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Research Article

Asymptotic Behaviour of the Iterates of Positive Linear Operators

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We present a general result concerning the limit of the iterates of positive linear operators acting on continuous functions defined on a compact set. As applications, we deduce the asymptotic behaviour of the iterates of almost all classic and new positive linear operators.

1. Introduction

The outstanding results of Kelisky and Rivlin [1] and Karlin and Ziegler [2] provided new insights into the study of the limit behavior of the iterates of linear operators defined on $C[0,1]$. They have attracted a lot of attention lately, and several alternative proofs and generalizations have been given in the last fifty years (see the references).

Nevertheless, the general problem concerning the overiterates of positive linear operators remained unsolved. This can be stated as follows.

Let X be a compact topological space, let $C(X)$ be the linear space of all continuous real-valued functions defined on X and endowed with the norm $\|f\| := \sup_{x\in X}|f(x)|$, and let $U : C(X) \to C(X)$ denote a positive linear operator, that is, $Uf \geq 0$ for all $f \geq 0$. The problem is to provide sufficient conditions for the convergence of the sequence of iterates $(U^k)_{k\in\mathbb{N}}$ and find its limit, which is the goal of this paper.

Various techniques from different areas such as spectral theory, probability theory, fixed point theory, and the theory of semigroups of operators, have been employed in the attempts to find a solution (see, in chronological order, [1–22] and the references therein).

However, although many useful contributions have been made, the problem, in its generality, remained unsolved. The limit remained unknown for a long while even for the case restricted to classical particular positive linear operators.

For the first time, a solution to the general problem of the asymptotic behavior of the iterates of positive linear operators defined on $C[0,1]$ was announced by the authors of this paper at the APPCOM08 conference held in Niš, Serbia, in August 2008. Related results appeared one year later in [19].

This paper describes the employment of a number of completely new methods in solving the general problem. To the best of our knowledge, this is the most general result known up to date. As an application of the main result, the asymptotic behavior of the iterates of many classical and new positive linear operators is deduced.

2. Notations and Preliminary Results

Throughout this paper, we will use the following notations:

X is a compact topological space;

$C(X)$ is the normed linear space of all continuous real-valued functions defined on X; $\|f\| := \max_{x\in X}|f(x)|$;

$\mathcal{V}$ is a linear subspace of $C(X)$ including the space $\mathcal{P}_0$ of constants;

$U : C(X) \to C(X)$ is a positive linear operator preserving the elements of $\mathcal{V}$;

$L : C(X) \to \mathcal{V}$ is an interpolation operator;

Y_L is the interpolation set of L,

$$Y_L = \{y \in X \mid Lf(y) = f(y),\ \forall f \in C(X)\} \neq \emptyset. \tag{2.1}$$

The existence of such an operator L is always assured. Indeed, for fixed $x_0 \in X$, the operator $L : C(X) \to \mathcal{P}_0$, $Lf := f(x_0)$ is an interpolation operator with interpolation set $Y_L = \{x_0\}$.

We also emphasize that if $U : C[0,1] \to C[0,1]$ is a positive linear operator preserving the affine functions, then Uf interpolates f at the end points for all $f \in C[0,1]$. This well-known result is a particular case of a theorem of Bauer, see [23] and [24, Proposition 1.4]. It follows that $Y_L \supseteq \{0,1\}$,

$e_i : [0,1] \to \mathbb{R}$ are the monomial functions $e_i(x) = x^i$, $i = 0,1,\ldots$,

$L_1 : C[0,1] \to C[0,1]$ is the Lagrange interpolation operator $L_1 f = f(0)e_0 + (f(1) - f(0))e_1$.

3. The Main Results

By using the notations presented in Section 2, the following theorem is the main result of the paper.

Theorem 3.1. *If there exists $\varphi \in C(X)$ such that*

$$U\varphi \geq \varphi \text{ on } X, \quad U\varphi \neq \varphi \text{ on } X \setminus Y_L, \tag{3.1}$$

then

$$\lim_{k} U^k f = Lf, \quad \forall f \in C(X). \tag{3.2}$$

Moreover, if X is a compact metric space, then the convergence is uniform.

