Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 582854, 11 pages
doi:10.1155/2012/582854

Research Article

Normality Criteria of Meromorphic Functions That Share a Holomorphic Function

Jie Ding,[1] Jianming Qi,[2] and Taiying Zhu[2]

[1] *School of Mathematics, Shandong University, Jinnan 250100, China*
[2] *Department of Mathematics and Physics, Shanghai Dianji University, Shanghai 201306, China*

Correspondence should be addressed to Jianming Qi, qijianmingdaxia@163.com

Received 11 October 2011; Accepted 29 November 2011

Academic Editor: Allan C Peterson

Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $P(f)$, $P(g)$ share ψ, then $\mathcal{F}$ is normal in D.

1. Introduction and Main Results

Let $\mathbb{C}$ be complex plane. Let D be a domain in $\mathbb{C}$. Let $\mathcal{F}$ be a family meromorphic functions defined in the domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

Let $f(z)$ and $g(z)$ be two meromorphic functions, let a be a finite complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros, then we say they share a or share a IM (ignoring multiplicity) (see [1–3]).

Definition 1.1. Let $a_i(z)$, $(i = 1, 2, \ldots, q-1)$, $b_j(z)$, $(j = 1, 2, \ldots, n)$ be analytic in D, let $n_0, n_1, \ldots, n_k$ be nonnegative integers, set

$$\begin{gathered} P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega, \\ M\left(f, f', \ldots, f^{(k)}\right) = f^{n_0}\left(f'\right)^{n_1} \cdots \left(f^{(k)}\right)^{n_k}, \\ \gamma_M = n_0 + n_1 + \cdots + n_k, \\ \Gamma_M = n_0 + 2n_1 + \cdots + (k+1)n_k, \end{gathered} \tag{1.1}$$

where $M(f, f', \ldots, f^{(k)})$ is called a differential monomial of f, γ_M the degree of $M(f, f', \ldots, f^{(k)})$, and Γ_M the weight of $M(f, f', \ldots, f^{(k)})$.

From Definition 1.1, we give Definition 1.2.

Definition 1.2. Let $M_j(f, f', \ldots, f^{(k)})$, $(j = 1, 2, \ldots, n)$ be differential monomials of f. Set

$$\begin{gathered}H\left(f, f', \ldots, f^{(k)}\right) = b_1(z) M_1\left(f, f', \ldots, f^{(k)}\right) + \cdots + b_n(z) M_n\left(f, f', \ldots, f^{(k)}\right), \\ \gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\}, \\ \Gamma_H = \max\{\Gamma_{M_1}, \Gamma_{M_2}, \ldots, \Gamma_{M_n}\},\end{gathered} \tag{1.2}$$

where $H(f, f', \ldots, f^{(k)})$ is called the differential polynomial of f, γ_H the degree of $H(f, f', \ldots, f^{(k)})$, and Γ_H the weight of $H(f, f', \ldots, f^{(k)})$,

$$\begin{gathered}\left.\frac{\Gamma}{\gamma}\right|_H = \max\left\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \ldots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\right\}, \\ G(f) = P\left(f^{(k)}\right) + H\left(f, f', \ldots, f^{(k)}\right).\end{gathered} \tag{1.3}$$

In 1979, Gu [4] proved the following result.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let a be a nonzero constant. If, for each function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq a$ in D, then $\mathcal{F}$ is normal in D.*

Yang [5] and Schwick [6] proved that Theorem A still holds if a is replaced by a holomorphic function $\psi(\not\equiv 0)$ in Theorem A.

Xu [7] improved Theorem A by the ideas of shared values and obtained the following result.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$ be a holomorphic functions and with only simple zeros in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, f has all multiple poles and $f \neq 0$. If, for every pair of functions f and g, $f^{(k)}$ and $g^{(k)}$ share ψ in D, then $\mathcal{F}$ is normal in D.*

Recently, Xu [7] did not know whether the condition ψ has only simple zero in D and f has all multiple poles are necessary or not in Theorem B.

In 2007, Fang and Chang considered the case $a = 0$ in Theorem A. In this note, Fang and Chang [8] proved the following result.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, and let k be a positive integer, and let b be a nonzero complex number. If, for each $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $(k+2)/k$, then $\mathcal{F}$ is normal in D.*

Remark 1.3. The number $(k+2)/k$ is sharp, as is shown by the examples in [8].

In 2009, Xia and Xu [9] replaced the constant 1 by a function $\psi(z) \not\equiv 0$ in Theorem C. They obtained the following result.

