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Research Article

Multiple Solutions for a Class of Fractional Boundary Value Problems

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We study the multiplicity of solutions for the following fractional boundary value problem: $(d/dt)((1/2)\,{}_0D_t^{-\beta}(u'(t)) + (1/2)\,{}_0D_T^{-\beta}(u'(t))) + \lambda\nabla F(t,u(t)) = 0$, a.e. $t \in [0,T]$, $u(0) = u(T) = 0$, where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, $\lambda > 0$ is a real number, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function, and $\nabla F(t,x)$ is the gradient of F at x. The approach used in this paper is the variational method. More precisely, the Weierstrass theorem and mountain pass theorem are used to prove the existence of at least two nontrivial solutions.

1. Introduction

In this paper, we consider the fractional boundary value problem of the following form:

$$
\begin{gathered}
\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_0D_T^{-\beta}(u'(t))\right) + \lambda\nabla F(t,u(t)) = 0, \quad \text{a.a. } t \in [0,T], \\
u(0) = u(T) = 0,
\end{gathered} \tag{P}
$$

where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, $\lambda > 0$ is a real number, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function, and $\nabla F(t,x)$ is the gradient of F at x.

In particular, if $\lambda = 1$, the problem (P_1) reduces to the standard second-order boundary value problem of the following form:

$$\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_0D_T^{-\beta}(u'(t))\right) + \nabla F(t, u(t)) = 0, \quad \text{a.a. } t \in [0, T], \qquad u(0) = u(T) = 0. \tag{P_1}$$

Fractional calculus and fractional differential equations can find many applications in various fields of physical science such as viscoelasticity, diffusion, control, relaxation processes, and modeling phenomena in engineering, see [1–12]. Recently, many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by use of techniques of nonlinear analysis, such as fixed-point theory (including Leray-Schauder nonlinear alternative) (see [13, 14]), topological degree theory (including coincidence degree theory) (see [15, 16]), and comparison method (including upper and lower solutions methods and monotone iterative method) (see [17, 18]). However, it seems that the popular methods mentioned above are not appropriate for discussing (P) and (P_1), as the equivalent integral equation is not easy to be obtained.

In the past, there were investigations of the eigenvalue problems for fractional differential equations. For more detailed information on this topic, we refer to Zhang et al. [19–21], Wang et al. [22, 23], and Jiang et al. [24].

Recently, there are many papers dealing with the existence of solutions for problem (P_1). In [25], Jiao and Zhou obtained the existence of solutions for (P_1) by mountain pass theorem under the Ambrosetti-Rabinowitz condition. Chen and Tang [26] studied the existence and multiplicity of solutions for the system (P_1) when the nonlinearity $F(t, \cdot)$ is superquadratic, asymptotically quadratic, and subquadratic, respectively.

But so far, few papers discuss the two solutions of the system (P) via critical point theory. The aim of the present paper is to study the existence of at least two solutions for the system (P) as the parameter $\lambda > \lambda_0$ for some constant λ_0.

The paper is organized as follows. We first introduce some basic preliminary results and a well-known lemma in Section 2, including the fractional derivative space E^α, where $\alpha \in (1/2, 1]$. In Section 3, we give the main result and its proof. In Section 4, we give the summary of this paper.

2. Preliminary

In this section, we recall some related preliminaries and display the variational setting which has been established for our problem.

Definition 2.1 (see [8]). Let $f(t)$ be a function defined on $[a, b]$ and $\tau > 0$. The left and right Riemann-Liouville fractional integrals of order τ for function $f(t)$ denoted by ${}_aD_t^{-\tau}f(t)$ and ${}_tD_b^{-\tau}f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\tau}f(t) &= \frac{1}{\Gamma(\tau)}\int_a^t (t-s)^{\tau-1}f(s)ds, \quad t \in [a, b], \\ {}_tD_b^{-\tau}f(t) &= \frac{1}{\Gamma(\tau)}\int_t^b (t-s)^{\tau-1}f(s)ds, \quad t \in [a, b], \end{aligned} \tag{2.1}$$

provided the right-hand sides are pointwise defined on $[a, b]$, where Γ is the gamma function.

Definition 2.2 (see [8]). Let $f(t)$ be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order τ for function $f(t)$ denoted by ${}_aD_b^\tau f(t)$ and ${}_tD_b^\tau f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^\tau f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\tau-n} f(t) = \frac{1}{\Gamma(n-\tau)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\tau-1} f(s)ds\right), \\ {}_tD_b^\tau f(t) &= (-1)^n \frac{d^n}{dt^n}\,{}_tD_b^{\tau-n} f(t) = \frac{1}{\Gamma(n-\tau)}\frac{d^n}{dt^n}\left(\int_t^b (t-s)^{n-\tau-1} f(s)ds\right), \end{aligned} \tag{2.2}$$

where $t \in [a, b]$, $n - 1 \le \tau < n$, and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives. In particular, they are defined for the function belonging to the space of absolutely continuous functions, which we denote by $AC([a, b], \mathbb{R}^N)$. $AC^k([a, b], \mathbb{R}^N)$ $(k = 1, 2, \ldots)$ is the space of functions f such that $f \in C^k([a, b], \mathbb{R}^N)$. In particular, $AC([a, b], \mathbb{R}^N) = AC^1([a, b], \mathbb{R}^N)$.

