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Research Article

On Certain Classes of Biharmonic Mappings Defined by Convolution

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We introduce a class of complex-valued biharmonic mappings, denoted by $BH^0(\phi_k;\sigma,a,b)$, together with its subclass $TBH^0(\phi_k;\sigma,a,b)$, and then generalize the discussions in Ali et al. (2010) to the setting of $BH^0(\phi_k;\sigma,a,b)$ and $TBH^0(\phi_k;\sigma,a,b)$ in a unified way.

1. Introduction

A four times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subset \mathbb{C}$ is *biharmonic* if ΔF, the Laplacian of F, is harmonic in D. Note that ΔF is *harmonic* in D if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$ in D, where Δ represents the Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z\partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.1}$$

It is known that, when D is simply connected, a mapping F is biharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{2} |z|^{2(k-1)} G_k(z), \tag{1.2}$$

where G_k are complex-valued harmonic mappings in D for $k \in \{1,2\}$ (cf. [1–6]). Also it is known that G_k can be expressed as the form

$$G_k = h_k + \overline{g_k} \tag{1.3}$$

for $k \in \{1,2\}$, where all h_k and g_k are analytic in D (cf. [7, 8]).

Biharmonic mappings arise in a lot of physical situations, particularly, in fluid dynamics and elasticity problems, and have many important applications in engineering and biology (cf. [9–11]). However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (cf. [1–6]).

In this paper, we consider the biharmonic mappings in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $BH^0(\mathbb{D})$ denote the set of all biharmonic mappings F in $\mathbb{D}$ with the following form:

$$\begin{aligned} F(z) &= \sum_{k=1}^{2} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right) \\ &= \sum_{k=1}^{2} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{k,j} z^j + \sum_{j=1}^{\infty} \overline{b_{k,j} z^j} \right), \end{aligned} \tag{1.4}$$

with $a_{1,1} = 1$, $a_{2,1} = 0$, $b_{1,1} = 0$, and $b_{2,1} = 0$.

In [12], Qiao and Wang proved that for each $F \in BH^0(\mathbb{D})$, if the coefficients of F satisfy the following inequality:

$$\sum_{k=1}^{2} \sum_{j=1}^{\infty} (2(k-1) + j) \left(|a_{k,j}| + |b_{k,j}| \right) \le 2, \tag{1.5}$$

then F is sense preserving, univalent, and starlike in $\mathbb{D}$ (see [12, Theorems 3.1 and 3.2]).

Let S_H denote the set of all univalent harmonic mappings f in $\mathbb{D}$, where

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j}, \tag{1.6}$$

with $|b_1| < 1$. In particular, we use S_H^0 to denote the set of all mappings in S_H with $b_1 = 0$. Obviously, $S_H^0 \subset BH^0(\mathbb{D})$.

In 1984, Clunie and Sheil-Small [7] discussed the class S_H and its geometric subclasses. Since then, there have been many related papers on S_H and its subclasses (see [13, 14] and the references therein). In 1999, Jahangiri [15] studied the class $S_H^*(\alpha)$ consisting of all mappings $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{j=2}^{\infty} |a_j| z^j, \qquad g(z) = \sum_{j=1}^{\infty} |b_j| z^j \tag{1.7}$$

and satisfy the condition

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) = \operatorname{Re}\left\{\frac{zh' - \overline{zg'}}{h+\overline{g}}\right\} > \alpha \tag{1.8}$$

in $\mathbb{D}$, where $0 \le \alpha < 1$.

For two analytic functions f_1 and f_2, if

$$f_1(z) = \sum_{j=1}^{\infty} a_j z^j, \qquad f_2(z) = \sum_{j=1}^{\infty} A_j z^j, \tag{1.9}$$

then the *convolution* of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = \sum_{j=1}^{\infty} a_j A_j z^j. \tag{1.10}$$

By using the convolution, in [16], Ali et al. introduced the class $S_H^0(\phi,\sigma,\alpha)$ of harmonic mappings in the form of (1.6) such that

$$\operatorname{Re}\left\{\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}}\right\} > \alpha \tag{1.11}$$

and the class $SP_H^0(\phi,\sigma,\alpha)$ such that

$$\operatorname{Re}\left\{\left(1+e^{i\gamma}\right)\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}} - e^{i\gamma}\right\} > \alpha, \tag{1.12}$$

where $\sigma \in \mathbb{R}$ and $\alpha \in [0,1)$ are constants, $\gamma \in \mathbb{R}$ and $\phi(z) = z + \sum_{n=2}^{\infty}\phi_n z^n$ is analytic in $\mathbb{D}$.