Proof. Let $f \in C(X)$. The case when $Lf = f$ is trivial. Indeed, in this case, since $Lf \in \mathcal{U}$ and U preserves the elements of $\mathcal{U}$, we have $Uf = ULf = Lf$, and hence $U^k f = Lf$, $k = 1, 2, \ldots$.

If $Lf \neq f$, for sufficiently small $\varepsilon > 0$, the inverse image of the open set $(-\varepsilon, \varepsilon)$ under the continuous function $Lf - f$ is an open set G, $Y_L \subseteq G \neq X$. It follows that

$$\left|Lf - f\right| < \varepsilon, \quad \text{on } G. \tag{3.3}$$

Since X is compact and G is open, it follows that $X \setminus G \subseteq X \setminus Y_L$ is a nonempty compact subset of X, and we obtain

$$m_\varepsilon := \inf_{x \in X \setminus G} \left(U\varphi(x) - \varphi(x)\right) > 0. \tag{3.4}$$

Consequently, the following decisive inequality

$$\left|Lf - f\right| < \varepsilon + \frac{\|f - Lf\|}{m_\varepsilon} \left(U\varphi - \varphi\right) \tag{3.5}$$

is satisfied. By applying the positive operator U^k to (3.5), we get

$$\left|Lf - U^k f\right| < \varepsilon + \frac{\|f - Lf\|}{m_\varepsilon} \left|U^{k+1}\varphi - U^k\varphi\right|. \tag{3.6}$$

Since $U\varphi \geq \varphi$, we obtain

$$\varphi \leq U^k\varphi \leq U^{k+1}\varphi \leq \|\varphi\|, \quad k = 1, 2, \ldots. \tag{3.7}$$

The sequence $(U^k\varphi)_{k\geq 1}$ is monotone and bounded. It follows that it is pointwise convergent. Since ε was chosen arbitrarily, by using (3.6) we deduce that $U^k f \xrightarrow{\text{pointwise}} Lf$.

In the particular case when X is a compact metric space, since

$$U^k\varphi \xrightarrow{\text{pointwise}} L\varphi \in C(X), \tag{3.8}$$

by Dini's Theorem, we obtain that $U^k\varphi \xrightarrow{\text{uniformly}} L\varphi$. From the inequalities

$$\left\|Lf - U^k f\right\| < \varepsilon + \frac{\|f - Lf\|}{m_\varepsilon} \left\|U^{k+1}\varphi - U^k\varphi\right\|, \tag{3.9}$$

we deduce that $U^k f \xrightarrow{\text{uniformly}} Lf$. □

In the following we give more information on the limit operator L.

Theorem 3.2. *The limit interpolation operator L is unique, positive, and satisfies the equalities*

$$UL = LU = L. \tag{3.10}$$

Proof. The unicity and positivity of the operator L follow from the existence of limit $\lim_k U^k = L$. Since U preserves the elements of $\mathcal{V}$, we obtain that $UL = L$. Taking into account the relations

$$\emptyset \subsetneq Y_L \subseteq \{y \in X \mid LUf(y) = Uf(y),\ \forall f \in C(X)\}, \tag{3.11}$$

we can repeat the proof of Theorem 3.1 by starting with Uf instead of f. We deduce that $\lim_k U^k f = LUf$, and hence $LU = L$. □

An immediate corollary of Theorem 3.1 is the following.