Definition 2.3 (see [8]). Let $\tau \ge 0$ and $n \in \mathbb{N}$. If $\tau \in [n-1, n)$ and $f(t) \in AC^n([a, b], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order τ for function $f(t)$ denoted by ${}_a^cD_t^\tau f(t)$ and ${}_t^cD_b^\tau f(t)$, respectively, which exist a.e. on $[a, b]$. ${}_a^cD_t^\tau f(t)$ and ${}_t^cD_b^\tau f(t)$ are represented by

$$\begin{aligned} {}_a^cD_t^\tau f(t) &= {}_aD_t^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)}\left(\int_a^t (t-s)^{n-\tau-1} f^{(n)}(s)ds\right), \\ {}_t^cD_b^\tau f(t) &= (-1)\,{}_t^nD_b^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)}\left(\int_t^b (t-s)^{n-\tau-1} f^{(n)}(s)ds\right), \end{aligned} \tag{2.3}$$

respectively, where $t \in [a, b]$.

Definition 2.4 (see [25]). Define $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p dt\right)^{1/p}, \quad \forall u \in E_0^{\alpha,p}, \tag{2.4}$$

where $C_0^\infty([0, T], \mathbb{R}^N)$ denotes the set of all functions $u \in C^\infty([0, T], \mathbb{R}^N)$ with $u(0) = u(T) = 0$. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0, T], \mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^p([0, T], \mathbb{R}^N)$ and $u(0) = u(T) = 0$.

Proposition 2.5 (see [25]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Proposition 2.6 (see [25]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, one has*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \left\| {}_0^c D_t^\alpha u \right\|_{L^p}. \tag{2.5}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \frac{T^{(\alpha-1)/p}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{1/q}} \left\| {}_0^c D_t^\alpha u \right\|_{L^p}. \tag{2.6}$$

According to (2.6)*, one can consider $E_0^{\alpha,p}$ with respect to the norm*

$$\|u\|_{\alpha,p} = \left\| {}_0^c D_t^\alpha u \right\|_{L^p} = \left(\int_0^T \left| {}_0^c D_t^\alpha u \right|^p dt \right)^{1/p}. \tag{2.7}$$

Proposition 2.7 (see [25]). *Define $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$, and the sequence u_k converges weakly to $u \in E_0^{\alpha,p}$, that is, $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, that is, $\|u_k - u\|_\infty \to 0$, as $k \to \infty$.*

Making use of Definition 2.3, for any $u \in AC([0,T],\mathbb{R}^N)$, problem (P) is equivalent to the following problem:

$$\begin{gathered} \frac{d}{dt}\left(\frac{1}{2}\, {}_0D_t^{\alpha-1}\left({}_0^c D_t^\alpha u(t)\right) - \frac{1}{2}\, {}_tD_T^{\alpha-1}\left({}_t^c D_T^\alpha u(t)\right) \right) + \lambda \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{P_2}$$

where $\alpha = 1 - \beta/2 \in (1/2, 1]$.

In the following, we will treat problem (P_2) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$. It follows from [25, Theorem 4.1] that the functional φ given by

$$\varphi(u) = \int_0^T \left[-\frac{1}{2}\left({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)\right) \right] dt - \lambda \int_0^T F(t,u(t))dt \tag{2.8}$$

is continuously differentiable on E^α. Moreover, for $u, v \in E^\alpha$, we have

$$\begin{aligned} \langle \varphi'(u), v\rangle = & -\int_0^T \frac{1}{2}\left[\left({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha v(t)\right) + \left({}_t^c D_T^\alpha u(t), {}_0^c D_t^\alpha v(t)\right) \right] dt \\ & - \lambda \int_0^T (\nabla F(t,u(t)), v(t)] dt. \end{aligned} \tag{2.9}$$

Definition 2.8 (see [25]). A function $u \in AC([0,T],\mathbb{R}^N)$ is called a solution of (P_2) if

(i) $D^\alpha(u(t))$ is derivative for a.e. $t \in [0,T]$,

(ii) u satisfies (P_2), where $D^\alpha(u(t)) := (1/2)\, {}_0D_t^{\alpha-1}({}_0^c D_t^\alpha u(t)) - (1/2)\ {}_tD_T^{\alpha-1}({}_t^c D_T^\alpha u(t))$.

Proposition 2.9 (see [25]). *If $1/2 < \alpha \leq 1$, then for any $u \in E^\alpha$, one has*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq -\int_0^T \left({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t)\right) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \tag{2.10}$$

Proposition 2.10 (see [25]). *Let $1/2 < \alpha \leq 1$ be satisfied. If $u \in E^\alpha$, then the functional $J : E^\alpha \to \mathbb{R}$ denoted by*

$$J(u) = -\frac{1}{2} \int_0^T \left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right) dt \tag{2.11}$$

is convex and continuous on E^α.

