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Research Article

Integral-Type Operators from Bloch-Type Spaces to Q_K Spaces

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The boundedness and compactness of the integral-type operator $I_{\varphi,g}^{(n)}f(z) = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta$, where $n \in \mathbb{N}_0$, φ is a holomorphic self-map of the unit disk $\mathbb{D}$, and g is a holomorphic function on $\mathbb{D}$, from α-Bloch spaces to Q_K spaces are characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, $\partial\mathbb{D}$ be its boundary, $D(w,r)$ be disk centered at w of radius r, and let $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$. Let

$$\eta_a(z) = \frac{a-z}{1-\overline{a}z}, \quad a \in \mathbb{D}, \tag{1.1}$$

be the involutive Möbius transformation which interchanges points 0 and a. If X is a Banach space, then by B_X we will denote the closed unit ball in X.

The α-Bloch space, $\mathcal{B}^\alpha(\mathbb{D}) = \mathcal{B}^\alpha$, $\alpha > 0$, consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \tag{1.2}$$

The little α-Bloch space $\mathcal{B}_0^\alpha(\mathbb{D}) = \mathcal{B}_0^\alpha$ consists of all functions f holomorphic on $\mathbb{D}$ such that $\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$. The norm on $\mathcal{B}^\alpha$ is defined by

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f'(z)|. \tag{1.3}$$

With this norm, $\mathcal{B}^\alpha$ is a Banach space, and the little α-Bloch space $\mathcal{B}_0^\alpha$ is a closed subspace of the α-Bloch space. Note that $\mathcal{B}^1 = \mathcal{B}$ is the usual Bloch space.

Given a nonnegative Lebesgue measurable function K on $(0,1]$ the space Q_K consists of those $f \in H(\mathbb{D})$ for which

$$b_{Q_K}^2(f) = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\left(1-|\eta_a(z)|^2\right) dm(z) < \infty, \tag{1.4}$$

where $dm(z) = (1/\pi)dx\,dy = (1/\pi)r\,dr\,d\theta$ is the normalized area measure on $\mathbb{D}$ [1]. It is known that b_{Q_K} is a seminorm on Q_K which is Möbius invariant, that is,

$$b_{Q_K}(f\circ\eta) = b_{Q_K}(f), \quad \eta \in \mathrm{Aut}(\mathbb{D}), \tag{1.5}$$

where $\mathrm{Aut}(\mathbb{D})$ is the group of all automorphisms of the unit disk $\mathbb{D}$. It is a Banach space with the norm defined by

$$\|f\|_{Q_K} = |f(0)| + b_{Q_K}(f). \tag{1.6}$$

The space $Q_{K,0}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 K\left(1-|\eta_a(z)|^2\right) dm(z) = 0. \tag{1.7}$$

It is known that $Q_{K,0}$ is a closed subspace of Q_K. For classical Q spaces, see [2].

It is clear that each Q_K contains all constant functions. If Q_K consists of just constant functions, we say that it is trivial. Q_K is nontrivial if and only if

$$\sup_{t\in(0,\,1)} \int_0^1 K(1-r)\frac{(1-t)^2}{(1-tr^2)^3} r\,dr < \infty. \tag{1.8}$$

Throughout this paper, we assume that condition (1.8) is satisfied, so that the space Q_K is nontrivial. An important tool in the study of Q_K spaces is the auxiliary function λ_K defined by

$$\lambda_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty, \tag{1.9}$$

where the domain of K is extended to $(0,\infty)$ by setting $K(t) = K(1)$ when $t > 1$. The next two conditions play important role in the study of Q_K spaces.

(a) There is a constant $C > 0$ such that for all $t > 0$

$$K(2t) \leq CK(t). \tag{1.10}$$

(b) The auxiliary function λ_K satisfies the following condition:

$$\int_0^1 \frac{\lambda_K(s)}{s} ds < \infty. \tag{1.11}$$

Let $\Omega(0,\infty)$ denote the class of all nondecreasing continuous functions on $(0,\infty)$ satisfying conditions (1.8), (1.10), and (1.11).

A positive Borel measure μ on $\mathbb{D}$ is called a K-Carleson measure [3] if

$$\sup_I \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty, \tag{1.12}$$

where the supermum is taken over all subarcs $I \subset \partial\mathbb{D}$, $|I|$ is the length of I, and $S(I)$ is the Carleson box defined by

$$S(I) = \left\{ z : 1-|I| < |z| < 1,\ \frac{z}{|z|} \in I \right\}. \tag{1.13}$$

A positive Borel measure μ is called a vanishing K-Carleson measure if

$$\lim_{|I|\to 0} \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) = 0. \tag{1.14}$$

We also need the following results of Wulan and Zhu in [3], in which Q_K spaces are characterized in terms of K-Carleson measures.

Theorem 1.1. *Let $K \in \Omega(0,\infty)$. Then a positive Borel measure μ on $\mathbb{D}$ is a K-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right) d\mu(z) < \infty. \tag{1.15}$$

Also, μ is a vanishing K-Carleson measure if and only if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right) d\mu(z) = 0. \tag{1.16}$$

From Theorem 1.1 and the definition of the spaces Q_K and $Q_{K,0}$, we see that when $K \in \Omega(0,\infty)$, then $f \in Q_K$ if and only if the measure $d\mu_f = |f'(z)|^2 dm(z)$ is a K-Carleson measure, while $f \in Q_{K,0}$ if and only if this measure is a vanishing K-Carleson measure.