Corollary 3.3. *If $U : C(X) \to C(X)$ is a positive linear operator possessing an interpolation point $x_0 \in X$ and there exists $\varphi \in C(X)$ such that $U\varphi \lneqq \varphi$ on $X \setminus \{x_0\}$, then*

$$\lim_k U^k f = f(x_0), \quad \forall f \in C(X). \tag{3.12}$$

Proof. In Theorem 3.1 we take $Lf = f(x_0)$. □

Remark 3.4. The existence of the function φ is essential here, in the sense that, if it is not satisfied, then the statement of Theorem 3.1 might not be true. Indeed, the positive linear operator $U : C[0,1] \to C[0,1]$ defined by

$$Uf(x) = xf(0) + (1-x)f(1) \tag{3.13}$$

preserves the space of constants $\mathcal{P}_0$, and the operator $L : C[0,1] \to \mathcal{P}_0$, $Lf(x) = f(0)$, is interpolator on $Y_L = \{0\}$. However, there exists no continuous function $\varphi \in C[0,1]$ such that $U\varphi(x) > \varphi(x)$, for all $x \in [0,1] \setminus \{0\}$(see $(U\varphi(0) - \varphi(0))(U\varphi(1) - \varphi(1)) = -(\varphi(1) - \varphi(0))^2 \le 0$) and the sequence $(U^n e_1)_{n\ge 1}$ has no limit.

4. Applications

In this section, as applications of Theorem 3.1 and Corollary 3.3, we rediscover known results and obtain new ones concerning the asymptotic behaviour of the iterates of positive linear operators.

4.1. Positive Operators on $C[0,1]$ *Preserving Linear Functions*

In the case of the particularisations, $X = [0,1]$, $\mathcal{V}$ is the space of all linear functions in $C[0,1]$, L is the Lagrange interpolation operator of degree one associated to f at the endpoints 0 and 1, and $Y_L = \{0,1\}$, by Theorem 3.1, we have that the following corollary holds.

Corollary 4.1. *Let $U : C[0,1] \to C[0,1]$ be a positive linear operator preserving the linear functions. If there exists $\varphi \in C[0,1]$ such that $U\varphi \lneqq \varphi$ on $(0,1)$, then the sequence of the iterates of U converges uniformly to the Lagrange operator L_1.*

4.2. The Meyer-König and Zeller Operators

In 1960 Meyer-König and Zeller, see [25], introduced a sequence of positive linear operators which were studied, modified, and generalized by several authors. The classical Meyer-König and Zeller operators $\mathbf{MKZ}_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, in the modified version of Cheney and Sharma, see [26], are defined by

$$\mathbf{MKZ}_n f(x) = \begin{cases} \sum_{k=0}^{\infty} \binom{n+k}{k} (1-x)^{n+1} x^k f\left(\frac{k}{n+k}\right), & x \in [0,1), \\ f(1), & x = 1. \end{cases} \tag{4.1}$$

Moreover, from [27, Equation (2.4)], see also [28], we have that

$$\mathbf{MKZ}_n e_2 - e_2 \geq (n+1)^{-1} \; e_1(1-e_1)^2 > 0 \quad \text{on } (0,1), \; n \geq 1. \tag{4.2}$$

For $\varphi = e_2$, we have, as a consequence of Corollary 3.3, that the following corollary holds.

Corollary 4.2. *The sequence of the iterates of the Meyer-König and Zeller operators* (4.1) *converges uniformly to the Lagrange interpolation operator L_1.*

4.3. The May Positive Linear Operators

The May operators, see [29], are defined by

$$\mathbf{M}_n f(x) := \int_0^1 f(t)\rho_n(x,t)\mathrm{d}t, \quad n \in \mathbb{N}, \tag{4.3}$$

where ρ_n denotes a kernel function. They satisfy

$$\mathbf{M}_n e_2 - e_2 = \lambda n^{-1} \; (e_1 - e_2) > 0 \quad \text{on } (0,1), \; n \geq 1, \tag{4.4}$$

for some $\lambda > 0$ and preserve linear functions. For $\varphi = e_2$, as a consequence of Corollary 3.3, the following corollary holds true.