In order to prove the existence of two solutions for problem (P), firstly, we recall some well-known results. Their proofs can be found in many books. Please refer to the references and its references therein.

Lemma 2.11 (see [27]). *If X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $e \in X$, and $r > 0$, such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e), \tag{2.12}$$

and if φ satisfies the PS condition, with

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}, \tag{2.13}$$

then c is a critical value of φ.

3. The Main Result and Proof of the Theorem

In this part, we will prove that for (P) there also exist two solutions for the general case.

Our hypotheses on nonsmooth potential $F(t, x)$ are as follows.

$\mathrm{H(F)}_1$: $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(t, 0) = 0$ a.e. on $[0, T]$ and satisfies the following facts:

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t, x)$ is measurable,

(ii) for a.a. $t \in [0, T]$, $x \mapsto F(t, x)$ is continuously differentiable,

(iii) there exist $c \in C([0, T], \mathbb{R})$ and $0 < \alpha_0 < 2$, such that

$$|\nabla F(t, x)|, |F(t, x)| \leq c(t)\left(1 + |x|^{\alpha_0}\right), \quad \text{for a.a. } t \in [0, T],\ \text{all } x \in \mathbb{R}^N, \tag{3.1}$$

(iv) there exist $\gamma > 2$ and $\mu \in L^\infty([0, T])$, such that

$$\limsup_{|x|\to 0} \frac{(\nabla F(t, x), x)}{|x|^\gamma} < \mu(t), \quad \text{uniformly for a.a. } t \in [0, T], \tag{3.2}$$

(v) there exist $\xi_0 \in \mathbb{R}^N$, $t_0 \in (0, T)$, and $r_0 > 0$, such that

$$F(t, \xi_0) > \delta_0 > 0, \quad \text{a.e. } t \in B_{r_0}(t_0), \tag{3.3}$$

where $B_{r_0}(t_0) := \{t \in [0, T] : |t - t_0| \leq r_0\} \subset [0, T]$.

Remark 3.1. It is easy to verify that $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the whole assumption in [25, Theorem 4.1]. So, $u \in E^\alpha$ is a solution of the corresponding Euler equation $\varphi'(u) = 0$, then u is a solution of problem (P_2) which, of course, corresponds to the solution of problem (P). For great details, please see [25, Theorem 4.2].

Theorem 3.2. *Suppose that $H(F)_1$ holds. Then there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, the problem (P_2) has at least two nontrivial solutions, which correspond to the two solutions of problem (P).*

Proof. The proof is divided into four steps as follows.
Step 1. We will show that φ is coercive in this step.

Firstly, by $\mathrm{H(F)}_1$(iii), (2.6), and (2.10), we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)\right]dt - \lambda\int_0^T F(t, u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - c_0\lambda T - c_0\lambda\int_0^T |u(t)|^{\alpha_0}dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - c_0\lambda T - c_0\lambda T\left(\frac{T^{(\alpha-1)/2}}{\sqrt{2\alpha-1}\Gamma(\alpha)}\right)^{\alpha_0}\|u\|_\alpha^{\alpha_0} \\
&\longrightarrow \infty, \quad \text{as } \|u\|_\alpha \longrightarrow \infty,
\end{aligned} \tag{3.4}$$

where $c_0 = \max_{t\in[0,T]}|c(t)|$.
Step 2. We will show that the φ is weakly lower semicontinuous.

Let $u_n \rightharpoonup u$ weakly in E^α, and by Proposition 2.7, we obtain the following results:

$$\begin{gathered}
E^\alpha \hookrightarrow C\left([0, T], \mathbb{R}^N\right), \\
u_n(t) \longrightarrow u(t) \quad \text{for a.e. } t \in [0, T], \\
F(t, u_n(t)) \longrightarrow F(t, u(t)) \quad \text{for a.e. } t \in [0, T].
\end{gathered} \tag{3.5}$$

By Fatou's lemma,

$$\limsup_{n\to\infty}\int_0^T F(t, u_n(t))dt \leq \int_0^T F(t, u(t))dt. \tag{3.6}$$

On the other hand, by Proposition 2.10, we have $\lim_{n\to\infty} J(u_n) = J(u)$, that is,

$$\lim_{n\to\infty}\int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)\right)\right]dt = \int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)\right]dt. \tag{3.7}$$

Thus,

$$\begin{aligned}\liminf_{n\to\infty}\varphi(u_n) &= \liminf_{n\to\infty}\int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^\alpha u_n(t),\, {}_t^cD_T^\alpha u_n(t)\right)\right]dt - \limsup_{n\to\infty}\lambda\int_0^T F(t,u_n(t))dt \\ &\geq \int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t),\, {}_t^cD_T^\alpha u(t)\right)\right]dt - \lambda\int_0^T F(t,u(t))dt = \varphi(u).\end{aligned}\tag{3.8}$$

Hence, by the Weierstrass theorem, we deduce that there exists a global minimizer $u_0 \in E^\alpha$ such that

$$\varphi(u_0) = \min_{u\in E^\alpha}\varphi(u).\tag{3.9}$$

Step 3. We will show that there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, $\varphi(u_0) < 0$.

