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Research Article

Sharp Bounds for Seiffert Mean in Terms of Contraharmonic Mean

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We find the greatest value α and the least value β in $(1/2, 1)$ such that the double inequality $C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a)$ holds for all $a, b > 0$ with $a \neq b$. Here, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ and $C(a,b) = (a^2+b^2)/(a+b)$ are the Seiffert and contraharmonic means of a and b, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Seiffert mean $T(a,b)$ and contraharmonic mean $C(a,b)$ are defined by

$$T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \tag{1.1}$$

$$C(a,b) = \frac{a^2+b^2}{a+b}, \tag{1.2}$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for these means can be found in the literature [1–12].

Let $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, $S(a,b) = \sqrt{(a^2+b^2)/2}$, and let $M_p(a,b) = ((a^p + b^p)/2)^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ be the arithmetic, geometric, square root, and pth power means of two positive numbers a and b, respectively. Then it is well known that

$M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$, and the inequalities

$$G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) < S(a,b) = M_2(a,b) < C(a,b) \tag{1.3}$$

hold for all $a,b > 0$ with $a \neq b$.

Seiffert [12] proved that the double inequality

$$A(a,b) = M_1(a,b) < T(a,b) < M_2(a,b) = S(a,b) \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$.

Hästö [13] proved that the function $T(1,x)/M_p(1,x)$ is increasing in $(0,\infty)$ if $p \leq 1$.

In [14], the authors found the greatest value p and the least value q such that the double inequality $H_p(a,b) < T(a,b) < H_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$. Here, $H_k(a,b) = ((a^k + (ab)^{k/2} + b^k)/3)^{1/k}$ $(k \neq 0)$, and $H_0(a,b) = \sqrt{ab}$ is the kth power-type Heron mean of a and b.

Wang et al. [15] answered the question: what are the best possible parameters λ and μ such that the double inequality $L_\lambda(a,b) < T(a,b) < L_\mu(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $L_r(a,b) = (a^{r+1} + b^{r+1})/(a^r + b^r)$ is the rth Lehmer mean of a and b.

In [16, 17], the authors proved that the inequalities

$$\begin{gathered} \alpha_1 T(a,b) + (1-\alpha_1)G(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)G(a,b), \\ \alpha_2 S(a,b) + (1-\alpha_2)A(a,b) < T(a,b) < \beta_2 S(a,b) + (1-\beta_2)A(a,b), \\ S^{\alpha_3}(a,b)A^{1-\alpha_3}(a,b) < T(a,b) < S^{\beta_3}(a,b)A^{1-\beta_3}(a,b) \end{gathered} \tag{1.5}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 3/5$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq (4-\pi)/[(\sqrt{2}-1)\pi]$, $\beta_2 \geq 2/3$, $\alpha_3 \leq 2/3$ and $\beta_3 \geq 4 - 2\log \pi/\log 2$.

For fixed $a,b > 0$ with $a \neq b$, let $x \in [1/2, 1]$ and

$$J(x) = C(xa + (1-x)b, xb + (1-x)a). \tag{1.6}$$

Then it is not difficult to verify that $J(x)$ is continuous and strictly increasing in $[1/2, 1]$. Note that $J(1/2) = A(a,b) < T(a,b)$ and $J(1) = C(a,b) > T(a,b)$. Therefore, it is natural to ask what are the greatest value α and the least value β in $(1/2, 1)$ such that the double inequality

$$C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \tag{1.7}$$

holds for all $a,b > 0$ with $a \neq b$. The main purpose of this paper is to answer this question. Our main result is the following Theorem 1.1.