Corollary 4.3. *The sequence $(\mathbf{M}_n^k)_{k\in\mathbb{N}}$ of the iterates of the May operators* (4.3) *converges uniformly to L_1.*

4.4. The Bernstein Operator on a Simplex

Consider the simplex S^p in $\mathbb{R}^p$, $p \geq 1$, given by

$$S^p = \{x := (x_1, \ldots, x_p) \mid x_i \geq 0,\ |x| := x_1 + \ldots + x_p \leq 1\}. \tag{4.5}$$

The $p+1$ vertices of the simplex S^p are the points $v_i \in \mathbb{R}^p$, where

$$\begin{gathered} v_0 = (0,0,\ldots,0), \\ v_1 = (1,0,\ldots,0), \\ \vdots \\ v_p = (0,0,\ldots,1). \end{gathered} \tag{4.6}$$

With

$$\mathcal{M} = \{m := (m_1, \ldots, m_p) \mid m_i \in \{0,1,\ldots,n\},\ |m| \leq n\}, \tag{4.7}$$

the Bernstein approximation operator $\mathbf{B}_n : C(S^p) \to C(S^p)$ is defined by

$$\mathbf{B}_n f(x) = \sum_{m \in \mathcal{M}} \frac{n!}{m_1! \cdots m_p!(n-|m|)!} x_1^{m1} \cdots x_p^{mp} (1-|x|)^{n-|m|} f\left(\frac{m}{n}\right). \tag{4.8}$$

The operator $\mathbf{B}_n$ preserves the subspace of linear functions

$$\mathcal{V} = \{f \mid f(x_1, \ldots, x_p) = a_0 + a_1 x_1 + \cdots + a_p x_p,\ a_0,\ a_1, \ldots,\ a_p \in \mathbb{R}\}. \tag{4.9}$$

The Lagrange interpolation operator $L : C(S^p) \to \mathcal{V}$ is defined by

$$Lf(x) = \left(1 - \sum_{k=1}^{p} x_k\right) f(v_0) + \sum_{k=1}^{p} x_k f(v_k) \tag{4.10}$$

and interpolates all functions in $C(S^p)$ on the set

$$Y_L = \{v_0, \ldots, v_p\}. \tag{4.11}$$

For $\varphi(x) = x_1^2 + \ldots + x_p^2$, we have

$$\mathbf{B}_n\varphi(x) - \varphi(x) = \frac{1}{n}\left(x_1(1-x_1) + \cdots + x_p(1-x_p)\right) > 0, \quad \forall x \in S^p - Y_L, \tag{4.12}$$

and, by using Theorem 3.1, we get the following.

Corollary 4.4. *The sequence of the iterates of the Bernstein operator associated with the simplex* S^p (4.8) *converges uniformly to the Lagrange interpolation operator* (4.10).

4.5. Positive Operators on $C[a,b]$ *Preserving* e_0 *and* e_2

In [15] Agratini introduced a sequence of positive linear operators $\Lambda_n : C[a,b] \to C[a,b]$ preserving e_0 and e_2. In the case of the particularisations, $D = [a,b]$, $Y_L = \{a,b\}$, $\mathcal{V} = \operatorname{Span}\{e_0, e_2\}$, $\varphi = e_4$, and L is the interpolation operator:

$$Lf(x) = \frac{1}{b^2 - a^2}\Big(f(a)b^2 - f(b)a^2 + \big(f(b) - f(a)\big)x^2\Big), \tag{4.13}$$

as a corollary of Theorem 3.1, we obtain a result of Agratini [15, Theorem 3.1].

4.6. Bernstein-Type Operators Preserving e_0 *and* e_j

Let $n, j \in \mathbb{N}$, $n > j > 1$. Aldaz et al. [30, Proposition 11] had recently considered the Bernstein-type operators $\mathcal{B}_{j,n} : C[0,1] \to C[0,1]$,

$$\mathcal{B}_{j,n}f(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\left(\frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}\right)^{1/j}\right) x^k (1-x)^{n-k}. \tag{4.14}$$

The operators $\mathcal{B}_{j,n}$ satisfy

$$\mathcal{B}_{j,n}e_0 = e_0, \qquad \mathcal{B}_{j,n}e_j = e_j. \tag{4.15}$$

Considering the particularisations, $D = [0,1]$, $Y_L = \{0,1\}$, $\mathcal{V} = \operatorname{Span}\{e_0, e_j\}$, $\varphi = e_{2j}$, and L_j is the interpolation operator:

$$L_j f = f(0)e_0 + \big(f(1) - f(0)\big)e_j, \tag{4.16}$$

as a corollary of Theorem 3.1, we obtain the following corollary.