By the condition by $\mathrm{H(F)}_1$ (v), there exists $\xi_0 \in \mathbb{R}^N$ such that $F(t,\xi_0) > \delta_0 > 0$, a.e. $t \in B_{r_0}(t_0)$. It is clear that

$$0 < M_1 := \max_{|x|\leq|\xi_0|}\left\{c_0 + c_0|x|^{\alpha_0}\right\} < +\infty.\tag{3.10}$$

Now we denote

$$\begin{gathered}\kappa_0 = \frac{M_1}{\delta_0 + M_1}, \\ \lambda_0 = \max_{\kappa\in[\kappa_1,\kappa_2]}\frac{|\xi_0|^2T^{3-2\alpha}}{2\Gamma^2(1-\alpha)r_0^3(1-\kappa)^2(1-\alpha)^2(3-2\alpha)(\delta_0\kappa - M_1 + \kappa M_1)|\cos(\pi\alpha)|},\end{gathered}\tag{3.11}$$

where $\kappa_0 < \kappa_1 < \kappa_2 < 1$, and δ_0 is given in the condition $\mathrm{H(F)}_1$ (v). A simple calculation shows that the function $\kappa \mapsto \delta_0\kappa - M_1 + \kappa M_1$ is positive whenever $\kappa > \kappa_0$ and $\delta_0\kappa_0 - M_1 + \kappa_0 M_1 = 0$. Thus, λ_0 is well defined and $\lambda_0 > 0$.

We will show that for each $\lambda > \lambda_0$, the problem (P_2) has two nontrivial solutions. In order to do this, for $t \in [t_1, t_2]$, let us define

$$\eta_\kappa(t) = \begin{cases} 0 & \text{if } t \in [0,T]/B_{r_0}(t_0), \\ \xi_0 & \text{if } t \in B_{\kappa r_0}(t_0), \\ \dfrac{\xi_0}{r_0(1-\kappa)}(r_0 - |t-t_0|) & \text{if } t \in B_{r_0}(t_0)/B_{\kappa r_0}(t_0). \end{cases}\tag{3.12}$$

Then $|\eta_\kappa'(t)| \leq |\xi_0|/r_0(1-\kappa)$ and

$$\left|{}_0^cD_t^\alpha\eta_\kappa(t)\right| \leq \frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left|\eta_\kappa'(s)\right|ds \leq \frac{1}{\Gamma(1-\alpha)}\frac{|\xi_0|}{r_0(1-\kappa)}\frac{t^{1-\alpha}}{1-\alpha}.\tag{3.13}$$

Hence,

$$\|\eta_\iota\|_\alpha^2 = \int_0^T\left|{}_0^cD_t^\alpha\eta_\kappa(t)\right|^2dt \leq \frac{|\xi_0|^2T^{3-2\alpha}}{\Gamma^2(1-\alpha)r_0^2(1-\kappa)^2(1-\alpha)^2(3-2\alpha)}.\tag{3.14}$$

By $\mathrm{H(F)}_1$ (iii) and (v), we have

$$\begin{aligned}\int_0^T F(t,\eta_\kappa(t))dt &= \int_{B_{\kappa r_0}(t_0)} F(t,\eta_\kappa(t))dt + \int_{B_{r_0}(t_0)\setminus B_{\kappa r_0}(t_0)} F(t,\eta_\kappa(t))dt \\ &\geq 2\kappa r_0\delta_0 - M_1(2r_0 - 2\kappa r_0) \\ &= 2r_0(\delta_0\kappa - M_1 + \kappa M_1).\end{aligned} \tag{3.15}$$

For $\kappa \in [\kappa_1,\kappa_2]$, by Proposition 2.9, we have

$$\begin{aligned}\varphi(\eta_\kappa) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha\eta_\kappa(t), {}_t^cD_T^\alpha\eta_\kappa(t)\right)\right]dt - \lambda\int_0^T F(t,\eta_\kappa(t))dt \\ &\leq \frac{1}{|\cos(\pi\alpha)|}\|\eta_\kappa\|_\alpha^2 - 2\lambda r_0(\delta_0\kappa - M_1 + \kappa M_1) \\ &\leq \frac{1}{|\cos(\pi\alpha)|}\frac{|\xi_0|^2T^{3-2\alpha}}{\Gamma^2(1-\alpha)r_0^2(1-\kappa)^2(1-\alpha)^2(3-2\alpha)} - 2\lambda r_0(\delta_0\kappa - M_1 + \kappa M_1),\end{aligned} \tag{3.16}$$

so that $\varphi(\eta_t) < 0$ whenever $\lambda > \lambda_0$.

Step 4. We will check the PS condition in the following.