Let $\varphi \in S(\mathbb{D})$ be the family of all holomorphic self-maps of $\mathbb{D}$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}_0$. We define an integral-type operator as follows:

$$I_{\varphi,g}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}. \tag{1.17}$$

Operator (1.17) extends several operators which has been introduced and studied recently (see, e.g., [4–9]). For related operators in n-dimensional case, see, for example, [10–19]. For some classical operators see, for example, [20, 21] and the related references therein. For other product-type operators, see [22] and the references therein.

Motivated by [23, 24] (see also [25–29]), we characterize when φ and g induce bounded and/or compact operators in (1.17) from α-Bloch to Q_K spaces.

Throughout this paper, constants are denoted by C; they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Auxiliary Results

Here, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma is folklore (see, e.g., [30]).

Lemma 2.1. *For any $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, the following inequalities hold*

$$|f(z)| \le C \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & \text{if } 0 < \alpha < 1, \\ \|f\|_{\mathcal{B}^\alpha} \ln \dfrac{e}{1-|z|^2}, & \text{if } \alpha = 1, \\ \dfrac{\|f\|_{\mathcal{B}^\alpha}}{\left(1-|z|^2\right)^{\alpha-1}}, & \text{if } \alpha > 1, \end{cases} \tag{2.1}$$

$$\begin{aligned} \left|f^{(n)}(z)\right| &\le C \frac{\sup_{w \in D(z,(1-|z|)/2)} \left(1-|w|^2\right)^\alpha |f'(w)|}{\left(1-|z|^2\right)^{\alpha+n-1}} \\ &\le C \frac{\|f\|_{\mathcal{B}^\alpha}}{\left(1-|z|^2\right)^{\alpha+n-1}}, \quad \text{if } n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

The next lemma is obtained in [31, 32].

Lemma 2.2. *Let $\alpha > 0$. Then there are two functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that*

$$|f_1'(z)| + |f_2'(z)| \ge \frac{C}{\left(1-|z|^2\right)^\alpha}, \quad z \in \mathbb{D}. \tag{2.3}$$

Also, if $\alpha \neq 1$, then there are two functions $f_3, f_4 \in \mathcal{B}^\alpha$ and $C > 0$, such that

$$|f_3(z)| + |f_4(z)| \geq \frac{C}{\left(1-|z|^2\right)^{\alpha-1}}, \quad z \in \mathbb{D}. \tag{2.4}$$

The next Schwartz-type lemma [33] is proved in a standard way, so we omit the proof.

Lemma 2.3. *Let $\alpha > 0, K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}_0$. Then $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to Q_K$ is compact if and only if for any bounded sequence $(f_m)_{m\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ converging to zero on compacts of $\mathbb{D}$, we have $\lim_{m\to\infty}\|I_{\varphi,g}^{(n)} f_m\|_{Q_K} = 0$.*

Lemma 2.4. *Let $\alpha > 0$, $K \in \Omega(0,\infty), \varphi \in S(\mathbb{D}), g \in H(\mathbb{D})$, and $n \in \mathbb{N}_0$. Then $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ (or $Q_{K,0}$) is weakly compact if and only if it is compact.*

Proof. By a known theorem $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ (or $Q_{K,0}$) is weakly compact if and only if $(I_{\varphi,g}^{(n)})^* : Q_K^*$ (or $Q_{K,0}^*$) $\to (\mathcal{B}_0^\alpha)^*$ is weakly compact. Since $(\mathcal{B}_0^\alpha)^* \cong A^1$ (the Bergman space) and A^1 has the Schur property, it follows that it is equivalent to $(I_{\varphi,g}^{(n)})^* : Q_K^*$ (or $Q_{K,0}^*$) $\to (\mathcal{B}_0^\alpha)^*$, is compact, which is equivalent to $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ (or $Q_{K,0}$), is compact, as claimed. □

Lemma 2.5. *Let $\alpha > 0$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}_0$. Then $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is compact if and only if $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ is bounded.*

Proof. By Lemma 2.4, $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is compact if and only if it is weakly compact, which, by Gantmacher's theorem ([34]), is equivalent to $(I_{\varphi,g}^{(n)})^{**}((\mathcal{B}_0^\alpha)^{**}) \subseteq Q_{K,0}$. Since $(\mathcal{B}_0^\alpha)^{**} = \mathcal{B}^\alpha$ and by a standard duality argument $(I_{\varphi,g}^{(n)})^{**} = I_{\varphi,g}^{(n)}$ on $\mathcal{B}^\alpha$, this can be written as $I_{\varphi,g}^{(n)}(\mathcal{B}^\alpha) \subseteq Q_{K,0}$, which by the closed graph theorem is equivalent to $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ is bounded. □

For $a \in \mathbb{D}$, set

$$\Phi_{\varphi,g,K}(a) = \int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z). \tag{2.5}$$

Lemma 2.6. *Let $\alpha > 0, K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}_0$. If $\Phi_{\varphi,g,K}$ is finite at some point $a \in \mathbb{D}$, then it is continuous on $\mathbb{D}$.*

Proof. We follow the lines of Lemma 2.3 in [24]. From the elementary inequality

$$\frac{(1-|a|)(1-|a_1|)}{4} \leq \frac{1-|\eta_a(z)|^2}{1-|\eta_{a_1}(z)|^2} \leq \frac{4}{(1-|a|)(1-|a_1|)}, \quad a,\ a_1,\ z \in \mathbb{D}, \tag{2.6}$$

and since K is nondecreasing and satisfies (1.10), we easily get

$$K\left(1-|\eta_{a_1}(z)|^2\right) \leq C^{[\log_2(4/(1-|a|)(1-|a_1|))]+1} K\left(1-|\eta_a(z)|^2\right). \tag{2.7}$$

From (2.7) and since $\Phi_{\varphi,g,K}(a)$ is finite, it follows that $\Phi_{\varphi,g,K}$ is finite at each point $a_1 \in \mathbb{D}$. Let $a \in \mathbb{D}$ be fixed, and let $(a_l)_{l\in\mathbb{N}} \subset \mathbb{D}$ be a sequence converging to a.