Corollary 4.5. *The sequence* $(\mathcal{B}^k_{j,n})_{k\in\mathbb{N}}$ *of the iterates of the Bernstein-type operators converges uniformly to the operator* L_j *in* (4.16).

4.7. The Cesàro Operator on $C[0,1]$

In the case when $X = [0,1]$, $\mathcal{V}$ is the space $\mathcal{P}_0$ of constants, $Lf = f(0)$, $Y_L = \{0\}$; $\varphi = e_1$, by Theorem 3.1, we generalize the following recent result of Galaz Fontes and Solís.

Corollary 4.6 (see [16, Theorem 3]). *Let* $\rho \in C[0,1]$ *be positive on* $(0,1)$ *such that* $\int_0^1 \rho(t)\,dt = 1$, *and let* $\mathcal{C} : C[0,1] \to C[0,1]$ *be the Cesàro mean operator,*

$$\mathcal{C}f(x) = \int_0^1 f(tx)\rho(t)dt, \quad x \in [0,1]. \tag{4.17}$$

Then,

$$\mathcal{C}^k f \xrightarrow{uniformly} f(0), \quad \forall f \in C[0,1]. \tag{4.18}$$

4.8. The Bernstein-Stancu Operators

Let $b > 0$. The Bernstein-Stancu operators $\mathbf{S}_{n,0,b} : C[0,1] \to C[0,1]$ (see, e.g., [31]),

$$\mathbf{S}_{n,0,b} f(x) = \sum_{k=0}^{n} \binom{n}{n} (1-x)^{n-k} x^k f\left(\frac{k}{n+b}\right), \tag{4.19}$$

satisfy the following:

$$\mathbf{S}_{n,0,b}\ e_0 = e_0\ , \qquad \mathbf{S}_{n,0,b}\ e_1 = \frac{n}{n+b}\ e_1. \tag{4.20}$$

For $X = [0,1]$, $x_0 = 0$ and $\varphi = e_1$, by Corollary 3.3, we obtain a result concerning the iterates of the Bernstein-Stancu operators.

Corollary 4.7. *The sequence of the iterates of Stancu's operators* (4.19) *converges uniformly to* $f(0)$.

4.9. The Cheney-Sharma Operator

Let $t_n \geq 0$, $n \in \mathbb{N}$ and let $\mathbf{CS}_n$ be the nth *Bernstein-Cheney-Sharma operator* on $C[0,1]$, defined by

$$\mathbf{CS}_n f(x) = (1+nt_n)^{-n} \sum_{k=0}^{n} \binom{n}{k} x(x+kt_n)^{k-1} (1-x+(n-k)t_n)^{n-k} f\left(\frac{k}{n}\right). \tag{4.21}$$

It is known that $0 \leq \mathbf{CS}_n\ e_1 \leq e_1/(1+t_n)$ (see, e.g., [32] and [3, (5.3.7)]. For $t_n > 0$ one has that

$$\mathbf{CS}_n\ e_0 = e_0, \qquad 0 \leq \mathbf{CS}_n\ e_1 \leq \frac{1}{1+t_n}\ e_1. \tag{4.22}$$

Taking $X = [0,1]$, $x_0 = 0$, and $\varphi = e_1$, in Corollary 3.3, we obtain the following application.

Corollary 4.8. *The sequence of the iterates of the Cheney-Sharma operators* (4.21) *converges uniformly to* $f(0)$.

