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Letter to the Editor

Comment on "Common Fixed Point Theorems for Commutating Mappings in Fuzzy Metric Spaces"

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In the recent paper "common fixed point theorems for commutating mappings in fuzzy metric spaces," the authors proved that a common fixed point theorem for commutating mappings in G-complete fuzzy metric spaces and gave an example to illustrate the main result. In this note, we point out that the above example is incorrect because it does not satisfy the condition of G-completeness, and then two appropriate examples are given. In addition, we prove that the theorem proposed by Zheng and Lian actually holds in an M-complete fuzzy metric space. Our results improve and extend some existing results in the relevant literature.

1. Introduction

In [1], Zheng and Lian extended Jungck's theorem in [2] to fuzzy metric spaces and obtained the following fixed point theorem for commutative mappings in fuzzy metric spaces in the sense of Kramosil and Michálek [3].

Theorem 1.1 (Zheng and Lian [1]). *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a continuous map and $g : X \to X$ a map. If*

(i) $g(X) \subseteq f(X)$,

(ii) *g commutes with f,*

(iii) *and $M(g(x), g(y), t) \geq M(f(x), f(y), \psi(t))$ for all $x, y \in X$ and $t > 0$, where $\psi : [0, +\infty) \to (0, +\infty)$ is an increasing and left-continuous function with $\psi(t) > t$ for all $t > 0$.*

Then f and g have a unique common fixed point.

Remark 1.2. It can be seen from the proof of Theorem 1.1 that the fuzzy metric space $(X, M, *)$ is complete in the sense of G-completeness.

Furthermore, the authors constructed the following example to illustrate the above theorem.

Example 1.3 (Zheng and Lian [1]). Let $X = [0, +\infty)$ be endowed with the usual metric $d(x, y) = |x - y|$. For all $x, y \in X$ and $t \geq 0$, define

$$M(x, y, t) = \begin{cases} 0, & t = 0, \\ e^{-|x-y|/t}, & t > 0. \end{cases} \tag{1.1}$$

Set $f(x) = A^{-1}(e^x - 1)$ and $g(x) = \ln(1 + Ax)$, $\psi(t) = A^{-2}t$, where $A \in (0, 1)$ is a constant.

In the Example 1.3, the authors claimed that $(X, M, *)$ is a complete fuzzy metric space in the sense of G-completeness (now known as a G-complete fuzzy metric space) with respect to t-norm $a * b = ab$, and then checked all the conditions of Theorem 1.1. Therefore, they concluded that f and g have the unique common fixed point 0.

However, we note that Example 1.3 is incorrect for Theorem 1.1, because $X = [0, +\infty)$ is not G-complete regarding the fuzzy metric M. For details, the reader can refer to [4–6].

In fact, G-completeness is a very strong kind of completeness. For instance, George and Veeramani [4] found that even $\mathbb{R}$ is not G-complete with respect to the standard fuzzy metric induced by Euclidean metric, and then proposed another kind of completeness (now known as an M-complete fuzzy metric space) by modifying the definition of Cauchy sequence. For these two types of completeness, it is easy to see that every G-complete fuzzy metric space is M-complete. Therefore, the construction of fixed point theorems in M-complete fuzzy metric spaces is more valuable and reasonable.

The main purpose of this note is to provide two appropriate examples for Theorem 1.1 and prove that this theorem does hold even if G-completeness of the fuzzy metric space is replaced by M-completeness. Our results not only improve and generalize Theorem 1.1, but also extend some main results of [2, 7].

2. Preliminaries

For completeness and clarity, in this section, some related concepts and conclusions are summarized below. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1 (Schweizer and Sklar [8]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *continuous triangular norm* (shortly, continuous t-norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative,

(TN-2) $*$ is continuous,

(TN-3) $a * 1 = a$ for every $a \in [0, 1]$,

(TN-4) and $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2 (Kramosil and Michálek [3]). The triple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) and $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left-continuous.

Remark 2.3. According to (FM-2) and (FM-4), it can easily be seen that $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$ (see Lemma 4 in [9]).