We have

$$\left|\Phi_{\varphi,g,K}(a) - \Phi_{\varphi,g,K}(a_l)\right| \leq \int_{\mathbb{D}} \frac{|g(z)|^2 \left|K\left(1-|\eta_a(z)|^2\right) - K\left(1-|\eta_{a_l}(z)|^2\right)\right|}{\left(1-|\varphi(z)|^2\right)^{2(\alpha+n-1)}} dm(z). \tag{2.8}$$

From (2.6), we have that for l such that $1-|a_l| \geq (1-|a|)/2$, say $l \geq l_0$, holds

$$1-|\eta_{a_l}(z)|^2 \leq \frac{8}{(1-|a|)^2}\left(1-|\eta_a(z)|^2\right), \tag{2.9}$$

and consequently for $l \geq l_0$, it holds

$$\left|K\left(1-|\eta_a(z)|^2\right) - K\left(1-|\eta_{a_l}(z)|^2\right)\right| \leq \left(1+C^{[\log_2(8/(1-|a|)^2)]+1}\right)K\left(1-|\eta_a(z)|^2\right). \tag{2.10}$$

From this and since $\Phi_{\varphi,g,K}$ is finite at a, by the Lebesgue dominated convergence theorem, we get that the integral in (2.8) converges to zero as $l \to \infty$ which implies that $\Phi_{\varphi,g,K}(a_l) \to \Phi_{\varphi,g,K}(a)$ as $l \to \infty$, from which the lemma follows. □

3. Boundedness and Compactness of $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) $\to Q_K$ (or $Q_{K,0}$)

In this section, we characterize the boundedness and compactness of the operators $I_{\varphi,g}^{(n)}$: $\mathcal{B}^\alpha$(or $\mathcal{B}_0^\alpha$) $\to Q_K$(or $Q_{K,0}$). Let

$$d\mu_{\varphi,g,n,\alpha}(z) = |g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z). \tag{3.1}$$

Theorem 3.1. *Let $\alpha > 0$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Then the following statements are equivalent.*

(a) $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ *is bounded.*

(b) $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ *is bounded.*

(c) $M := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K(1-|\eta_a(z)|^2)|g(z)|^2(1-|\varphi(z)|^2)^{2(1-\alpha-n)} dm(z) < \infty$.

(d) $d\mu_{\varphi,g,n,\alpha}(z)$ *is a K-Carleson measure.*

Moreover, if $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ is bounded, then the next asymptotic relations hold

$$\left\|I_{\varphi,g}^{(n)}\right\|_{\mathcal{B}^\alpha\to Q_K} \asymp \left\|I_{\varphi,g}^{(n)}\right\|_{\mathcal{B}_0^\alpha\to Q_K} \asymp M^{1/2}. \tag{3.2}$$

Proof. By Theorem 1.1, it is clear that (c) and (d) are equivalent.

(c) ⇒ (a). Let $f \in B_{\mathcal{B}^\alpha}$. First note that $I_{\varphi,g}^{(n)} f(0) = 0$ for each $f \in H(\mathbb{B})$ and $n \in \mathbb{N}_0$. From this and by Lemma 2.1, we have

$$\begin{aligned}
\left\| I_{\varphi,g}^{(n)} f \right\|_{Q_K}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left| \left(I_{\varphi,g}^{(n)} f \right)'(z) \right|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left| f^{(n)}(\varphi(z)) \right|^2 |g(z)|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) \\
&\le C\|f\|_{\mathcal{B}^\alpha}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z),
\end{aligned} \tag{3.3}$$

from which the boundedness of $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ follows, and moreover

$$\left\| I_{\varphi,g}^{(n)} \right\|_{\mathcal{B}^\alpha \to Q_K} \le CM^{1/2}. \tag{3.4}$$

(a) ⇒ (b). This implication is obvious.

(b) ⇒ (c). By Lemma 2.2, if $n \in \mathbb{N}$, there are two functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that (2.3) holds, and if $n = 0$ and $\alpha > 1$ such that (2.4) holds. Let

$$h_1(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k, \qquad h_2(z) = f_2(z) - \sum_{k=1}^{n-1} \frac{f_2^{(k)}(0)}{k!} z^k. \tag{3.5}$$

It is known (see [30]) that for each $f \in H(\mathbb{D})$ and $n \in \mathbb{N}$, we have

$$\left(1 - |z|^2\right)^{\alpha+n-1} \left| f^{(n)}(z) \right| + \sum_{k=1}^{n-1} \left| f^{(k)}(0) \right| \asymp \left(1 - |z|^2\right)^{\alpha} \left| f'(z) \right|. \tag{3.6}$$

From this, Lemma 2.2, and since $h_1^{(k)}(0) = h_2^{(k)}(0) = 0$, $k = 0, 1, \ldots, n-1$, we have that there is a $\delta > 0$ such that

$$C\left(1 - |z|^2\right)^{-(\alpha+n-1)} \le \left| h_1^{(n)}(z) \right| + \left| h_2^{(n)}(z) \right|, \quad \text{for } |z| > \delta. \tag{3.7}$$

Now note that for any $f \in \mathcal{B}^\alpha$, the functions $f_r(z) = f(rz)$, $r \in (0,1)$ belong to $\mathcal{B}^\alpha$, and moreover, $\sup_{0<r<1} \|f_r\|_{\mathcal{B}^\alpha} \le \|f\|_{\mathcal{B}^\alpha}$.

