$$

From (3.17) and by Lemma 2.3, we derive $\lim_{n\to\infty}|g_n| = 0$. Hence

$$\lim_{n\to\infty} \ln y_n = \lim_{n\to\infty}(z_n \ln C_n) = \lim_{n\to\infty} z_n \cdot \ln\left(\lim_{n\to\infty} C_n\right) = \frac{1}{\alpha\omega+1}\cdot \ln 1 = 0, \tag{3.18}$$

and consequently

$$\lim_{n\to\infty} x_n = A^{1/\alpha+1}\lim_{n\to\infty} y_n = A^{1/\alpha+1}. \tag{3.19}$$

Case 2. Let $f_i > A$, $i = -k, \ldots, -1$.

By Remark 2.7, we have $C_n > 1$, $n \in \mathbb{N}_0$, and (3.10) is transformed into

$$\begin{aligned}z_n &= \max\left\{1 - \alpha\left(\log_{C_n} C_{n-k}\right)z_{n-k}, \log_{C_n}\lambda - \beta\left(\log_{C_n} C_{n-m}\right)z_{n-m}\right\}\\ &= \max\left\{1 - \alpha\omega z_{n-k}, \log_{C_n}\lambda - \beta\left(\log_{C_n} C_{n-m}\right)z_{n-m}\right\},\end{aligned} \tag{3.20}$$

for all $n \ge \max\{k, m\}$. Then employing the following change

$$z_n = -g_n + \frac{1}{\alpha\omega+1}, \tag{3.21}$$

(3.20) is transformed into

$$g_n = \min\left\{-\alpha\omega g_{n-k}, T_n - \beta\left(\log_{C_n} C_{n-m}\right)g_{n-m}\right\}, \quad n \ge \max\{k, m\}, \tag{3.22}$$

where $T_n = -\log_{C_n}\lambda + (\beta\log_{C_n} C_{n-m} + 1)/(\alpha\omega + 1)$. In this case, $T_n > 0$ obviously holds. The rest of the proof is similar to that of *Case* 1 so is omitted. □

To illustrate Theorem 3.2, we present the following example.

Example 3.3. Consider the difference equation

$$x_n = \max\left\{\frac{f_n}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-m}^{\beta}}\right\}, \quad n \in \mathbb{N}_0, \tag{3.23}$$

where $B > 0$, $m \ge 2$, and $f_n = Aq^{1/\omega^{n+1}}$, $\omega > 1$, $A, q > 0$, $q \ne 1$, $n \in \mathbb{N}_0$.

By Theorem 3.2 and through some calculations, we obtain

$$\Gamma = \sup_{n\geq m}\left\{\frac{\ln f_{n-m} - \ln A}{\ln f_n - \ln A}\right\} = \sup_{n\geq m}\left\{\frac{\omega^{n+1}}{\omega^{n-m+1}}\right\} = \omega^m. \tag{3.24}$$

Hence if $0 < \alpha < 1/\omega$, $|\beta| < 1/\omega^m$, $B < A^{(\beta+1)/(\alpha+1)}$, then every positive solution to (3.23) converges to $A^{1/(\alpha+1)}$.

Theorem 3.4. *Consider* (1.7)*. If* $(f_n)_{n\in\mathbb{N}_0}$ *is an increasing sequence converging to* $A, |\alpha|, |\beta| < 1$ *and* $B = A^{(\beta+1)/(\alpha+1)}$*, then every positive solution to* (1.7) *converges to* $A^{1/(\alpha+1)}$.

Proof. By the change $x_n = y_n A^{1/(\alpha+1)}$, (1.7) becomes

$$y_n = \max\left\{\frac{C_n}{y_{n-k}^{\alpha}}, \frac{1}{y_{n-m}^{\beta}}\right\}, \quad n \in \mathbb{N}_0, \tag{3.25}$$

where $C_n = f_n/A < 1$, $n \in \mathbb{N}_0$. The rest of the proof is analogous to that of Theorem 3.1 and thus is omitted. □

4. Explicit Solutions

Recently, Stević and Iričanin in [21] proved the following theorem.