4.10. The Schurer Operator

For $p, n \in \mathbb{N}$, $n \geq p$, the *Bernstein-Schurer-type operator* [3, (5.3.1)], $\mathcal{B}_n : C[0,1] \to C[0,1]$ is defined by

$$\mathcal{B}_n f(x) = \sum_{k=0}^{n-p} f\left(\frac{k}{n}\right)\binom{n-p}{k} x^k (1-x)^{n-p-k}. \tag{4.23}$$

One can prove that this operator satisfies

$$\mathcal{B}_n\ e_0 = e_0, \qquad \mathcal{B}_n\ e_1 = \frac{n-p}{n}\ e_1. \tag{4.24}$$

In the case, when $X = [0,1]$, $x_0 = 0$, and $\varphi = e_1$, by Corollary 3.3, we obtain the following result.

Corollary 4.9. *The sequence of the iterates of the Schurer operators* (4.23) *converges uniformly to* $f(0)$.

4.11. Piecewise Bernstein Operators

Let $f \in C[a,b]$. Consider the Bernstein operator $B_{n,a,b} : C[a,b] \to C[a,b]$:

$$B_{n,a,b} f(x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (b-x)^{n-k} (x-a)^k f\left(a + \frac{k(b-a)}{n}\right), \quad n = 0, 1, \ldots. \tag{4.25}$$

In the case of the particularisations, $X = [a,c]$, $\mathcal{U}$ is the linear space of polygonal lines with vertices possessing abscissae at a, b, and c, Lf is the polygonal line with vertices at $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $Y_L = \{a, b, c\}$, and $\varphi(x) = x^2$, by Theorem 3.1, we have that the following corollary holds.

Corollary 4.10. *For $a < b < c$, consider the composite Bernstein operator $B : C[a,c] \to C[a,c]$,*

$$Bf(x) = \begin{cases} B_{n,a,b} f(x), & x \in [a,b], \\ B_{n,b,c} f(x), & x \in [b,c]. \end{cases} \tag{4.26}$$

Then,

$$B^k f \xrightarrow{\text{uniformly}} \begin{cases} L[a,b;f], & \text{on}[a,b], \\ L[b,c;f], & \text{on}[b,c] \end{cases} \qquad \forall f \in C[0,1]. \tag{4.27}$$