Applying (3.7), using an elementary inequality, the boundedness of $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$, and the last inequality, we obtain

$$\begin{aligned}
&\int_{|r\varphi(z)|>\delta} r^{2n} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-(r|\varphi(z)|)^2\right)^{2(1-\alpha-n)} dm(z) \\
&\quad \le C\int_{\mathbb{D}} r^{2n} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(\left|h_1^{(n)}(r\varphi(z))\right|^2+\left|h_2^{(n)}(r\varphi(z))\right|^2\right) dm(z) \\
&\quad = C\int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right)\left|\left(I_{\varphi,g}^{(n)}(h_1)_r\right)'(z)\right|^2 dm(z) \\
&\qquad + C\int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right)\left|\left(I_{\varphi,g}^{(n)}(h_2)_r\right)'(z)\right|^2 dm(z) \\
&\quad \le \left\|I_{\varphi,g}^{(n)}\right\|^2_{\mathcal{B}_0^\alpha\to Q_K}\left(\|h_1\|^2_{\mathcal{B}^\alpha}+\|h_2\|^2_{\mathcal{B}^\alpha}\right).
\end{aligned} \tag{3.8}$$

Letting $r \to 1$ in (3.8) and using the monotone convergence theorem, we get

$$\int_{|\varphi(z)|>\delta} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z) \le C\left\|I_{\varphi,g}^{(n)}\right\|^2_{\mathcal{B}_0^\alpha\to Q_K}. \tag{3.9}$$

On the other hand, for $f_0(z) = z^n/n! \in \mathcal{B}_0^\alpha$, we get $I_{\varphi,g}^{(n)} f_0 \in Q_K$ which implies

$$\sup_{\alpha\in\mathbb{D}} \int_{|\varphi(z)|\le\delta} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z) \le \frac{\left\|I_{\varphi,g}^{(n)}\right\|^2_{\mathcal{B}_0^\alpha\to Q_K}\left\|f_0\right\|^2_{\mathcal{B}^\alpha}}{(1-\delta^2)^{2(\alpha+n-1)}}. \tag{3.10}$$

From (3.9) and (3.10), (c) follows. Moreover we get $M^{1/2} \le C\|I_{\varphi,g}^{(n)}\|_{\mathcal{B}_0^\alpha\to Q_K}$. From this, (3.4) and since $\|I_{\varphi,g}^{(n)}\|_{\mathcal{B}_0^\alpha\to Q_K} \le \|I_{\varphi,g}^{(n)}\|_{\mathcal{B}^\alpha\to Q_K}$ the asymptotic relations in (3.2) follow, finishing the proof of the theorem. □

Theorem 3.2. *Let $\alpha > 0$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Let $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ be bounded. Then the following statements are equivalent.*

(a) *$I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ is compact.*

(b) *$I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is compact.*

(c) *$I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is weakly compact.*

(d) *$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K(1-|\eta_a(z)|^2)|g(z)|^2 dm(z) < \infty$, and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z) = 0. \tag{3.11}$$

Proof. By Lemma 2.4, we have that (b) is equivalent to (c).

(d) $\Rightarrow$ (a). Let $(f_l)_{l\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$, say by L, converging to zero uniformly on compacts of $\mathbb{D}$. Then $f_l^{(n)}$ also converges to zero uniformly on compacts of $\mathbb{D}$. From (3.11) we have that for every $\varepsilon > 0$ there is an $r_1 \in (0,1)$ such that for $r \in (r_1, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z) < \varepsilon. \tag{3.12}$$

Therefore, by Lemma 2.1 and (3.12), we have that for $r \in (r_1, 1)$

$$\begin{aligned}\left\|I_{\varphi,g}^{(n)} f_l\right\|_{Q_K}^2 &= \left(\int_{|\varphi(z)|\le r} + \int_{|\varphi(z)|>r}\right)\left|f_l^{(n)}(\varphi(z))\right|^2 K\left(1-|\eta_a(z)|^2\right)|g(z)|^2 dm(z)\\ &< \sup_{|\varphi(z)|\le r}\left|f_l^{(n)}(\varphi(z))\right|^2\int_{\mathbb{D}} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2 dm(z) + CL^2\varepsilon.\end{aligned} \tag{3.13}$$

Letting $l \to \infty$ in (3.13), using the first condition in (d) and $\sup_{|w|\le r}|f_l^{(n)}(w)| \to 0$ as $l \to \infty$, it follows that $\lim_{l\to\infty}\|I_{\varphi,g}^{(n)} f_l\|_{Q_K} = 0$. Thus, by Lemma 2.3, $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ is compact.