Theorem 1.1. *If $\alpha, \beta \in (1/2, 1)$, then the double inequality*

$$C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \tag{1.8}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 + \sqrt{4/\pi - 1})/2$ and $\beta \geq (3 + \sqrt{3})/6$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (1 + \sqrt{4/\pi - 1})/2$ and $\mu = (3 + \sqrt{3})/6$. We first proof that the inequalities

$$T(a,b) > C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a), \tag{2.1}$$

$$T(a,b) < C(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.2) we clearly see that both $T(a,b)$ and $C(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $p \in (1/2, 1)$, then from (1.1) and (1.2) one has

$$\begin{aligned} &C(pa + (1-p)b, pb + (1-p)a) - T(a,b) \\ &\quad = b\frac{[pt + (1-p)]^2 + [(1-p)t + p]^2}{2(t+1)\arctan((t-1)/(t+1))} \\ &\quad\quad \times \left\{2\arctan\left(\frac{t-1}{t+1}\right) - \frac{t^2-1}{[pt + (1-p)]^2 + [(1-p)t + p]^2}\right\}. \end{aligned} \tag{2.3}$$

Let

$$f(t) = 2\arctan\left(\frac{t-1}{t+1}\right) - \frac{t^2-1}{[pt + (1-p)]^2 + [(1-p)t + p]^2}. \tag{2.4}$$

Then simple computations lead to

$$f(1) = 0, \tag{2.5}$$

$$\lim_{t \to +\infty} f(t) = \frac{\pi}{2} - \frac{1}{p^2 + (1-p)^2}, \tag{2.6}$$

$$f'(t) = \frac{2f_1(t)}{\left\{[pt + (1-p)]^2 + [(1-p)t + p]^2\right\}^2 (t^2+1)}, \tag{2.7}$$

where

$$\begin{aligned} f_1(t) &= (4p^4 - 8p^3 + 10p^2 - 6p + 1)t^4 - 2(2p-1)^2(2p^2 - 2p + 1)t^3 \\ &\quad + 2(12p^4 - 24p^3 + 18p^2 - 6p + 1)t^2 \\ &\quad - 2(2p-1)^2(2p^2 - 2p + 1)t + 4p^4 - 8p^3 + 10p^2 - 6p + 1, \end{aligned} \tag{2.8}$$

$$f_1(1) = 0. \tag{2.9}$$

Let $f_2(t) = f_1'(t)/2$, $f_3(t) = f_2'(t)/2$, $f_4(t) = f_3'(t)/3$. Then from (2.8) we get

$$\begin{aligned} f_2(t) &= 2\left(4p^4 - 8p^3 + 10p^2 - 6p + 1\right)t^3 - 3(2p-1)^2\left(2p^2 - 2p + 1\right)t^2 \\ &\quad + 2\left(12p^4 - 24p^3 + 18p^2 - 6p + 1\right)t - (2p-1)^2\left(2p^2 - 2p + 1\right), \end{aligned} \tag{2.10}$$

$$f_2(1) = 0, \tag{2.11}$$

$$\begin{aligned} f_3(t) &= 3\left(4p^4 - 8p^3 + 10p^2 - 6p + 1\right)t^2 - 3(2p-1)^2\left(2p^2 - 2p + 1\right)t \\ &\quad + 12p^4 - 24p^3 + 18p^2 - 6p + 1, \end{aligned} \tag{2.12}$$

$$f_3(1) = 6p^2 - 6p + 1, \tag{2.13}$$

$$f_4(t) = 2\left(4p^4 - 8p^3 + 10p^2 - 6p + 1\right)t - (2p-1)^2\left(2p^2 - 2p + 1\right), \tag{2.14}$$

$$f_4(1) = 6p^2 - 6p + 1. \tag{2.15}$$

We divide the proof into two cases.

Case 1 ($p = \lambda = (1 + \sqrt{4/\pi - 1})/2$). Then (2.6), (2.13), and (2.15) lead to

$$\lim_{t \to +\infty} f(t) = 0, \tag{2.16}$$

$$f_3(1) = -\frac{2(\pi - 3)}{\pi} < 0, \tag{2.17}$$

$$f_4(1) = -\frac{2(\pi - 3)}{\pi} < 0. \tag{2.18}$$

Note that

$$4p^4 - 8p^3 + 10p^2 - 6p + 1 = \frac{4 + 2\pi - \pi^2}{\pi^2} > 0. \tag{2.19}$$