Now we consider a class of biharmonic mappings, denoted by $BH^0(\phi_k;\sigma,a,b)$, as follows: $F \in BH^0(\mathbb{D})$ with the form (1.4) is said to be in $BH^0(\phi_k;\sigma,a,b)$ if and only if

$$\operatorname{Re}\left\{a\frac{\Phi(z)}{\Psi(z)} - b\right\} > 0, \tag{1.13}$$

where

$$\begin{aligned}\Phi(z) &= z\left[\left(\sum_{k=1}^{2}|z|^{2(k-1)}(h_k*\phi_k)(z)\right)' + \sigma\left(\sum_{k=1}^{2}|z|^{2(k-1)}\overline{(g_k*\phi_k)(z)}\right)'\right],\\ \Psi(z) &= z'\sum_{k=1}^{2}|z|^{2(k-1)}\left((h_k*\phi_k)(z)+\sigma\overline{(g_k*\phi_k)(z)}\right),\end{aligned} \tag{1.14}$$

$\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ are analytic in $\mathbb{D}$ for $k \in \{1,2\}$, $\sigma \in \mathbb{R}$ is a constant, $a = p + \rho e^{i\gamma}$, $b = q + \rho e^{i\gamma}$, p, q, $\rho \in [0,+\infty)$ are constants with $a - b > 0$, $\gamma \in \mathbb{R}$, and $z = re^{i\theta}$. Here and in what follows, "$'$" always stands for "$\partial/\partial\theta$".

Obviously, if $\phi_2 = 0$, $a = 1$ and $b = \alpha$, then $BH^0(\phi_k; \sigma, a, b)$ reduces to $S_H^0(\phi, \sigma, \alpha)$, and if $\phi_2 = 0$, $a = 1 + e^{i\gamma}$ and $b = \alpha + e^{i\gamma}$, then $BH^0(\phi_k; \sigma, a, b)$ reduces to $SP_H^0(\phi, \sigma, \alpha)$.

Further, we use $TBH^0(\phi_k; \sigma, a, b)$ to denote the class consisting of all mappings F in $BH^0(\phi_k; \sigma, a, b)$ with the form

$$F(z) = \sum_{k=1}^{2} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right), \tag{1.15}$$

where

$$\begin{aligned} h_k(z) &= a_{k,1} z - \sum_{j=2}^{\infty} a_{k,j} z^j, \quad a_{k,j} \geq 0,\ a_{1,1} = 1,\ a_{2,1} = 0, \\ g_k(z) &= \sigma \sum_{j=1}^{\infty} b_{k,j} z^j, \quad b_{k,j} \geq 0,\ b_{1,1} = b_{2,1} = 0. \end{aligned} \tag{1.16}$$

The object of this paper is to generalize the discussions in [16] to the setting of $BH^0(\phi_k; \sigma, a, b)$ and $TBH^0(\phi_k; \sigma, a, b)$ in a unified way. The organization of this paper is as follows. In Section 2, we get a convolution characterization for $BH^0(\phi_k; \sigma, a, b)$. As a corollary, we derive a sufficient coefficient condition for mappings in $BH^0(\mathbb{D})$ to belong to $BH^0(\phi_k; \sigma, a, b)$. The main results are Theorems 2.1 and 2.3. In Section 3, first, we get a coefficient characterization for $TBH^0(\phi_k; \sigma, a, b)$, and then find the extreme points of $TBH^0(\phi_k; \sigma, a, b)$. The corresponding results are Theorems 3.1 and 3.6.

2. A Convolution Characterization

We begin with a convolution characterization for $BH^0(\phi_k; \sigma, a, b)$.