(a) $\Rightarrow$ (b). The implication is trivial since $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(b) $\Rightarrow$ (d). By choosing $f(z) = z^n/n! \in \mathcal{B}_0^\alpha$, $n \in \mathbb{N}_0$, we have that the first condition in (d) holds. Let $f_l(z) = z^l/l$, $l \in \mathbb{N}$. It is easy to see that $(f_l)_{l\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}_0^\alpha$ converging to zero uniformly on compacts of $\mathbb{D}$. Hence, by Lemma 2.3, it follows that $\|I_{\varphi,g}^{(n)}(f_l)\|_{Q_K} \to 0$ as $l \to \infty$. Thus, for every $\varepsilon > 0$, there is an $l_0 \in \mathbb{N}$, $l_0 > n$ such that for $l \ge l_0$

$$\left(\prod_{j=1}^{n-1}(l-j)\right)^2 \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |\varphi(z)|^{2(l-n)} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2 dm(z) < \varepsilon. \tag{3.14}$$

From (3.14) we have that for each $r \in (0,1)$ and $l \ge l_0$

$$r^{2(l-n)}\left(\prod_{j=1}^{n-1}(l-j)\right)^2 \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2 dm(z) < \varepsilon. \tag{3.15}$$

Hence, for $r \in [(\prod_{j=1}^{n-1}(l_0-j))^{-1/(l_0-n)}, 1)$, we have that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} K\left(1-|\eta_a(z)|^2\right)|g(z)|^2 dm(z) < \varepsilon. \tag{3.16}$$

Let $f \in B_{\mathcal{B}_0^\alpha}$, and let $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1}\|f_t\|_{\mathcal{B}^\alpha} \le \|f\|_{\mathcal{B}^\alpha}$, $f_t \in \mathcal{B}_0^\alpha$, $t \in (0,1)$, and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ implies

$$\lim_{t\to 1}\left\| I_{\varphi,g}^{(n)} f_t - I_{\varphi,g}^{(n)} f \right\|_{Q_K} = 0. \tag{3.17}$$

Hence, for every $\varepsilon > 0$, there is a $t \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left| f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) < \varepsilon. \tag{3.18}$$

From this and (3.16), we have that for such t and each $r \in [(\prod_{j=1}^{n-1}(l_0 - j))^{-1/(l_0-n)}, 1)$

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) \\
&\quad \le 2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) \\
&\quad\quad + 2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f_t^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) \\
&\quad < 2\varepsilon\left(1 + \left\| f_t^{(n)} \right\|_\infty^2\right).
\end{aligned} \tag{3.19}$$

From (3.19) we conclude that for every $f \in B_{\mathcal{B}_0^\alpha}$, there is a $\delta_0 \in (0,1)$ and $\delta_0 = \delta_0(f,\varepsilon)$ such that for $r \in (\delta_0, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) < \varepsilon. \tag{3.20}$$

Since $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is compact, we have that for every $\varepsilon > 0$ there is a finite collection of functions $f_1, f_2, \ldots, f_k \in B_{\mathcal{B}_0^\alpha}$ such that, for each $f \in B_{\mathcal{B}_0^\alpha}$, there is a $j \in \{1, \ldots, k\}$, such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left| f^{(n)}(\varphi(z)) - f_j^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) < \varepsilon. \tag{3.21}$$

On the other hand, from (3.20), it follows that if $\widehat{\delta} := \max_{1\le j\le k}\delta_j(f_j, \varepsilon)$, then for $r \in (\widehat{\delta}, 1)$ and all $j \in \{1, \ldots, k\}$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f_j^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) < \varepsilon. \tag{3.22}$$

From (3.21) and (3.22), we have that for $r \in (\widehat{\delta}, 1)$ and every $f \in B_{\mathcal{B}_0^\alpha}$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \left| f^{(n)}(\varphi(z)) \right|^2 K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 dm(z) < 4\varepsilon. \tag{3.23}$$

If we apply (3.23) to the delays of the functions in (3.5) which are normalized so that they belong to $B_{\mathcal{B}^\alpha}$ and then use the monotone convergence theorem, we easily get that for $r > \max\{\delta, \widehat{\delta}\}$ where δ is chosen as in (3.7)

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z) < C\varepsilon, \tag{3.24}$$

from which (3.11) follows, as desired. □

Theorem 3.3. *Let $\alpha > 0, K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$ and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Then the next statements are equivalent.*

(a) $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ *is bounded.*

(b) $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ *is compact.*

(c) $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ *is compact.*

(d) $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ *is weakly compact.*

(e) $\lim_{|a|\to 1} \int_{\mathbb{D}} K(1 - |\eta_a(z)|^2)|g(z)|^2(1 - |\varphi(z)|^2)^{2(1-\alpha-n)} dm(z) = 0.$

(f) $d\mu_{\varphi,g,n,\alpha}(z)$ *is a vanishing K-Carleson measure.*

Proof. By Theorem 1.1, (e) and (f) are equivalent; by Lemma 2.4, (c) is equivalent to (d), while, by Lemma 2.5, (a) is equivalent to (c). Also (b) obviously implies (a).