Suppose that $\{u_n\}_{n\geq 1} \subseteq E^\alpha$ such that $\varphi(u_n) \to c$ and $\|\varphi'(u_n)\|_\alpha \to 0$.

Since φ is coercive and $\{u_n\}_{n\geq 1}$ is bounded in E^α and passed to a subsequence, which still denote $\{u_n\}_{n\geq 1}$, we may assume that there exists $u \in E^\alpha$, such that $u_n \rightharpoonup u$ weakly in E^α; thus, we have

$$\begin{aligned}\langle\varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \langle\varphi'(u_n), u_n - u\rangle - \langle\varphi'(u), u_n - u\rangle \\ &\leq \|\varphi'(u_n)\|\|u_n\|_\alpha - \langle\varphi'(u), u_n - u\rangle \longrightarrow 0,\end{aligned} \tag{3.17}$$

as $n \to \infty$. Moreover, according to Proposition 2.7, we have $\|u_n - u\|_\infty \to 0$, as $n \to \infty$. Observing that

$$\begin{aligned}&\langle\varphi'(u_n) - \varphi'(u), u_n - u\rangle \\ &\quad= -\int_0^T \left({}_0^cD_t^\alpha(u_n(t) - u(t)), {}_t^cD_T^\alpha(u_n(t) - u(t))\right)dt \\ &\qquad- \int_0^T (\nabla F(t,u_n(t)) - F(t,u(t)), u_n(t) - u(t))dt \\ &\quad\geq |\cos(\pi\alpha)|\|u_n - u\|_\alpha^2 - \int_0^T |\nabla F(t,u_n(t)) - F(t,u(t))|dt\|u_n - u\|_\infty,\end{aligned} \tag{3.18}$$

combining this with (3.17), it is easy to verify that $\|u_n - u\|_\alpha \to 0$, as $n \to \infty$, and hence that $u_n \to u$ in E^α. Thus, φ satisfies the PS condition. □

Step 5. We will show that there exists another nontrivial weak solution of problem (P_2).

From the mean value theorem and $\mathrm{H(F)}_1$ (v), we have

$$
\begin{aligned}
F(t,x) - F(t,0) &= \int_0^1 (\nabla F(t,\theta x), x) d\theta \\
&= \int_0^1 \frac{(\nabla F(t,\theta x), \theta x)}{\theta} d\theta \\
&\leq \int_0^1 \frac{\mu(t)}{\theta} |\theta x|^{\gamma} d\theta \\
&= \mu(t)|x|^{\gamma} \int_0^1 \theta^{\gamma-1} d\theta \\
&= \frac{\mu(t)|x|^{\gamma}}{\gamma},
\end{aligned}
\tag{3.19}
$$

for all $|x| < \beta$, a.e. $t \in [0,T]$.

It follows from the conditions $\mathrm{H(F)}_1$ (iii) and all $|x| \geq \beta$ and a.e. $t \in [0,T]$ that

$$
\begin{aligned}
|F(t,x)| &\leq c_0|t| + c_0|x|^{\alpha_0} \\
&\leq c_0\left|\frac{x}{\beta}\right| + c_0|x|^{\alpha_0} \\
&\leq \left(\frac{c_0}{\beta^{\alpha_0}} + c_0\right)|x|^{\alpha_0-\gamma}|x|^{\gamma} \\
&\leq \left(\frac{c_0}{\beta^{\gamma}} + \frac{c_0}{\beta^{\gamma-\alpha_0}}\right)|x|^{\gamma},
\end{aligned}
\tag{3.20}
$$

and this together with (3.19) yields that for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$,

$$
|F(t,x)| \leq \left(\frac{\mu(t)}{\gamma} + \frac{c_0}{\beta^{\gamma}} + \frac{c_0}{\beta^{\gamma-\alpha_0}}\right)|x|^{\gamma(x)} \leq c_1|x|^{\gamma},
\tag{3.21}
$$

for some positive constant c_1.

For all $\lambda > \lambda_0$, $\|u\|_{\alpha} < 1$, and $|u|_{\infty} < 1$, we have

$$
\begin{aligned}
\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^{\alpha}u(t),\ {}_t^cD_T^{\alpha}u(t)\right)\right]dt - \lambda\int_0^T F(t,u(t))dt \\
&\geq |\cos(\pi\alpha)|\|u\|_{\alpha}^2 - \lambda c_1 \int_0^T |u(t)|^{\gamma} dt \\
&\geq |\cos(\pi\alpha)|\|u\|_{\alpha}^2 - \lambda c_1 \left[\frac{T^{(\alpha-1)/2}}{\Gamma\alpha(\alpha)\sqrt{2\alpha-1}}\right]^{\gamma} |\ \|u\|_{\alpha}^{\gamma}.
\end{aligned}
\tag{3.22}
$$

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u) > \nu, \quad \text{for } \|u\|_\alpha = \rho\ , \tag{3.23}$$

and $\|u_0\|_\alpha > \rho$. So by the mountain pass theorem (cf. Lemma 2.11), we can get $u_1 \in E^\alpha$ which satisfies

$$\varphi(u_1) = c > 0, \qquad \varphi'(u_1) = 0. \tag{3.24}$$

Therefore, u_1 is another nontrivial critical point of φ.