Theorem 2.1. *Let $F \in BH^0(\mathbb{D})$. Then $F \in BH^0(\phi_k; \sigma, a, b)$ if and only if*

$$\begin{aligned} &\sum_{k=1}^{2} |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{z + ((ax - a + 2b)/(2a - 2b)) z^2}{(1-z)^2} \right) \\ &\quad - \sigma \sum_{k=1}^{2} |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{((ax + b)/(a - b))\overline{z} - ((ax - a + 2b)/(2a - 2b))\overline{z}^2}{(1 - \overline{z})^2} \right) \neq 0, \end{aligned} \tag{2.1}$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$.

Proof. By definition, a necessary and sufficient condition for a mapping F in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k;\sigma,a,b)$ is given by (1.13). Let

$$G(z)=\frac{1}{a-b}\left(a\frac{\Phi(z)}{\Psi(z)}-b\right). \tag{2.2}$$

Then $G(0)=1$, and so the condition (1.13) is equivalent to

$$G(z)\neq\frac{x-1}{x+1}, \tag{2.3}$$

for all $z\in\mathbb{D}\setminus\{0\}$ and all $x\in\mathbb{C}$ with $|x|=1$ and $x\neq-1$. Obviously, (2.3) holds if and only if

$$a(x+1)\Phi(z)-b(x+1)\Psi(z)-(a-b)(x-1)\Psi(z)\neq 0. \tag{2.4}$$

Straightforward computations show that

$$\begin{aligned}
&a(x+1)\Phi(z)-b(x+1)\Psi(z)-(a-b)(x-1)\Psi(z)\\
&=a(x+1)z'\sum_{k=1}^{2}|z|^{2(k-1)}\left(z+\sum_{j=2}^{\infty}ja_{k,j}\phi_{k,j}z^j-\sigma\sum_{j=2}^{\infty}j\overline{b_{k,j}\phi_{k,j}z^j}\right)\\
&\quad-(ax-a+2b)z'\sum_{k=1}^{2}|z|^{2(k-1)}\left(z+\sum_{j=2}^{\infty}a_{k,j}\phi_{k,j}z^j+\sigma\sum_{j=2}^{\infty}\overline{b_{k,j}\phi_{k,j}z^j}\right)\\
&=z'\sum_{k=1}^{2}|z|^{2(k-1)}(h_k*\phi_k)(z)*\left(\frac{2(a-b)z+(ax-a+2b)z^2}{(1-z)^2}\right)\\
&\quad-\sigma z'\sum_{k=1}^{2}|z|^{2(k-1)}\overline{(g_k*\phi_k)(z)}*\left(\frac{2(ax+b)\overline{z}-(ax-a+2b)\overline{z}^2}{(1-\overline{z})^2}\right),
\end{aligned} \tag{2.5}$$

from which we see that (2.3) is true if and only if so is (2.1). The proof is complete. □

Remark 2.2. If $h_2=g_2=0$, $a=1$ and $b=\alpha$, then Theorem 2.1 coincides with Theorem 2.1 in [16], and if $h_2=g_2=0$, $a=1+e^{i\gamma}$, and $b=\alpha+e^{i\gamma}$, then Theorem 2.1 coincides with Theorem 2.3 in [16].

As an application of Theorem 2.1, we derive a sufficient condition for mappings in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k;\sigma,a,b)$ in terms of their coefficients.

Theorem 2.3. *Let $F\in BH^0(\mathbb{D})$. Then $F\in BH^0(\phi_k;\sigma,a,b)$ if*

$$\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}-\|b\|_{\max}}{a-b}\left|\phi_{k,j}a_{k,j}\right|+|\sigma|\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}+\|b\|_{\max}}{a-b}\left|\phi_{k,j}b_{k,j}\right|\leq 1, \tag{2.6}$$

here and in the following, $\|z\|_{\max} = \max_{\gamma\in R}\{|x+ye^{i\gamma}|\} = x+y$, *where* $z = x+ye^{i\gamma}$, x *and* $y \in [0,+\infty)$ *are constants.*