Theorem 4.1 (see [21, Theorem 2.8]). *Consider*

$$x_n = \max\{x_{n-1}^{a_1}, \ldots, x_{n-k}^{a_k}\}, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $k \in \mathbb{N}, a_i \in \mathbb{R}, i = 1, \ldots, k$*. Then every well-defined solution of the equation has the following form:*

$$x_n = d_n^{\prod_{j=1}^{k} a_j^{i_n^{(j)}}}, \tag{4.2}$$

where $[(n+k)/k] \leq i_n^{(1)} + \cdots + i_n^{(k)} \leq n+1$, $n \in \mathbb{N}_0$, $i_n^{(j)} \geq 0$, $j = 1, \ldots, k$*, and where* d_n *is equal to one of the initial values* $x_{-k}, \ldots, x_{-1}$.

The result is interesting since (4.2) holds for all real a_j's and for all nonzero initial values if one of these exponents is negative. However, (4.2) does not give explicit solutions to (4.1) since d_n's and $i_n^{(j)}$'s in (4.2) are uncertain. Thus the problem of finding more explicit expressions of solutions to (4.1) is of interest.

In this section we find explicit solutions to the next particular cases of (4.1)

$$x_n = \max\left\{\frac{1}{x_{n-1}^p}, \frac{1}{x_{n-2}}\right\}, \quad n \in \mathbb{N}_0, \tag{4.3}$$

$$x_n = \max\left\{\frac{1}{x_{n-1}}, \frac{1}{x_{n-2}^p}\right\}, \quad n \in \mathbb{N}_0, \tag{4.4}$$

with $p > 1$ and positive initial values x_{-2}, x_{-1}. First we prove a useful lemma.

Lemma 4.2. *Let $(x_n)_{n\in\mathbb{N}_0}$ be a positive solution to (4.3) or (4.4). If there exists an $N \in \mathbb{N}_0$ such that*

$$x_{N+3} = x_N^p, \qquad x_{N+4} = x_{N+1}^p, \qquad x_{N+5} = x_{N+2}^p, \tag{4.5}$$

then for each $k \in \mathbb{N}$ the following equalities hold:

$$x_{N+3k} = x_N^{p^k}, \qquad x_{N+3k+1} = x_{N+1}^{p^k}, \qquad x_{N+3k+2} = x_{N+2}^{p^k}. \tag{4.6}$$

Proof. We will only consider (4.3), because similar proof can be given to (4.4). The case $k = 1$ obviously holds due to (4.5). Next assume that (4.6) holds for $1 \leq k \leq m$ for some $m \in \mathbb{N}$. Then by (4.3) we derive

$$\begin{aligned}
x_{N+3(m+1)} &= \max\left\{\frac{1}{x_{N+3m+2}^p}, \frac{1}{x_{N+3m+1}}\right\} = \max\left\{\frac{1}{x_{N+2}^{p^{m+1}}}, \frac{1}{x_{N+1}^{p^m}}\right\} \\
&= \max\left\{\frac{1}{x_{N+2}^p}, \frac{1}{x_{N+1}}\right\}^{p^m} = x_{N+3}^{p^m} = x_N^{p^{m+1}}, \\
x_{N+3(m+1)+1} &= \max\left\{\frac{1}{x_{N+3(m+1)}^p}, \frac{1}{x_{N+3m+2}}\right\} = \max\left\{\frac{1}{x_{N+3}^{p^{m+1}}}, \frac{1}{x_{N+2}^{p^m}}\right\} \\
&= \max\left\{\frac{1}{x_{N+3}^p}, \frac{1}{x_{N+2}}\right\}^{p^m} = x_{N+4}^{p^m} = x_{N+1}^{p^{m+1}}, \\
x_{N+3(m+1)+2} &= \max\left\{\frac{1}{x_{N+3(m+1)+1}^p}, \frac{1}{x_{N+3(m+1)}}\right\} = \max\left\{\frac{1}{x_{N+4}^{p^{m+1}}}, \frac{1}{x_{N+3}^{p^m}}\right\} \\
&= \max\left\{\frac{1}{x_{N+4}^p}, \frac{1}{x_{N+3}}\right\}^{p^m} = x_{N+5}^{p^m} = x_{N+2}^{p^{m+1}}.
\end{aligned} \tag{4.7}$$