Remark 3.3. We can find a potential function satisfying the hypothesis of our Theorem 3.2. For great details, please see Section 4(B) in Summary.

So far, the results involved potential functions exhibiting sublinear. The next theorem concerns problems where the potential function is superlinear.

Our hypotheses on nonsmooth potential $F(t,x)$ are as follows.

$\mathrm{H(F)}_2$: $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(t,0) = 0$ a.e. on $[0,T]$ and satisfies the following facts:

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable,

(ii) for a.a. $t \in [0,T]$, $x \mapsto F(t,x)$ is continuously differentiable,

(iii) there exist $\overline{c} \in C([0,T],\mathbb{R})$ and $\alpha_0 > 2$, such that

$$|\nabla F(t,x)|, |F(t,x)| \le \overline{c}(t)\bigl(1 + (t)|x|^{\alpha_0}\bigr), \quad \text{for a.a. } t \in [0,T],\ \text{all } x \in \mathbb{R}^N, \tag{3.25}$$

(iv) there exist $\gamma > 2$ and $\mu \in L^\infty([0,T])$, such that

$$\limsup_{|x|\to 0} \frac{(\nabla F(t,x), x)}{|x|^\gamma} < \mu(t), \quad \text{uniformly for a.a. } t \in [0,T], \tag{3.26}$$

(v) there exist $\xi_0 \in \mathbb{R}^N$, $t_0 \in (0,T)$, and $r_0 > 0$, such that $F(t,\xi_0) > \delta_0 > 0$, a.e. $t \in B_{r_0}(t_0)$,where $B_{r_0}(t_0) := \{t \in [0,T] : |t - t_0| \le r_0\} \subset [0,T]$,

(vi) for a.a. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, we have

$$F(t,x) \le \nu(t) \text{ with } \nu \in L^\beta([0,T],\mathbb{R}), \quad 1 \le \beta < 2. \tag{3.27}$$

Theorem 3.4. *Suppose that $H(F)_2$ holds. Then there exists a $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, the problem (P_2) has at least two nontrivial solutions, which correspond to the two solutions of problem (P).*

Proof. The steps are similar to those of Theorem 3.2. In fact, we only need to modify Step 1 and Step 4 as follows: $1'$ shows that φ is coercive under the condition $\mathrm{H(F)}_2$ (vi); $4'$ shows that

there exists a second nontrivial solution under the conditions H(F)$_2$ (iii) and H(F)$_2$ (iv). Then from Steps 1$'$, 2, 3, and 4$'$ above, problem (P) has at least two nontrivial solutions.
Step 1$'$. By H(F)$_2$ (vi), for all $u \in E^\alpha$, $\|u\|_\alpha > 1$, we have

$$\begin{aligned}\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)\right]dt - \lambda \int_0^T F(t,u(t))dt \\ &\geq |\cos(\pi\alpha)|\|u\|_\alpha^2 - \lambda \overline{c}_0 \int_0^T \nu(t)dt \longrightarrow \infty, \quad \text{as } \|u\|_\alpha \longrightarrow \infty,\end{aligned} \tag{3.28}$$

where $\overline{c}_0 = \max_{t\in[0,T]}|\overline{c}(t)|$.
Step 4$'$. Because of hypothesis H(F)$_2$ (iii), we have

$$\begin{aligned}F(t,x) &\leq \overline{c}_0 + \overline{c}_0|x|^{\alpha_0} \\ &\leq \overline{c}_0\left|\frac{x}{\beta}\right|^{\alpha_0} + \overline{c}_0|x|^{\alpha_0} \\ &= \overline{c}_0\left|\frac{1}{\beta}\right|^{\alpha_0}|x|^{\alpha_0} + \overline{c}_0|x|^{\alpha_0} \\ &= c_2|x|^{\alpha_0},\end{aligned} \tag{3.29}$$

for a.e. $t \in [0,T]$ and all $|x| \geq \beta$ with $c_2 > 0$.

Combining (3.19) and (3.29), it follows that

$$|F(t,x)| \leq \frac{\mu(t)}{\gamma}|x|^\gamma + c_2|x|^{\alpha_0}, \tag{3.30}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

Thus, for all $\lambda > \lambda_0$, $\|u\|_\alpha < 1$ and $|u|_\infty < 1$, we have

$$\begin{aligned}\varphi(u) &= \int_0^T \left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)\right]dt - \lambda \int_0^T F(t,u(t))dt \\ &\geq |\cos(\pi\alpha)|\|u\|_\alpha^2 - \lambda \int_0^T \frac{\mu(t)}{\gamma}|u(t)|^\gamma dt - \lambda c_2 \int_0^T |u(t)|^{\alpha_0}dt \\ &\geq |\cos(\pi\alpha)|\|u\|_\alpha^2 - \lambda c_3\left|\left[\frac{T^{(\alpha-1)/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\right]^\gamma\right|\|u\|_\alpha^\gamma - \lambda c_4\left|\left[\frac{T^{(\alpha-1)/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\right]^{\alpha_0}\right|\|u\|_\alpha^{\alpha_0},\end{aligned} \tag{3.31}$$

where c_3 and c_4 are positive constants.