Proof. For F given by (1.4), we see that

$$\begin{aligned}
L(z) \triangleq & \left| \sum_{k=1}^{2} |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{z + ((ax-a+2b)/(2a-2b))z^2}{(1-z)^2} \right) \right. \\
& \left. -\sigma \sum_{k=1}^{2} |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{((ax+b)/(a-b))\overline{z} - ((ax-a+2b)/(2a-2b))\overline{z}^2}{(1-\overline{z})^2} \right) \right| \\
= & \left| z + \sum_{k=1}^{2} |z|^{2(k-1)} \sum_{j=2}^{\infty} \left(j + (j-1)\frac{ax-a+2b}{2a-2b} \right) \phi_{k,j} a_{k,j} z^j \right. \\
& \left. -\sigma \sum_{k=1}^{2} |z|^{2(k-1)} \sum_{j=2}^{\infty} \left(j\frac{ax+b}{a-b} - (j-1)\frac{ax-a+2b}{2a-2b} \right) \overline{\phi_{k,j} b_{k,j} z^j} \right|.
\end{aligned} \tag{2.7}$$

If F is the identity, obviously, $L(z) = |z|$.

If F is not the identity, then

$$L(z) > |z| \left(1 - \sum_{k=1}^{2}\sum_{j=2}^{\infty} \frac{j\|a\|_{\max} - \|b\|_{\max}}{a-b} \left|\phi_{k,j} a_{k,j}\right| - |\sigma| \sum_{k=1}^{2}\sum_{j=2}^{\infty} \frac{j\|a\|_{\max} + \|b\|_{\max}}{a-b} \left|\phi_{k,j} b_{k,j}\right| \right). \tag{2.8}$$

Hence the assumption implies that $L(z) > 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$. It follows from Theorem 2.1 that $F \in BH^0(\phi_k; \sigma, a, b)$. □

Remark 2.4. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 2.3 coincides with Theorem 2.2 in [16], and if $h_2 = g_2 = 0$, $a = 1 + e^{i\gamma}$ and $b = \alpha + e^{i\gamma}$, then Theorem 2.3 coincides with Theorem 2.4 in [16].

3. A Coefficient Characterization and Extreme Points

We start with a coefficient characterization for $TBH^0(\phi_k; \sigma, a, b)$.

Theorem 3.1. *Let* $\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ *with* $\phi_{k,j} \geq 0$, *and let* F *be of the form* (1.15). *Then* $F \in TBH^0(\phi_k; \sigma, a, b)$ *if and only if*

$$\sum_{k=1}^{2}\sum_{j=2}^{\infty} \frac{j\|a\|_{\max} - \|b\|_{\max}}{a-b} \phi_{k,j} a_{k,j} + \sigma^2 \sum_{k=1}^{2}\sum_{j=2}^{\infty} \frac{j\|a\|_{\max} + \|b\|_{\max}}{a-b} \phi_{k,j} b_{k,j} \leq 1. \tag{3.1}$$

Proof. By similar arguments as in the proof of Theorem 2.3, we see that it suffices to prove the "only if" part. For $F \in TBH^0(\phi_k;\sigma,a,b)$, obviously, (1.13) is equivalent to

$$\operatorname{Re}\left\{\frac{P(z)-Q(z)}{z-\sum_{k=1}^{2}|z|^{2(k-1)}\left(\sum_{j=2}^{\infty}a_{k,j}\phi_{k,j}z^j-\sigma^2\sum_{j=2}^{\infty}b_{k,j}\phi_{k,j}\overline{z}^j\right)}\right\}>0 \tag{3.2}$$

in $\mathbb{D}$, where

$$\begin{aligned} P(z) &= (a-b)z-\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}(aj-b)a_{k,j}\phi_{k,j}z^j, \\ Q(z) &= \sigma^2\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}(aj+b)b_{k,j}\phi_{k,j}\overline{z}^j. \end{aligned} \tag{3.3}$$

Letting $z \to 1^-$ through real values leads to the desired inequality. So the proof is complete. □

Remark 3.2. If $h_2 = g_2 = 0$, $a = 1$, and $b = \alpha$, then Theorem 3.1 coincides with Theorem 3.1 in [16].

It follows from Theorem 3.1 that we have the following.