Similar to the case in [1], in this note, we suppose that $(X, M, *)$ is a fuzzy metric space with the following additional condition:

(FM-6) $\lim_{t \to +\infty} M(x, y, t) = 1$, for all $x, y \in X$.

Definition 2.4 (Grabiec [9], George and Veeramani [4]). Let $(X, M, *)$ be a fuzzy metric space. Then

(i) a sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$, denoted by $\lim_{n \to \infty} x_n = x$, if $\lim_{n \to \infty} M(x_n, x, t) = 1$, for any $t > 0$;

(ii) a sequence $\{x_n\}$ in X is called a *G-Cauchy sequence* if and only if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for any $t > 0$ and $p > 0$;

(iii) a sequence $\{x_n\}$ in X is called an *M-Cauchy sequence* if and only if for each $\epsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \epsilon$, for any $m, n \geq n_0$;

(iv) a fuzzy metric space $(X, M, *)$ is said to be *G-complete (M-complete)* if every G-Cauchy sequence (M-Cauchy sequence) is convergent;

(v) a map $f : X \to X$ is said to be *continuous* at $x_0 \in X$ if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converging to x_0.

The authors have proved the following conclusion (see the proof of Theorem 2.2 in [1]).

Lemma 2.5 (Zheng and Lian [1]). *Let $\psi : [0, +\infty) \to (0, +\infty)$ be an increasing and left-continuous function with $\psi(t) > t$ for all $t > 0$. Then*

$$\lim_{n \to +\infty} \psi^n(t) = +\infty, \tag{2.1}$$

for any $t > 0$, where $\psi^n(t)$ denotes the composition of $\psi(t)$ with itself n times.

3. Two Appropriate Examples

In this section, we will construct two appropriate examples for Theorem 1.1.

Example 3.1. Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ be equipped with the usual metric $d(x,y) = |x-y|$, $\varphi(t) = k \cdot t$, for all $t \in [0,+\infty)$, where $k \in (1,4]$ is a constant. For all $x,y \in X$ and $t \ge 0$, define

$$M(x,y,t) = \begin{cases} 0, & t = 0, \\ \dfrac{t}{t + |x-y|}, & t > 0. \end{cases} \tag{3.1}$$

Clearly, $(X, M, *)$ is a G-complete fuzzy metric space with regard to t-norm $a * b = ab$.

Set $f(x) = x/2$ and $g(x) = x/8$ for all $x \in X$. It is obvious that $g(X) \subseteq f(X)$. For any $x, y \in X$ and $t > 0$, we have

$$\begin{aligned} M(f(x), f(y), \varphi(t)) &= \frac{k \cdot t}{k \cdot t + |x-y|/2} \\ &= \frac{2k \cdot t}{2k \cdot t + |x-y|} \\ &\le \frac{8 \cdot t}{8 \cdot t + |x-y|} = M(g(x), g(y), t). \end{aligned} \tag{3.2}$$

Thus, all the conditions of Theorem 1.1 are satisfied and f and g have a unique fixed point, that is, $x = 0$.

Example 3.2. Let X be the subset of $\mathbb{R}^2$ defined by

$$X = \{A, B, C, D, E, F\}, \tag{3.3}$$

where $A = (0,0)$, $B = (1,0)$, $C = (2,0)$, $D = (0,2)$, $E = (1,1)$, and $F = (2,2)$. $\varphi(t) = \sqrt{2}t$, for all $t \in [0,+\infty)$. For all $x, y \in X$ and $t \ge 0$, define

$$M(x,y,t) = \begin{cases} 0, & t = 0, \\ e^{-d(x,y)/t}, & t > 0, \end{cases} \tag{3.4}$$

where $d(x,y)$ denotes the Euclidean distance of $\mathbb{R}^2$.

Clearly, $(X, M, *)$ is also a G-complete fuzzy metric space with regard to t-norm $a * b = ab$.