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u) > \nu, \quad \text{for } \|u\|_\alpha = \rho, \tag{3.32}$$

and $\|u_0\|_\alpha > \rho$.

Arguing as in proof of Step 4 of Theorem 3.2, we conclude that φ satisfies the PS condition. So by the mountain pass theorem (cf. Lemma 2.11), we can get that $u_1 \in E^\alpha$ satisfies

$$\varphi(u_1) = c > 0, \qquad \varphi'(u_1) = 0. \tag{3.33}$$

Therefore, u_1 is another nontrivial critical point of φ. □

Remark 3.5. We will give some examples, which satisfy the hypothesis of our Theorem 3.4. For great details, please see Section 4(C) in Summary.

4. Summary

(A) If $\beta = 0$, then $\alpha = 1 - \beta/2 = 1$. Therefore, by Theorems 3.2 and 3.4, we actually obtain the existence of two weak solutions of the following eigenvalue problem:

$$\begin{gathered} u''(t) + \lambda \nabla F(t, u(t)) = 0, \quad \text{a.a. } t \in [0, T], \\ u(0) = u(T) = 0, \end{gathered} \tag{P_3}$$

where $\lambda > 0$ is a real number, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function, and $\nabla F(t, x)$ is the gradient of F at x. Although many excellent results have been worked out on the existence of solutions for second-order system (P_3) (e.g., [28, 29]), it seems that no similar results were obtained in the literature for fractional system (P).

(B) We give an example in the following to illustrate our viewpoint in Remark 3.1. We consider

$$F(t, x) = \begin{cases} a(t) \sin\left(\dfrac{\pi}{2}|x|^{\gamma}\right), & 0 \le |x| < 1, \\ a(t), & |x| \ge 1, \end{cases} \tag{4.1}$$

where $\gamma > 2$, $a \in C([0, T], \mathbb{R})$, $a(t) > 0$ for all $t \in [0, T]$.

Obviously, hypotheses $\mathrm{H(F)}_1$ (i), (ii), and (v) are satisfied. Moreover,

$$\begin{gathered} |F(t, x)| \le \begin{cases} a(t)\dfrac{\pi}{2}|x|^{\gamma} \le a(t)\dfrac{\pi}{2}|x|^{\gamma - \alpha_0}|x|^{\alpha_0} \le a(t)\dfrac{\pi}{2}|x|^{\alpha_0} \le a(t)\dfrac{\pi\gamma}{2}|x|^{\alpha_0}, & 0 \le |x| < 1, \\ a(t) \le a(t)\dfrac{\pi\gamma}{2}, & |x| \ge 1, \end{cases} \\ |\nabla F(t, x)| = \begin{cases} \left|a(t)\dfrac{\pi\gamma}{2}\cos\left(\dfrac{\pi}{2}|x|^{\gamma}\right)|x|^{\gamma-2}x\right| \le a(t)\dfrac{\pi\gamma}{2}|x|^{\gamma-1} \le a(t)\dfrac{\pi\gamma}{2}, & 0 \le |x| < 1, \\ 0, & |x| \ge 1. \end{cases} \end{gathered} \tag{4.2}$$

Therefore,

$$|\nabla F(t,x)|, |F(t,x)| \leq \frac{\pi\gamma a(t)}{2} + \frac{\pi\gamma a(t)}{2}|x|^{\alpha_0}. \tag{4.3}$$

So, condition $H(F)_1$(iii) holds.

On the other hand,

$$\limsup_{|x|\to 0} \frac{\langle \nabla F(t,x), x\rangle}{|x|^{\gamma}} = \limsup_{|x|\to 0} \frac{\gamma(\pi/2)a(t)\cos((\pi/2)|x|^{\gamma})|x|^{\gamma}}{|x|^{\gamma}} = \frac{\pi}{2}\gamma a(t), \tag{4.4}$$

uniformly for a.a. $t \in [0,T]$, so condition $H(F)_1$ (iv) holds.

(C) We can find the following potential functions satisfying the conditions stated in Theorem 3.4:

$$F_1(t,x) = \begin{cases} -\dfrac{a(t)}{\gamma}|x|^{\gamma}, & |x| < 1, \\ \dfrac{a(t)}{\pi}\cos\left(\dfrac{\pi}{2}|x|^2\right) - \dfrac{a(t)}{\gamma}, & |x| \geq 1, \end{cases}$$
$$F_2(t,x) = \begin{cases} a(t)\sin\left(\dfrac{\pi}{2}|x|^{\gamma}\right), & |x| < 1, \\ a(t)\left[\dfrac{1}{|x|} + \dfrac{1}{2}|x|^2 - \dfrac{1}{2}\right], & |x| \geq 1, \end{cases} \tag{4.5}$$
$$F_3(t,x) = \begin{cases} \dfrac{a(t)}{\gamma}|x|^{\gamma}, & |x| < 1, \\ -\dfrac{2a(t)}{\sqrt{|x|}} + a(t)\left(2 + \dfrac{1}{\gamma}\right), & |x| \geq 1, \end{cases}$$

where $a \in C([0,T],\mathbb{R})$, $a(t) > 0$ for all $t \in [0,T]$.