It follows from (2.8), (2.10), (2.12), (2.14), and (2.19) that

$$\lim_{t \to +\infty} f_1(t) = +\infty, \tag{2.20}$$

$$\lim_{t \to +\infty} f_2(t) = +\infty, \tag{2.21}$$

$$\lim_{t \to +\infty} f_3(t) = +\infty, \tag{2.22}$$

$$\lim_{t \to +\infty} f_4(t) = +\infty. \tag{2.23}$$

From (2.14) and inequality (2.19), we clearly see that $f_4(t)$ is strictly increasing in $[1, +\infty)$. Then (2.18) and (2.23) lead to the conclusion that there exists $t_0 > 1$ such that $f_4(t) < 0$ for $t \in [1, t_0)$ and $f_4(t) > 0$ for $t \in (t_0, +\infty)$. Hence, $f_3(t)$ is strictly decreasing in $[1, t_0]$ and strictly increasing in $[t_0, +\infty)$.

It follows from (2.17) and (2.22) together with the piecewise monotonicity of $f_3(t)$ that there exists $t_1 > t_0 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

From (2.11) and (2.21) together with the piecewise monotonicity of $f_2(t)$, we conclude that there exists $t_2 > t_1 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$.

Equations (2.7), (2.9), and (2.20) together with the piecewise monotonicity of $f_1(t)$ imply that there exists $t_3 > t_2 > 1$ such that $f(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and (2.16) together with the piecewise monotonicity of $f(t)$.

Case 2 ($p = \mu = (3+\sqrt{3})/6$). Then (2.8) leads to

$$f_1(t) = \frac{(t-1)^4}{9} > 0 \tag{2.24}$$

for $t > 1$.

Inequality (2.24) and (2.7) imply that $f(t)$ is strictly increasing in $[1, +\infty)$. Therefore, inequality (2.2) follows from (2.3)–(2.5) together with the monotonicity of $f(t)$.

From inequalities (2.1) and (2.2) together with the monotonicity of $J(x) = C(xa + (1-x)b, xb + (1-x)a)$ in $[1/2, 1]$, we know that inequality (1.8) holds for all $\alpha \le (1+\sqrt{4/\pi - 1})/2$, $\beta \ge (3+\sqrt{3})/6$, and all $a, b > 0$ with $a \ne b$.

Next, we prove that $\lambda = (1+\sqrt{4/\pi - 1})/2$ is the best possible parameter in $[1/2, 1]$ such that inequality (2.1) holds for all $a, b > 0$ with $a \ne b$.

For any $1 > p > \lambda = (1+\sqrt{4/\pi - 1})/2$, from (2.6) one has

$$\lim_{t \to +\infty} f(t) = \frac{\pi}{2} - \frac{1}{p^2 + (1-p)^2} > 0. \tag{2.25}$$

Equations (2.3) and (2.4) together with inequality (2.25) imply that for any $1 > p > \lambda = (1+\sqrt{4/\pi - 1})/2$ there exists $T_0 = T_0(p) > 1$ such that

$$C(pa + (1-p)b, pb + (1-p)a) > T(a, b) \tag{2.26}$$

for $a/b \in (T_0, +\infty)$.

Finally, we prove that $\mu = (3+\sqrt{3})/6$ is the best possible parameter such that inequality (2.2) holds for all $a, b > 0$ with $a \ne b$.

For any $1/2 < p < \mu = (3+\sqrt{3})/6$, from (2.13) one has

$$f_3(1) = 6p^2 - 6p + 1 < 0. \tag{2.27}$$

From inequality (2.27) and the continuity of $f_3(t)$, we know that there exists $\delta = \delta(p) > 0$ such that

$$f_3(t) < 0 \tag{2.28}$$

for $t \in (1, 1+\delta)$.

Equations (2.3)–(2.5), (2.7), (2.9), and (2.11) together with inequality (2.28) imply that for any $1/2 < p < \mu = (3+\sqrt{3})/6$ there exists $\delta = \delta(p) > 0$ such that

$$T(a,b) > C(pa + (1-p)b, pb + (1-p)a) \tag{2.29}$$

for $a/b \in (1, 1+\delta)$. □

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