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and all zeros of $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f - \psi(z)$ have multiplicity at least $(k+2)/k$. If, for $k = 1$, ψ has only zeros with multiplicities at most 2 and, for $k \geq 2$, ψ has only simple zeros, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Theorem D can be improved by the ideas of shared values. In this paper, we investigate the problem and obtain the following results.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $P(f)$ and $P(g)$ share ψ, then $\mathcal{F}$ is normal in D.*

By Theorem 1.4, we immediately deduce.

Corollary 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, , \ldots, a_{k-1}$ be holomorphic functions in D, and k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share ψ, then $\mathcal{F}$ is normal in D.*

Remark 1.6. By the ideas of sharing values, Theorem 1.4 and Corollary 1.5 yield the number $(k+2)/k$ can be omitted.

Remark 1.7. Obviously, Corollary 1.5 omitted the conditions ψ with only simple zeros, and, for every function $f \in \mathcal{F}$, f has all multiple poles in Theorem D. But the condition for every function $f \in \mathcal{F}$, $f^{(k)} \neq 0$ is additional. Hence, Corollary 1.5 improves Theorem B in some sense.

The condition $\psi \not\equiv 0$ in Theorem 1.4 is necessary. For example, we consider the following families.

Example 1.8. $\mathcal{F} = \{f_m(z) = e^{mz},\ m = 1, 2, \ldots\}$, obviously, any $f \in F$ satisfies $f \neq 0$, $f^{(k)} \neq 0$. For distinct positive integers m, l, $f_m^{(k)}$, and $f_l^{(k)}$ share 0 IM. However, the families $\mathcal{F}$ are not normal at $z = 0$.

Remark 1.9. Some ideas of this paper are based on [7, 9, 10].

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version, which is due to Pang and Zaclman [11].

Lemma 2.1 (see [11, 12]). *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ with the property that, for each $f \in \mathcal{F}$, all zeros are of multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f = 0$. If $\mathcal{F}$ is not normal in Δ, then, for $0 \leq \alpha \leq k$, there exist*

(1) *a number $r \in (0, 1)$;*

(2) *a sequence of complex numbers z_n, $|z_n| < r$;*

(3) *a sequence of functions* $f_n \in \mathcal{F}$;

(4) *a sequence of positive numbers* $\rho_n \to 0^+$;

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ *converge locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function* $g(\xi)$ *on* $\mathbb{C}$, *and, moreover, the zeros of* $g(\xi)$ *are of multiplicity at least* k, $g^\sharp(\xi) \le g^\sharp(0) = kA + 1$. *In particular,* g *has order at most 2.*

Here, as usual, $g^\sharp(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

Lemma 2.2 (see [1]). *Let* $f(z)$ *be a transcendental meromorphic function in* $\mathbb{C}$, *let* $k(\ge 1)$ *be a integer, and let* b *be a nonzero finite value, then* f *or* $f^{(k)} - b$ *has infinite zeros.*

Lemma 2.3 (see [7]). *Let* $f(z)$ *be a nonconstant rational function. Let* $k \ge 1$ *be an integer, and let* b *be a non-zero finite value. If* $f \neq 0$, *then* $f^{(k)}(z) - b$ *has at least two distinct zeros in the plane.*

Lemma 2.4. *Let* $f(z)$ *be a nonconstant rational function. Let* $k \ge 1$ *be an integer, and let* l *be a positive integer. If* $f \neq 0$, $f^{(k)} \neq 0$, *then* $f^{(k)}(z) - z^l$ *has at least two distinct zeros in the plane.*

Proof. Since $f \neq 0$ and $f^{(k)} \neq 0$, then f is a nonpolynomial rational function and has the form

$$f(z) = \frac{A}{(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_t)^{m_t}}, \tag{2.1}$$

where $A \neq 0$ is a constant, and $m_1, m_2, \dots, m_t$ are positive integers. Set $m = m_1 + m_2 + \cdots + m_t$. Then,

$$f'(z) = \frac{-A\left(mz^{t-1} + b_{t-2}z^{t-2} + \cdots + b_0\right)}{(z - z_1)^{m_1+1}(z - z_2)^{m_2+1} \cdots (z - z_t)^{m_t+1}}, \tag{2.2}$$

where $b_{t-2}, \dots, b_0$ are constants. For $k \ge 2$, by mathematical induction, we have

$$f^{(k)}(z) = \frac{Bz^{kt-k} + c_{kt-k-1}z^{kt-k-1} + \cdots + c_0}{(z - z_1)^{m_1+k}(z - z_2)^{m_2+k} \cdots (z - z_t)^{m_t+k}}, \tag{2.3}$$