(a) ⇒ (e) Let h_1 and h_2 be as in (3.5). Then from (3.7) and an elementary inequality, we get

$$\begin{aligned}\int_{|\varphi(z)|>\delta} &K\left(1 - |\eta_a(z)|^2\right)\left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} |g(z)|^2 dm(z) \\ &\le C\int_{\mathbb{D}} K\left(1 - |\eta_a(z)|^2\right)\left|\left(I_{\varphi,g}^{(n)} h_1\right)'(z)\right|^2 dm(z) \\ &\quad + C\int_{\mathbb{D}} K\left(1 - |\eta_a(z)|^2\right)\left|\left(I_{\varphi,g}^{(n)} h_2\right)'(z)\right|^2 dm(z).\end{aligned} \tag{3.25}$$

For $f_0(z) = z^n/n! \in \mathcal{B}^\alpha$, we get $I_{\varphi,g}^{(n)} f_0 \in Q_{K,0}$. From this and since $I_{\varphi,g}^{(n)}(h_j) \in Q_{K,0}$, $j = 1, 2$, by letting $|a| \to 1$, we get that (e) holds.

(e) ⇒ (b). We have that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ so that for $|a| > \delta$

$$\Phi_{\varphi,g,K}(a) < \varepsilon. \tag{3.26}$$

On the other hand, by Lemma 2.6, $\Phi_{\varphi,g,K}$ is continuous on $|a| \le \delta$, so uniformly bounded therein, which along with (3.26) gives the boundedness of $\Phi_{\varphi,g,K}$ on $\mathbb{D}$. Hence, by Theorem 3.1, $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_K$ is bounded. By Lemma 2.1, we have

$$\begin{aligned} &\lim_{|a|\to 1} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \int_{\mathbb{D}} \left| \left(I_{\varphi,g}^{(n)} f \right)'(z) \right|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) \\ &\quad \le C \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \|f\|_{\mathcal{B}^\alpha}^2 \lim_{|a|\to 1} \Phi_{\varphi,g,K}(a) = C \lim_{|a|\to 1} \Phi_{\varphi,g,\mathrm{K}}(a) = 0, \end{aligned} \tag{3.27}$$

so $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ is bounded.

Now assume that $(f_l)_{l\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha$, say by L, converging to zero uniformly on compacta of $\mathbb{D}$ as $l \to \infty$. To show that the operator $I_{\varphi,g}^{(n)} : \mathcal{B}^\alpha \to Q_{K,0}$ is compact, it is enough to prove that there is a subsequence $(f_{l_k})_{k\in\mathbb{N}}$ of $(f_l)_{l\in\mathbb{N}}$ such that $I_{\varphi,g}^{(n)} f_{l_k}$ converges in $Q_{K,0}$ as $k \to \infty$. By Lemma 2.1 and Montel's theorem, it follows that there is a subsequence, which we may denote again by $(f_l)_{l\in\mathbb{N}}$ converging uniformly on compacta of $\mathbb{D}$ to an $f \in \mathcal{B}^\alpha$, such that $\|f\|_{\mathcal{B}^\alpha} \le L$. Since $I_{\varphi,g}^{(n)}(\mathcal{B}^\alpha) \subseteq Q_{K,0}$, then clearly $I_{\varphi,g}^{(n)} f \in Q_{K,0}$. We show that

$$\lim_{l\to\infty} \left\| I_{\varphi,g}^{(n)} f_l - I_{\varphi,g}^{(n)} f \right\|_{Q_K} = 0. \tag{3.28}$$

From (3.26), Lemma 2.1, and some simple calculation, we obtain

$$\sup_{\delta<|a|<1} \int_{\mathbb{D}} \left| \left(I_{\varphi,g}^{(n)} f_l(z) - I_{\varphi,g}^{(n)} f(z) \right)' \right|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) < 4CL^2\varepsilon. \tag{3.29}$$

For $a \in \mathbb{D}$ and $t \in (0,1)$, let

$$\Psi_t(a) = \int_{\mathbb{D}\backslash t\mathbb{D}} K\left(1 - |\eta_a(z)|^2\right) |g(z)|^2 \left(1 - |\varphi(z)|^2\right)^{2(1-\alpha-n)} dm(z). \tag{3.30}$$

Lemma 2.6 essentially shows that Ψ_t is continuous on $\mathbb{D}$. Hence, for each $a \in \mathbb{D}$, there is a $t(a) \in (r,1)$ such that $\Psi_{t(a)}(a) < \varepsilon/2$. Moreover, there is a neighborhood $\mathcal{O}(a)$ of a such that, for every $b \in \mathcal{O}(a)$, $\Psi_{t(a)}(b) < \varepsilon$. From this and since the set $|a| \le \delta$ is compact, it follows that there is a $t_0 \in (0,1)$ such that $\Psi_{t_0}(a) < \varepsilon$ when $|a| \le \delta$. This along with Lemma 2.1 implies that

$$\begin{aligned} &\sup_{|a|\le\delta} \int_{\mathbb{D}\backslash t_0\mathbb{D}} \left| \left(I_{\varphi,g}^{(n)} f_l(z) - I_{\varphi,g}^{(n)} f(z) \right)' \right|^2 K\left(1 - |\eta_a(z)|^2\right) dm(z) \\ &\quad \le C\|f_l - f\|_{\mathcal{B}^\alpha}^2 \sup_{|a|\le\delta} \Psi_{t_0}(a) < 4CL^2\varepsilon. \end{aligned} \tag{3.31}$$