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References

[1] R. P. Kelisky and T. J. Rivlin, "Iterates of Bernstein polynomials," *Pacific Journal of Mathematics*, vol. 21, pp. 511–520, 1967.
[2] S. Karlin and Z. Ziegler, "Iteration of positive approximation operators," *Journal of Approximation Theory*, vol. 3, pp. 310–339, 1970.
[3] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and Its Applications*, vol. 17 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1994, Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff.
[4] H. H. Gonska and X. L. Zhou, "Approximation theorems for the iterated Boolean sums of Bernstein operators," *Journal of Computational and Applied Mathematics*, vol. 53, no. 1, pp. 21–31, 1994.
[5] J. A. Adell, F. G. Badía, and J. de la Cal, "On the iterates of some Bernstein-type operators," *Journal of Mathematical Analysis and Applications*, vol. 209, no. 2, pp. 529–541, 1997.
[6] H. Oruç and N. Tuncer, "On the convergence and iterates of q-Bernstein polynomials," *Journal of Approximation Theory*, vol. 117, no. 2, pp. 301–313, 2002.
[7] S. Ostrovska, "q-Bernstein polynomials and their iterates," *Journal of Approximation Theory*, vol. 123, no. 2, pp. 232–255, 2003.
[8] J. P. King, "Positive linear operators which preserve x^2," *Acta Mathematica Hungarica*, vol. 99, no. 3, pp. 203–208, 2003.
[9] I. A. Rus, "Iterates of Bernstein operators, via contraction principle," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 1, pp. 259–261, 2004.
[10] U. Itai, "On the eigenstructure of the Bernstein kernel," *Electronic Transactions on Numerical Analysis*, vol. 25, pp. 431–438, 2006.
[11] H. Gonska, D. Kacsó, and P. Piţul, "The degree of convergence of over-iterated positive linear operators," *Journal of Applied Functional Analysis*, vol. 1, no. 4, pp. 403–423, 2006.
[12] H. Gonska and I. Raşa, "The limiting semigroup of the Bernstein iterates: degree of convergence," *Acta Mathematica Hungarica*, vol. 111, no. 1-2, pp. 119–130, 2006.
[13] H. Gonska, P. Piţul, and I. Raşa, "Over-iterates of Bernstein-Stancu operators," *Calcolo*, vol. 44, no. 2, pp. 117–125, 2007.
[14] H.-J. Wenz, "On the limits of (linear combinations of) iterates of linear operators," *Journal of Approximation Theory*, vol. 89, no. 2, pp. 219–237, 1997.
[15] O. Agratini, "On the iterates of a class of summation-type linear positive operators," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1178–1180, 2008.
[16] F. Galaz Fontes and F. J. Solís, "Iterating the Cesàro operators," *Proceedings of the American Mathematical Society*, vol. 136, no. 6, pp. 2147–2153, 2008.
[17] U. Abel and M. Ivan, "Over-iterates of Bernstein's operators: a short and elementary proof," *American Mathematical Monthly*, vol. 116, no. 6, pp. 535–538, 2009.
[18] C. Badea, "Bernstein polynomials and operator theory," *Results in Mathematics*, vol. 53, no. 3-4, pp. 229–236, 2009.
[19] I. Rasa, "Asymptotic behaviour of certain semigroups generated by differential operators," *Jaen Journal on Approximation*, vol. 1, no. 1, pp. 27–36, 2009.
[20] I. Gavrea and M. Ivan, "On the iterates of positive linear operators preserving the affine functions," *Journal of Mathematical Analysis and Applications*, vol. 372, no. 2, pp. 366–368, 2010.
[21] I. Rasa, "C_0 semigroups and iterates of positive linear operators: asymptotic behaviour," *Rendiconti del Circolo Matematico di Palermo*, vol. 2, supplement 82, pp. 123–142, 2010.
[22] I. Gavrea and M. Ivan, "On the iterates of positive linear operators," *Journal of Approximation Theory*. In press.
[23] H. Bauer, "Šilovscher Rand und Dirichletsches Problem," *Université de Grenoble. Annales de l'Institut Fourier*, vol. 11, pp. 89–136, 1961.
[24] R. R. Phelps, *Lectures on Choquet's Theorem*, D. Van Nostrand, Princeton, NJ, USA, 1966.

[25] W. Meyer-König and K. Zeller, "Bernsteinsche potenzreihen," *Studia Mathematica*, vol. 19, pp. 89–94, 1960.

[26] E. W. Cheney and A. Sharma, "Bernstein power series," *Canadian Journal of Mathematics*, vol. 16, pp. 241–252, 1964.

[27] M. Becker and R. J. Nessel, "A global approximation theorem for Meyer-König and Zeller operators," *Mathematische Zeitschrift*, vol. 160, no. 3, pp. 195–206, 1978.

[28] P. C. Sikkema, "On the asymptotic approximation with operators of Meyer-König and Zeller," *Indagationes Mathematicae*, vol. 32, pp. 428–440, 1970.

[29] C. P. May, "Saturation and inverse theorems for combinations of a class of exponential-type operators," *Canadian Journal of Mathematics*, vol. 28, no. 6, pp. 1224–1250, 1976.

[30] J. M. Aldaz, O. Kounchev, and H. Render, "Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces," *Numerische Mathematik*, vol. 114, no. 1, pp. 1–25, 2009.

[31] D. D. Stancu, "Approximation of functions by a new class of linear polynomial operators," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 13, pp. 1173–1194, 1968.

[32] E. W. Cheney and A. Sharma, "On a generalization of Bernstein polynomials," *Rivista di Matematica della Universita' Degli Studi di Parma*, vol. 5, pp. 77–84, 1964.