Let $f : X \to X$ and $g : X \to X$ be given by

$$\begin{aligned} &f(A) = f(B) = D, \quad f(C) = F, \quad f(D) = A, \quad f(E) = E, \quad f(F) = B, \\ &g(A) = g(B) = B, \quad g(C) = g(D) = g(E) = g(F) = E. \end{aligned} \tag{3.5}$$

Obviously, $g(X) \subseteq f(X)$. Furthermore, it is easy to see that $M(f(x), f(y), \varphi(t)) \le M(g(x), g(y), t)$ for any $x, y \in X$ and $t > 0$. Hence, all the conditions of Theorem 1.1 are satisfied and $x = E$ is the unique common fixed point of f and g.

4. Main Results

Now, we will prove that Theorem 1.1 does hold even if G-completeness is replaced by M-completeness.

Theorem 4.1. *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $f : X \to X$ be a continuous map and $g : X \to X$ a map. If*

(i) $g(X) \subseteq f(X)$,

(ii) *g commutes with f,*

(iii) *and $M(g(x), g(y), t) \geq M(f(x), f(y), \psi(t))$ for all $x, y \in X$ and $t > 0$, where $\psi : [0, +\infty) \to (0, +\infty)$ is an increasing and left-continuous function with $\psi(t) > t$ for all $t > 0$.*

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (i), we can find x_1 such that $f(x_1) = g(x_0)$. By induction, we can find a sequence $\{x_n\} \subseteq X$ such that $f(x_n) \subseteq g(x_{n-1})$. For any $t > 0$, we have

$$\begin{aligned} M(f(x_n), f(x_{n+1}), t) &= M(g(x_{n-1}), g(x_n), t) \\ &\geq M(f(x_{n-1}), f(x_n), \psi(t)) \\ &\geq \cdots \geq M(f(x_0), f(x_1), \psi^n(t)). \end{aligned} \tag{4.1}$$

As $\lim_{n\to+\infty}\psi^n(t) = +\infty$, for any $t > 0$, it follows by (FM-6) that $\lim_{n\to+\infty}M(f(x_0), f(x_1), \psi^n(t)) = 1$. Hence, $\lim_{n\to+\infty}M(f(x_n), f(x_{n+1}), t) = 1$, for any $t > 0$.

Next, we claim that $\{f(x_n)\}$ is an M-Cauchy sequence. Suppose that it is not. Then there exist $\epsilon \in (0, 1)$ and two sequences $\{p(n)\}$, $\{q(n)\}$ such that for every $n \in \mathbb{N}$ and $t > 0$, and then we can obtain that

$$p(n) > q(n) \geq n, \qquad M(f(x_{p(n)}), f(x_{q(n)}), t) \leq 1 - \epsilon. \tag{4.2}$$

Moreover, for every $n \in \mathbb{N}$, we can choose the two smallest numbers $p(n)$ and $q(n)$ such that

$$M(f(x_{p(n)-1}), f(x_{q(n)-1}), t) > 1 - \epsilon. \tag{4.3}$$

For every $n \in \mathbb{N}$, we can obtain

$$\begin{aligned} 1 - \epsilon &\geq M(f(x_{p(n)}), f(x_{q(n)}), t) \\ &= M(g(x_{p(n)-1}), g(x_{q(n)-1}), t) \\ &\geq M(f(x_{p(n)-1}), f(x_{q(n)-1}), \psi(t)) \\ &\geq M(f(x_{p(n)-1}), f(x_{q(n)-1}), t) > 1 - \epsilon. \end{aligned} \tag{4.4}$$

Clearly, this leads to a contradiction.