By the Weierstrass theorem $f_l^{(n)} \to f^{(n)}$ uniformly on compacta as $l \to \infty$, from which along with (2.2) and since $\varphi(t_0\mathbb{D})$ is compact, for $r = \sup_{w\in\varphi(t_0\mathbb{D})}|w|$, it follows that

$$\begin{aligned}
&\sup_{|a|\le\delta}\int_{t_0\mathbb{D}}\left|\left(I_{\varphi,g}^{(n)}f_l(z) - I_{\varphi,g}^{(n)}f(z)\right)'\right|^2 K\left(1-|\eta_a(z)|^2\right)dm(z)\\
&\quad\le C\sup_{|z|\le r}\left|(f_l-f)^{(n)}(z)\right|^2\sup_{|a|\le\delta}\Phi_{\varphi,g,K}(a)\longrightarrow 0,\quad \text{as } l\longrightarrow\infty.
\end{aligned}\tag{3.32}$$

From (3.29)–(3.32) and since $I_{\varphi,g}^{(n)}f(0) = 0$ for each $f \in H(\mathbb{D})$, we easily get (3.28), from which (b) follows, finishing the proof of this theorem. □

Theorem 3.4. *Let $\alpha > 0$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, $g \in H(\mathbb{D})$, and $n \in \mathbb{N}$, or $n = 0$ and $\alpha > 1$. Then the following statements are equivalent.*

(a) $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ *is bounded,*

(b) $\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g(z)|^2K(1-|\eta_a(z)|^2)(1-|\varphi(z)|^2)^{2(1-\alpha-n)}dm(z) < \infty$, *and*

$$\lim_{|a|\to 1}\int_{\mathbb{D}}|g(z)|^2K\left(1-|\eta_a(z)|^2\right)dm(z) = 0.\tag{3.33}$$

Proof. Suppose (b) holds and $f \in \mathcal{B}_0^\alpha$. Then by Theorem 3.1, $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is bounded. We show $I_{\varphi,g}^{(n)}f \in Q_{K,0}$, for every $f \in \mathcal{B}_0^\alpha$. Since $f \in \mathcal{B}_0^\alpha$, we have that, for every $\varepsilon > 0$, there is an $r \in (0,1)$ such that (see, e.g., the idea in [35, Lemma 2.4])

$$\left|f^{(n)}(\varphi(z))\right|^2\left(1-|\varphi(z)|^2\right)^{2(\alpha+n-1)} < \varepsilon \quad \text{for } |\varphi(z)| > r.\tag{3.34}$$

Thus,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left|\left(I_{\varphi,g}^{(n)}f(z)\right)'\right|^2K\left(1-|\eta_a(z)|^2\right)dm(z)\\
&\quad<\varepsilon\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}K\left(1-|\eta_a(z)|^2\right)\left(1-|\varphi(z)|^2\right)^{2(1-\alpha-n)}|g(z)|^2dm(z).
\end{aligned}\tag{3.35}$$

We also have

$$\begin{aligned}
&\lim_{|a|\to 1}\int_{|\varphi(z)|\le r}\left|\left(I_{\varphi,g}^{(n)}f(z)\right)'\right|^2K\left(1-|\eta_a(z)|^2\right)dm(z)\\
&\quad\le C\frac{\|f\|_{\mathcal{B}^\alpha}^2}{(1-r^2)^{2(\alpha+n-1)}}\lim_{|a|\to 1}\int_{|\varphi(z)|\le r}K\left(1-|\eta_a(z)|^2\right)|g(z)|^2dm(z)\\
&\quad\le C\frac{\|f\|_{\mathcal{B}^\alpha}^2}{(1-r^2)^{2(\alpha+n-1)}}\lim_{|a|\to 1}\int_{\mathbb{D}}K\left(1-|\eta_a(z)|^2\right)|g(z)|^2dm(z) = 0.
\end{aligned}\tag{3.36}$$

Combining (3.35) and (3.36), we get $I_{\varphi,g}^{(n)} f \in Q_{K,0}$. Hence, $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is bounded.

Conversely, if $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is bounded, then $I_{\varphi,g}^{(n)} : \mathcal{B}_0^\alpha \to Q_K$ is bounded too. Thus, by Theorem 3.1, we get the first condition in (b). For $f_0(z) = z^n/n! \in \mathcal{B}_0^\alpha$, we get $I_{\varphi,g}^{(n)} f_0 \in Q_{K,0}$, which is equivalent to (3.33), finishing the proof of the theorem. □

If $n = 0$, we simply denote the operator $I_{\varphi,g}^{(0)}$ by $I_{\varphi,g}$.

Theorem 3.5. *Let $\alpha \in (0,1)$, $K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(a) *$I_{\varphi,g} : \mathcal{B}^\alpha \to Q_K$ is bounded.*

(b) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_K$ is bounded.*

(c) *$M_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K(1 - |\eta_a(z)|^2)|g(z)|^2 dm(z) < \infty$.*

(d) *$d\mu_1(z) = |g(z)|^2 dm(z)$ is a K-Carleson measure.*

(e) *$I_{\varphi,g} : \mathcal{B}^\alpha \to Q_K$ is compact.*

(f) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_K$ is compact.*

(g) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_K$ is weakly compact.*

Moreover, if $I_{\varphi,g} : \mathcal{B}^\alpha \to Q_K$ is bounded, then the next asymptotic relations hold

$$\left\| I_{\varphi,g} \right\|_{\mathcal{B}^\alpha \to Q_K} \asymp \left\| I_{\varphi,g} \right\|_{\mathcal{B}_0^\alpha \to Q_K} \asymp M_1^{1/2}. \tag{3.37}$$

Proof. The proof of the equivalence of statements (a)–(d) of this theorem is similar to the proof of Theorem 3.1; moreover, the implication (b) ⇒ (c) is much simpler since it follows by using the test function $f_0(z) \equiv 1$. That (c) is equivalent to (e)–(g) is proved similarly as in Theorem 3.2, by using the well-known fact that if a bounded sequence $(f_l)_{l\in\mathbb{N}}$ in $\mathcal{B}^\alpha$, $\alpha \in (0,1)$ converges to zero uniformly on compacts of $\mathbb{D}$, then it converges to zero uniformly on the whole $\mathbb{D}$. The details are omitted. □

The proof of the next theorem is similar to the proofs of Theorems 3.3 and 3.4 and will be omitted.