where $B = (-1)^k m(m+1)(m+2)\cdots(m+k-1)A \neq 0$, $c_{kt-k-1}, \dots, c_0$ are constants. Since $f^{(k)} \neq 0$, we deduce that $t = 1$, and thus

$$f(z) = \frac{A}{(z - z_1)^{m_1}}, \tag{2.4}$$

$$f^{(k)}(z) = \frac{B}{(z - z_1)^{m_1+k}}. \tag{2.5}$$

Case 1 (if $f^{(k)} - z^l$ has exactly one zero z_0). From (2.5), we set

$$f^{(k)}(z) - z^l = \frac{B}{(z - z_1)^{m_1+k}} - z^l = \frac{B'(z - z_0)^{m_1+k+l}}{(z - z_1)^{m_1+k}}. \tag{2.6}$$

Obviously, B' is a nonzero constant and $l \ge 1$.

From (2.6), we obtain

$$f^{(k+l+1)}(z) = \frac{(z-z_0)^{m_1+k-1} P_1(z)}{(z-z_1)^{m_1+k+l+1}}, \tag{2.7}$$

where $P_1(z) \not\equiv 0$. By (2.4), we deduce

$$f^{(k+l+1)}(z) = \frac{A'}{(z-z_1)^{m_1+k+1+1}}, \tag{2.8}$$

where A' is nonzero constant.

Comparing (2.7) and (2.8), we obtain that $\deg A' = 0 \geq m_1 + k - 1$ is impossible.

Case 2 (if $f^{(k)}(z) - z^l \neq 0$). By (2.5), clearly Case 2 is impossible.

Lemma 2.4 is proved. □

Lemma 2.5 (see [7]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let $\varphi(\neq 0)$ be a holomorphic function in D. If, for any $f \in \mathcal{F}$ satisfying $f \neq 0$ and if $f^{(k)}, g^{(k)}$ share φ IM for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In this paper, by the same method of [7], we consider the differential polynomial in Lemma 2.5 and prove a more general result.

Lemma 2.6. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let $\varphi(\neq 0)$ be a holomorphic function in D. If, for any $f \in \mathcal{F}$ satisfying $f \neq 0$ and if $G(f), G(g)$ share φ IM for every pair of functions $f, g \in \mathcal{F}$, where $G(f)$ is a differential polynomial of f as the definition 1 satisfying $q \geq \gamma_H$, and $\Gamma/\gamma|_H < k+1$, then $\mathcal{F}$ is normal in D, where q, $\Gamma/\gamma|_H$ are as in Definitions 1.1 and 1.2.*

Proof. We may assume that $D = \Delta = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exists a number $r \in (0,1)$; a sequence of complex numbers z_j, $z_j \to 0$ $(j \to \infty)$; a sequence of functions $f_j \in \mathcal{F}$; a sequence of positive numbers $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi)$ converges uniformly with respect to the spherical metric to a nonconstant meromorphic functions $\overline{g}(\xi)$ in C. Moreover, $\overline{g}(\xi)$ is of order at most 2. Hurwitz's theorem implies that $\overline{g}(\xi) \neq 0$.

We have

$$\begin{aligned} G(f_j)(z_j + \rho_j\zeta) &= P\Big(f_j^{(k)}(z_j + \rho_j\zeta)\Big) + H\Big(f_j, f_j', \ldots, f_j^{(k)}\Big)(z_j + \rho_j\zeta), \\ H\Big(f_j, f_j', \ldots, f_j^{(k)}\Big)(z_j + \rho_j\zeta) &= \sum_{i=1}^{n} b_i(z_j + \rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i\Big(g_j, g_j', \ldots, g_j^{(k)}\Big)(\zeta). \end{aligned} \tag{2.9}$$

Considering $b_i(z)$ is analytic on $D(i = 1, 2, \ldots, n)$, we have

$$\left|b_i(z_j + \rho_j\zeta)\right| \leq M\left(\frac{1+r}{2}, b_i(z)\right) < \infty, \quad (i = 1, 2, \ldots, n) \tag{2.10}$$

for sufficiently large j.

Hence, we deduce from $\Gamma/\gamma|_H < k+1$ that

$$\sum_{i=1}^{n} b_i(z_j + \rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} M_i\left(g_j, g_j', \ldots, g_j^{(k)}\right)(\zeta) \tag{2.11}$$

converges uniformly to 0 on every compact subset of $\mathbb{C}$ which contains no poles of $\overline{g}(\xi)$.