Here, we also consider another particular case. That is, for each $t > 0$, there exist $\epsilon \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $M(f(x_m), f(x_n), t) \le 1 - \epsilon$, for all $m, n \ge n_0$. Then, for any $p \in \mathbb{N}$, we know that $M(f(x_{n_0+p+2}), f(x_{n_0+p+1}), t) \le 1 - \epsilon$. Since

$$\begin{aligned} M(f(x_{n_0+p+2}), f(x_{n_0+p+1}), t) &= M(g(x_{n_0+p+1}), g(x_{n_0+p}), t) \\ &\ge M(f(x_{n_0+p+1}), f(x_{n_0+p}), \psi(t)) \\ &\ge M(f(x_{n_0+p+1}), f(x_{n_0+p}), t), \end{aligned} \tag{4.5}$$

we can conclude that $\{M(f(x_{n_0+p+2}), f(x_{n_0+p+1}), t)\}$ is a monotone and bounded sequence with respect to p. Therefore, there exists $\gamma \in [0, 1-\epsilon]$ such that $\lim_{p\to+\infty} M(f(x_{n_0+p+2}), f(x_{n_0+p+1}), t) = \gamma$. In addition, according to the foregoing inequality, we can obtain

$$M(f(x_{n_0+p+2}), f(x_{n_0+p+1}), t) \ge M\Big(f(x_{n_0+1}), f(x_{n_0}), \psi^{p+1}(t)\Big). \tag{4.6}$$

By supposing that $p \to +\infty$, it follows that $\gamma \ge 1$, which is also a contradiction.

Hence, $\{f(x_n)\}$ is an M-Cauchy sequence in the M-complete fuzzy metric space X. Furthermore, we conclude that there exists a point $y \in X$ such that $\lim_{n\to+\infty} f(x_n) = y$. So $\lim_{n\to+\infty} g(x_n) = \lim_{n\to+\infty} f(x_{n+1}) = y$.

By (iii), it can be seen that the continuity of f implies that of g. Consequently, we obtain that $\lim_{n\to+\infty} g(f(x_n)) = g(y)$. According to the commutativity of f and g, we know that $\lim_{n\to+\infty} f(g(x_n)) = \lim_{n\to+\infty} g(f(x_n)) = g(y)$. Because of the uniqueness of limits, it follows immediately that $f(y) = g(y)$. So $f(f(y)) = f(g(y)) = g(f(y)) = g(g(y))$. Thus, we have

$$\begin{aligned} M(g(y), g(g(y)), t) &\ge M(f(y), f(g(y)), \psi(t)) \\ &= M(g(y), g(g(y)), \psi(t)) \\ &\ge \cdots \ge M(g(y), g(g(y)), \psi^n(t)). \end{aligned} \tag{4.7}$$

Letting $n \to +\infty$, we obtain that $\lim_{n\to+\infty} M(g(y), g(g(y)), \psi^n(t)) = 1$, for any $t > 0$. So $M(g(y), g(g(y)), t) = 1$. By (FM-2), we conclude that $g(g(y)) = g(y)$. Hence, $g(y) = g(g(y)) = f(g(y))$, that is, $g(y)$ is a common fixed point of f and g.

Furthermore, we show that $g(y)$ is the unique common fixed point of f and g. Assume that x and z are two common fixed point of f and g, for any $t > 0$, we then obtain

$$\begin{aligned} M(x, z, t) = M(g(x), g(z), t) &\ge M(f(x), f(z), \psi(t)) \\ &= M(x, z, \psi(t)) \ge \cdots \ge M(x, z, \psi^n(t)). \end{aligned} \tag{4.8}$$

As $n \to +\infty$, we have $\lim_{n\to+\infty} M(x, z, \psi^n(t)) = 1$. Thus $M(x, z, t) = 1$ for any $t > 0$. Furthermore, we can obtain $x = z$. This completes the proof. □

Remark 4.2. It should be pointed out that the foregoing three examples are suitable for Theorem 4.1.

Corollary 4.3. *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $f : X \to X$ be a continuous map and $g : X \to X$ a map. If*

(i) $g(X) \subseteq f(X)$,

(ii) g *commutes with* f,

(iii) *and* $M(g(x), g(y), kt) \geq M(f(x), f(y), t)$ *for all* $x, y \in X$ *and* $t > 0$, *where* $0 < k < 1$.

Then f *and* g *have a unique common fixed point.*

Remark 4.4. Corollary 4.3 is the immediate consequence of Theorem 4.1, which can be regarded as an improvement of Theorem 2 in [7].