Theorem 3.6. *Let $\alpha \in (0,1), K \in \Omega(0,\infty)$, $\varphi \in S(\mathbb{D})$, and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(a) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is bounded.*

(b) *$I_{\varphi,g} : \mathcal{B}^\alpha \to Q_{K,0}$ is bounded.*

(c) *$I_{\varphi,g} : \mathcal{B}^\alpha \to Q_{K,0}$ is compact.*

(d) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is compact.*

(e) *$I_{\varphi,g} : \mathcal{B}_0^\alpha \to Q_{K,0}$ is weakly compact.*

(f) $\lim_{|a|\to 1} \int_{\mathbb{D}} K(1 - |\eta_a(z)|^2)|g(z)|^2 dm(z) = 0$.

(g) *$d\mu_1(z) = |g(z)|^2 dm(z)$ is a vanishing K-Carleson measure.*

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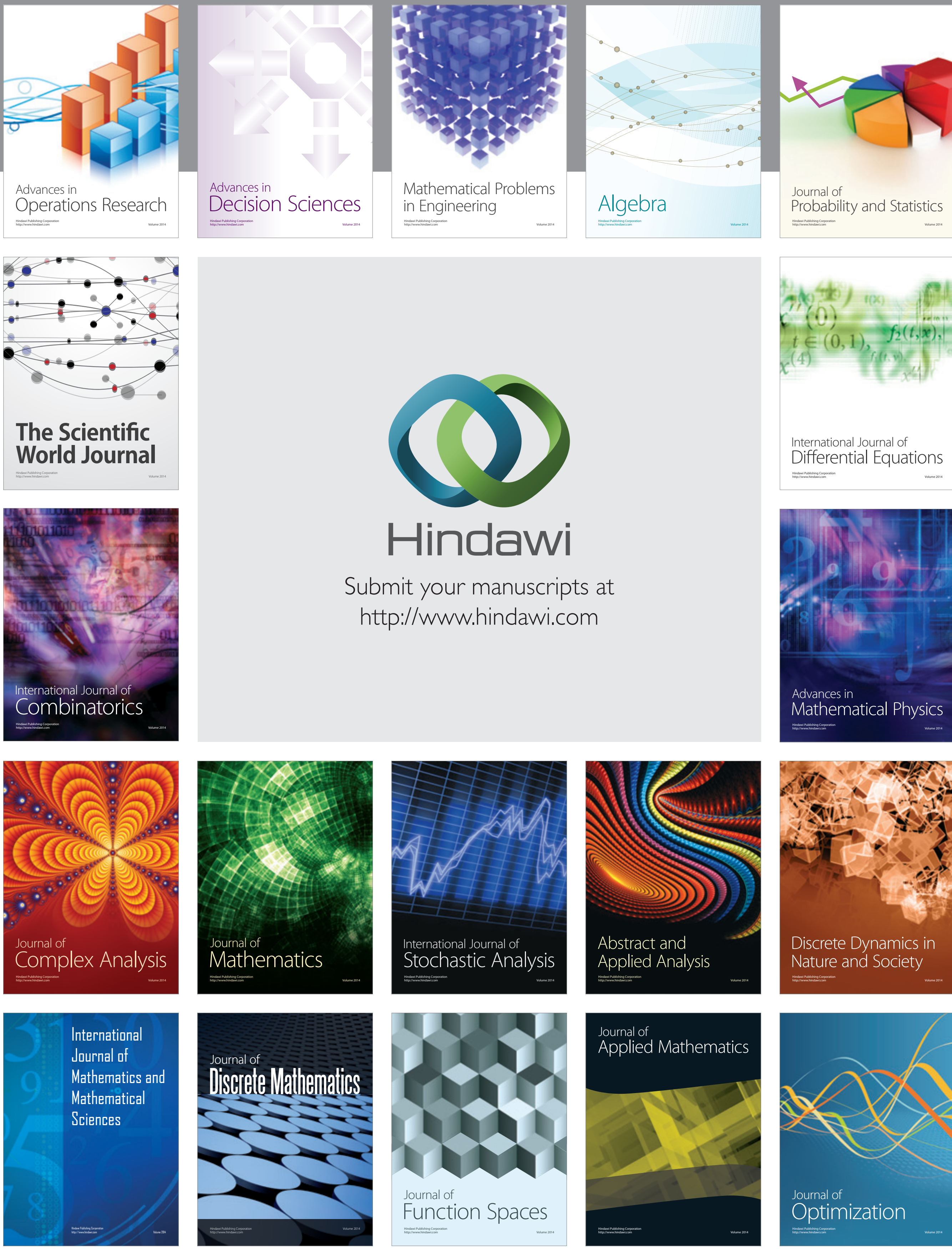
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