Thus (4.6) holds for $k = m + 1$, finishing the inductive proof of the lemma. □

Proposition 4.3. *Let $(x_n)_{n\in\mathbb{N}_0}$ be a solution to (4.3) with $p > 1$ and positive initial values x_{-2}, x_{-1}, then for each $k \in \mathbb{N}_0$ the following statements hold true.*

(1) *If* $x_{-2} > x_{-1}^p, x_{-1} \geq 1$, *then* $x_{3k} = 1/x_{-1}^{p^{k+1}}, x_{3k+1} = x_{-1}^{p^{k+2}}, x_{3k+2} = x_{-1}^{p^{k+1}}$.

(2) *If* $x_{-2} > x_{-1}^p, x_{-1} < 1$, *then* $x_{3k} = 1/x_{-1}^{p^{k+1}}, x_{3k+1} = 1/x_{-1}^{p^k}, x_{3k+2} = x_{-1}^{p^{k+1}}$.

(3) *If* $x_{-2} \leq x_{-1}^p, x_{-2}^p x_{-1} \geq 1$ *then* $x_{3k} = 1/x_{-2}^{p^k}, x_{3k+1} = x_{-2}^{p^{k+1}}$ *and*

$$x_{3k+2} = x_{-2}^{p^k} \text{ if } x_{-2} \geq 1 \quad \text{or} \quad = \frac{1}{x_{-2}^{p^{k+2}}} \text{ if } x_{-2} < 1. \tag{4.8}$$

(4) *If* $x_{-2} \leq x_{-1}^p, x_{-2}^p x_{-1} < 1$ *then* $x_{3k+1} = 1/x_{-1}^{p^k}, x_{3k+2} = x_{-1}^{p^{k+1}}$ *and*

$$x_{3k+3} = x_{-1}^{p^k} \text{ if } x_{-1} \geq 1 \quad \text{or} \quad = \frac{1}{x_{-1}^{p^{k+2}}} \text{ if } x_{-1} < 1. \tag{4.9}$$

Proof. (1) By the assumption $x_{-2} > x_{-1}^p$ and (4.3) it follows that

$$x_0 = \max\left\{\frac{1}{x_{-1}^p}, \frac{1}{x_{-2}}\right\} = \frac{1}{x_{-1}^p}. \tag{4.10}$$

Then by $x_{-1} > 1$ and (4.3), we have the following equalities:

$$\begin{aligned}
x_1 &= \max\left\{\frac{1}{x_0^p}, \frac{1}{x_{-1}}\right\} = \max\left\{x_{-1}^{p^2}, \frac{1}{x_{-1}}\right\} = x_{-1}^{p^2}, \\
x_2 &= \max\left\{\frac{1}{x_1^p}, \frac{1}{x_0}\right\} = \max\left\{\frac{1}{x_{-1}^{p^3}}, x_{-1}^p\right\} = x_{-1}^p, \\
x_3 &= \max\left\{\frac{1}{x_2^p}, \frac{1}{x_1}\right\} = \max\left\{\frac{1}{x_{-1}^{p^2}}, \frac{1}{x_{-1}^{p^2}}\right\} = \frac{1}{x_{-1}^{p^2}} = x_0^p, \\
x_4 &= \max\left\{\frac{1}{x_3^p}, \frac{1}{x_2}\right\} = \max\left\{x_{-1}^{p^3}, \frac{1}{x_{-1}^p}\right\} = x_{-1}^{p^3} = x_1^p, \\
x_5 &= \max\left\{\frac{1}{x_4^p}, \frac{1}{x_3}\right\} = \max\left\{\frac{1}{x_{-1}^{p^4}}, x_{-1}^{p^2}\right\} = x_{-1}^{p^2} = x_2^p.
\end{aligned} \tag{4.11}$$