Corollary 3.3. *Let $\phi_k(z) = z + \sum_{j=2}^{\infty}\phi_{k,j}z^j$ with $\phi_{k,j} \geq \phi_{1,2} > 0$ $(k \in \{1,2\},\ j \geq 2)$ and $|\sigma| \geq (2\|a\|_{\max} - \|b\|_{\max})/(2\|a\|_{\max} + \|b\|_{\max})$. If $F \in TBH^0(\phi_k;\sigma,a,b)$, then for $|z| = r < 1$, one has*

$$r-\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}r^2 \leq |F(z)| \leq r+\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}r^2. \tag{3.4}$$

The result is sharp with equality for mappings

$$F(z) = z-\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}z^2. \tag{3.5}$$

Theorem 3.1 and Corollary 3.3 imply the following

Corollary 3.4. *Under the hypotheses of Corollary 3.3, one has that $TBH^0(\phi_k;\sigma,a,b)$ is closed under the convex combination.*

Definition 3.5. Let X be a topological vector space over the field of complex numbers, and let E be a subset of X. A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1-t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in E (cf. [17]).

We now determine the extreme points of $TBH^0(\phi_k;\sigma,a,b)$.

Theorem 3.6. *Let*

(1) $h_{11}(z) = z$,

(2) $h_{21}(z) = g_{11}(z) = g_{21}(z) = 0$,

(3) $h_{kj}(z) = z - |z|^{2(k-1)}((a-b)/(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j})z^j$ *for* $k \in \{1,2\}$ *and all* $j \geq 2$,

(4) $g_{kj}(z) = z + |z|^{2(k-1)}((a-b)/\sigma(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j})\overline{z}^j$ *for* $k \in \{1,2\}$ *and all* $j \geq 2$.

Under the hypotheses of Corollary 3.3*, one has that* $F \in TBH^0(\phi_k; \sigma, a, b)$ *if and only if it can be expressed as*

$$F(z) = \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)\right), \tag{3.6}$$

where $x_{21} = y_{11} = y_{21} = 0$*, all other* x_{kj} *and* y_{kj} *are nonnegative, and* $\sum_{k=1}^{2}\sum_{j=1}^{\infty}(x_{kj} + y_{kj}) = 1$.

In particular, the extreme points of $TBH^0(\phi_k; \sigma, a, b)$ *are all mappings* h_{kj} *and* g_{kj} *listed in* (1), (3), *and* (4) *above.*

Proof. It follows from the assumptions that

$$\begin{aligned}
F(z) &= \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)\right) \\
&= z - \sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}\frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}}x_{kj}z^j \\
&\quad + \sigma\sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}\frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}}y_{kj}\overline{z}^j,
\end{aligned} \tag{3.7}$$

whence

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{(j\|a\|_{\max} - \|b\|_{\max})}{a-b}\phi_{k,j}\cdot\frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}}x_{kj} \\
&\quad + \sigma^2\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{(j\|a\|_{\max} + \|b\|_{\max})}{a-b}\phi_{k,j}\cdot\frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}}y_{kj} \\
&= \sum_{k=1}^{2}\sum_{j=2}^{\infty}x_{kj} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}y_{kj} \\
&\leq 1,
\end{aligned} \tag{3.8}$$

and so Theorem 3.1 implies that $F \in TBH^0(\phi_k; \sigma, a, b)$.

Conversely, assume $F \in TBH^0(\phi_k; \sigma, a, b)$, and let

$$x_{21} = y_{11} = y_{21} = 0, \qquad x_{11} = 1 - \sum_{k=1}^{2}\sum_{j=2}^{\infty} x_{kj} - \sum_{k=1}^{2}\sum_{j=2}^{\infty} y_{kj},$$
$$x_{kj} = \frac{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}a_{k,j}}{a-b}, \tag{3.9}$$
$$y_{kj} = \frac{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}b_{k,j}}{a-b},$$

for $k \in \{1,2\}$ and all $j \geq 2$. Then

$$F(z) = z - \sum_{k=1}^{2} |z|^{2(k-1)} \sum_{j=2}^{\infty} a_{k,j} z^j + \sigma \sum_{k=1}^{2} |z|^{2(k-1)} \sum_{j=2}^{\infty} b_{k,j} \overline{z}^j. \tag{3.10}$$

The proof of the theorem is complete. □

Remark 3.7. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 3.6 coincides with Theorem 3.2 in [16].

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