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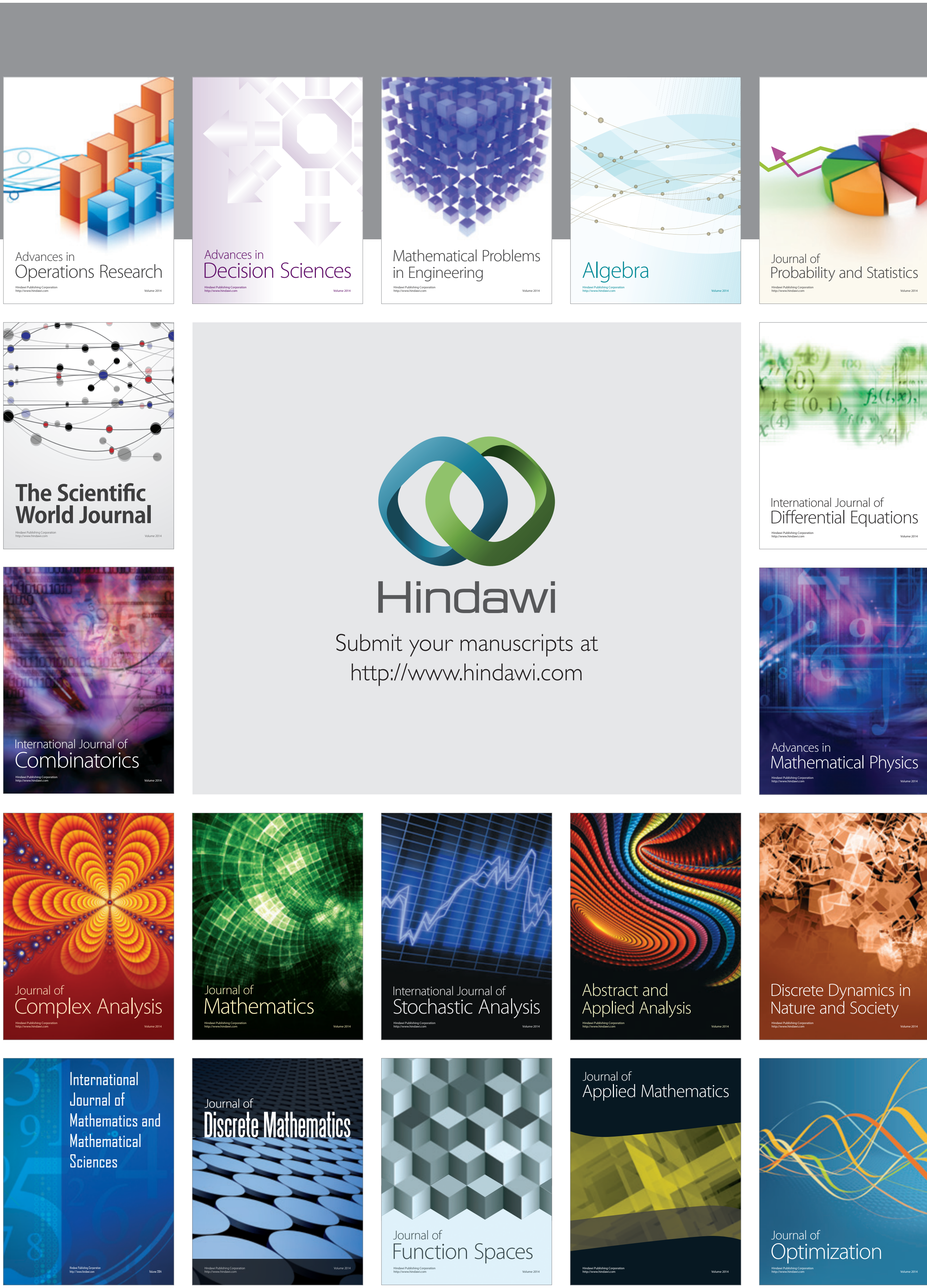
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