Hence (4.5) is satisfied for $N = 0$. Then by Lemma 4.2 we have that

$$x_{3k} = 1/x_{-1}^{p^{k+1}}, \quad x_{3k+1} = x_{-1}^{p^{k+2}}, \quad x_{3k+2} = x_{-1}^{p^{k+1}}, \quad k \in \mathbb{N}_0, \tag{4.12}$$

as desired.

(2) By similar calculations as in (1), the following equalities hold:

$$x_0 = \frac{1}{x_{-1}^p}, \qquad x_1 = \frac{1}{x_{-1}}, \qquad x_2 = x_{-1}^p, \qquad x_3 = \frac{1}{x_{-1}^{p^2}} = x_0^p,$$
$$x_4 = \frac{1}{x_{-1}^p} = x_1^p, \qquad x_5 = x_{-1}^{p^2} = x_2^p. \tag{4.13}$$

Thus (4.5) holds for $N = 0$, and the result follows again by Lemma 4.2.

(3) By the assumption $x_{-2} \le x_{-1}^p$ and (4.3) it follows that

$$x_0 = \max\left\{\frac{1}{x_{-1}^p}, \frac{1}{x_{-2}}\right\} = \frac{1}{x_{-2}}. \tag{4.14}$$

Then from $x_{-2}^p x_{-1} \ge 1$ and (4.3), we get

$$x_1 = \max\left\{\frac{1}{x_0^p}, \frac{1}{x_{-1}}\right\} = \max\left\{x_{-2}^p, \frac{1}{x_{-1}}\right\} = x_{-2}^p. \tag{4.15}$$

Now we will consider two cases $x_{-2} \ge 1$ and $x_{-2} < 1$. When $x_{-2} \ge 1$, the following equalities hold:

$$\begin{aligned}
x_2 &= \max\left\{\frac{1}{x_1^p}, \frac{1}{x_0}\right\} = \max\left\{\frac{1}{x_{-2}^{p^2}}, x_{-2}\right\} = x_{-2}, \\
x_3 &= \max\left\{\frac{1}{x_2^p}, \frac{1}{x_1}\right\} = \max\left\{\frac{1}{x_{-2}^p}, \frac{1}{x_{-2}^p}\right\} = \frac{1}{x_{-2}^p} = x_0^p, \\
x_4 &= \max\left\{\frac{1}{x_3^p}, \frac{1}{x_2}\right\} = \max\left\{x_{-2}^{p^2}, \frac{1}{x_{-2}}\right\} = x_{-2}^{p^2} = x_1^p, \\
x_5 &= \max\left\{\frac{1}{x_4^p}, \frac{1}{x_3}\right\} = \max\left\{\frac{1}{x_{-2}^{p^3}}, x_{-2}^p\right\} = x_{-2}^p = x_2^p.
\end{aligned} \tag{4.16}$$

On the other hand, the case $x_{-2} < 1$ leads to

$$x_2 = \frac{1}{x_{-2}^{p^2}}, \qquad x_3 = \frac{1}{x_{-2}^p} = x_0^p, \qquad x_4 = x_{-2}^{p^2} = x_1^p, \qquad x_5 = \frac{1}{x_{-2}^{p^3}} = x_2^p. \tag{4.17}$$

Hence (4.5) holds for $N = 0$ no matter the value of x_{-2} is bigger or less than one. From this the result follows by Lemma 4.2.