It is clear that $F_i(x,0) = 0$ $(i = 1,2,3)$ for a.e. $t \in [0,T]$, and hypotheses $H(F)_2$ (i) and $H(F)_2$ (ii) are satisfied. A direct verification shows that conditions $H(F)_2$ (v) and $H(F)_2$ (vi) are satisfied. Note that

$$|\nabla F_1(t,x)| = \begin{cases} \left|-a(t)|x|^{\gamma-2}x\right| \leq a(t)|x|^{\gamma-1} \leq 2a(t), & 0 \leq |x| < 1, \\ \left|-a(t)\sin\left(\dfrac{\pi}{2}|x|^2\right)x\right| \leq a(t)|x| \leq 2a(t)|x|^{\gamma}, & |x| \geq 1, \end{cases}$$

$$
\begin{gathered}
|\nabla F_2(t,x)| = \begin{cases} \left|\gamma a(t)\cos\left(\frac{\pi}{2}|x|^{\gamma}\right)\frac{\pi}{2}|x|^{\gamma-2}x\right| \le \gamma a(t)\frac{\pi}{2}|x|^{\gamma-1} \le a(t)\frac{\pi\gamma}{2}, & 0 \le |x| < 1, \\ \left|a(t)\left(x - \frac{x}{|x|^3}\right)\right| \le a(t)\left(|x| + \frac{1}{|x|^2}\right) \le a(t) + a(t)|x|^{\gamma}, & |x| \ge 1, \end{cases} \\
|\nabla F_3(t,x)| = \begin{cases} \left|a(t)|x|^{\gamma-2}x\right| \le \left|a(t)|x|^{2(5/2)}\right| \le a(t), & 0 \le |x| < 1, \\ \left|a(t)|x|^{-5/2}x\right| \le a(t)|x| \le a(t)|x|^{\gamma}, & |x| \ge 1, \end{cases} \\
|F_1(t,x)| \le a(t)|x|^{\gamma} + a(t)(1+1) = 2a(t) + 2a(t)|x|^{\gamma}, \\
|F_2(t,x)| \le a(t)\left|\sin\left(\frac{\pi}{2}|x|^{\gamma}\right)\right| + a(t)\left[1 + \frac{1}{2}|x|^{\gamma} - \frac{1}{2}\right] \le \frac{3a(t) + a(t)|x|^{\gamma}}{2}, \\
|F_3(t,x)| \le a(t)\frac{1}{\gamma}|x|^{\gamma} + \frac{2a(t)}{\sqrt{|x|}} + a(t)\left(2 + \frac{1}{\gamma}\right) \le 5a(t) + a(t)|x|^{\gamma}.
\end{gathered}
\tag{4.6}
$$

So,

$$
\begin{aligned}
&|\nabla F_1(t,x)|, |F_1(t,x)| \le 2a(t)\left(1 + |x|^{\gamma}\right), \\
&|\nabla F_2(t,x)|, |F_2(t,x)| \le a(t)\frac{\pi\gamma}{2}\left(1 + |x|^{\gamma}\right), \\
&|\nabla F_3(t,x)|, |F_3(t,x)| \le 5a(t)\left(1 + |x|^{\gamma}\right), \\
&\limsup_{|x|\to 0}\frac{(\nabla F_1(t,x), x)}{|x|^{\gamma}} = \lim_{|x|\to 0}\frac{-a(t)|x|^{\gamma}}{|x|^{\gamma}} = -a(t), \\
&\limsup_{|x|\to 0}\frac{(\nabla F_2(t,x), x)}{|x|^{\gamma}} = \lim_{|x|\to 0}\frac{a(t)\cos\left((\pi/2)|x|^{\gamma}\right)(\pi/2)\gamma|x|^{\gamma}}{|x|^{\gamma}} \\
&\qquad = \lim_{|x|\to 0} a(t)\cos\left(\frac{\pi}{2}|x|^{\gamma}\right)\frac{\pi}{2}\gamma = \frac{\gamma\pi a(t)}{2}, \\
&\limsup_{|x|\to 0}\frac{(\nabla F_3(t,x), x)}{|x|^{\gamma}} = \lim_{|x|\to 0}\frac{a(t)|x|^{\gamma}}{|x|^{\gamma}} = a(t),
\end{aligned}
\tag{4.7}
$$

which shows that assumptions $H(F)_2$ (iii) and $H(F)_2$ (iv) are fulfilled.

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