(4) Through analogous calculations to (3), if $x_{-1} \geq 1$ then

$$
\begin{gathered}
x_0 = \frac{1}{x_{-2}}, \qquad x_1 = \frac{1}{x_{-1}}, \qquad x_2 = x_{-1}^{p}, \qquad x_3 = x_{-1}, \\
x_4 = \frac{1}{x_{-1}^{p}} = x_1^{p}, \qquad x_5 = x_{-1}^{p^2} = x_2^{p}, \qquad x_6 = x_{-1}^{p} = x_3^{p}.
\end{gathered} \tag{4.18}
$$

If $x_{-1} < 1$ then

$$
\begin{gathered}
x_0 = \frac{1}{x_{-2}}, \qquad x_1 = \frac{1}{x_{-1}}, \qquad x_2 = x_{-1}^{p}, \qquad x_3 = \frac{1}{x_{-1}^{p^2}}, \\
x_4 = \frac{1}{x_{-1}^{p}} = x_1^{p}, \qquad x_5 = x_{-1}^{p^2} = x_2^{p}, \qquad x_6 = \frac{1}{x_{-1}^{p^3}} = x_3^{p}.
\end{gathered} \tag{4.19}
$$

Hence (4.5) holds for $N = 1$ and any $x_{-1} > 0$. Hence, the results also follow from Lemma 4.2, finishing the proof of the proposition. □

The next proposition can be similarly proved as the proof of Proposition 4.3, hence the proof is omitted here.

Proposition 4.4. *Let $(x_n)_{n\in\mathbb{N}_0}$ be a solution to (4.4) with $p > 1$ and positive initial values x_{-2}, x_{-1}, then for each $k \in \mathbb{N}_0$ the following statements hold true.*

(1) *If $x_{-2}^{p} \geq x_{-1}$, $x_{-1} \geq 1$, then $x_{3k} = 1/x_{-1}^{p^k}$, $x_{3k+1} = x_{-1}^{p^k}$, $x_{3k+2} = {}_{-1}^{p^{k+1}}$.*

(2) *If $x_{-2}^{p} \geq x_{-1}$, $x_{-1} < 1$, then $x_{3k} = 1/x_{-1}^{p^k}$, $x_{3k+1} = 1/x_{-1}^{p^{k+1}}$, $x_{3k+2} = x_{-1}^{p^{k+1}}$.*

(3) *If $x_{-2}^{p} < x_{-1}$, $x_{-2}x_{-1} \geq 1$ then $x_{3k} = 1/x_{-2}^{p^{k+1}}$, $x_{3k+1} = x_{-2}^{p^{k+1}}$, and*

$$
x_{3k+2} = \begin{cases} x_{-2}^{p^{k+2}} & \text{if } x_{-2} \geq 1, \\ \dfrac{1}{x_{-2}^{p^{k+1}}} & \text{if } x_{-2} < 1. \end{cases} \tag{4.20}
$$

(4) *If $x_{-2}^{p} < x_{-1}$, $x_{-2}x_{-1} < 1$ then $x_{3k+1} = 1/x_{-1}^{p^{k+1}}$, $x_{3k+2} = x_{-1}^{p^{k+1}}$, and*

$$
x_{3k+3} = \begin{cases} x_{-1}^{p^{k+2}} & \text{if } x_{-1} \geq 1, \\ \dfrac{1}{x_{-1}^{p^{k+1}}} & \text{if } x_{-1} < 1. \end{cases} \tag{4.21}
$$

Remark 4.5. From the above propositions, we know that any positive solution $(x_n)_{n\in\mathbb{N}_0}$ to (4.3) or (4.4) can be divided into three subsequences which have explicit expressions. If we regard the sequence $(\ldots, 0, \infty, \infty, 0, \infty, \infty, \ldots)$ as a *general* periodic solution to (4.3), then the solution $(x_n)_{n\in\mathbb{N}_0}$ eventually converges to the general period-three solution $(0, \infty, \infty)$.

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