Thus, we have

$$\begin{gathered} G(f_j)(z_j + \rho_j\zeta) \longrightarrow P\left(\overline{g}^{(k)}\right)(\zeta), \\ G(f_j)(z_j + \rho_j\zeta) - \psi(z_j + \rho_j\zeta) \longrightarrow P\left(\overline{g}^{(k)}\right)(\zeta) - \psi(z_0) \end{gathered} \tag{2.12}$$

on every compact subset of $\mathbb{C}$ which contains no poles of $\overline{g}(\zeta)$.

Next, we will prove that $G(f_j)(\zeta) - \psi(z_0)$ has just a unique zero. By way of contradiction, let ζ_0 and ζ_0^* be two distinct solutions of $G(f_j)(\zeta) - \psi(z_0)$, and choose $\delta(>0)$ small enough such that $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$ where $D(\zeta_0, \delta) = \{\zeta : |\zeta - \zeta_0| < \delta\}$ and $D(\zeta_0^*, \delta) = \{\zeta : |\zeta - \zeta_0^*| < \delta\}$. By Hurwitz's theorem, there exist points $\zeta_j \in D(\zeta_0, \delta)$, $\zeta_j^* \in D(\zeta_0^*, \delta)$ such that, for sufficiently large j,

$$\begin{gathered} G(f_j)(z_j + \rho_j\zeta_j) - \psi(z_0) = 0, \\ G(f_j)\left(z_j + \rho_j\zeta_j^*\right) - \psi(z_0) = 0. \end{gathered} \tag{2.13}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $G(f)$ and $G(g)$ share $\psi(z_0)$ in D, we know that, for any positive integer m,

$$\begin{gathered} G(f_m)(z_j + \rho_j\zeta_j) - \psi(z_0) = 0, \\ G(f_m)\left(z_j + \rho_j\zeta_j^*\right) - \psi(z_0) = 0. \end{gathered} \tag{2.14}$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\zeta_j \to 0$, $z_j + \rho_j\zeta_j^* \to 0$, then

$$G(f_m)(0) - \psi(z_0) = 0. \tag{2.15}$$

Since the zeros of $G(f_m)(0) - \psi(z_0) = 0$ have no accumulation point, so $z_j + \rho_j\zeta_j = 0$, $z_j + \rho_j\zeta_j^* = 0$.

Hence,

$$\zeta_j = -\frac{z_j}{\rho_j}, \qquad \zeta_j^* = -\frac{z_j}{\rho_j}. \tag{2.16}$$

This contradicts with $\zeta_j \in D(\zeta_0, \delta)$, $\zeta_j^* \in D(\zeta_0^*, \delta)$, and $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$. So $G(f_j) - \psi(z_0)$ has just a unique zero. By Hurwitz's theorem, we know $P(\overline{g}^{(k)})(\zeta) - \psi(z_0)$ has just a unique zero.

By Lemmas 2.2 and 2.3, we know $\overline{g}^{(k)}(\zeta) - \psi(z_0)$ has at least two distinct zeros. From the definition of $P(w)$, we deduce that $P(\overline{g}^{(k)}(\zeta)) - \psi(z_0)$ has more than two distinct zeros, a contradiction.

So $\mathcal{F}$ is normal in D. Lemma 2.6 is proved. □

By Lemma 2.6, we immediately deduce the following lemma.

Lemma 2.7. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\neq 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f$, $g^{(k)} + a_{k-1}g^{(k-1)} + \cdots + a_1 g' + a_0 g$ share ψ, then $\mathcal{F}$ is normal in D.*

Lemma 2.8 (see [1]). *Let $f(z)$ be a meromorphic function. Let k be a positive integer. If $f(z) \neq 0$, then $f^{(k)}(z) \neq 1$, then f is a constant.*

Lemma 2.9 (see [13, 14]). *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$, and let $P(\not\equiv 0)$ be a polynomial. Let k be a positive integer. If all zeros (except at most finite zeros) of $f(z)$ have the multiplicity at least 3, then $f^{(k)}(z) - P(z)$ has infinite zeros.*

3. Proof of Theorem 1.4

Proof. Since normality is a local property, without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and

$$\psi(z) = z^l \varphi(z) \quad (z \in \Delta), \tag{3.1}$$

where l is a positive integer, $\varphi(0) = 1$, $\varphi(z) \neq 0$ on $\Delta' = \{z : 0 < |z| < 1\}$. By Lemma 2.6, we only need to prove that $\mathcal{F}$ is normal at $z = 0$.

If $f \in \mathcal{F}$, $P(f)(0) \neq \psi(0)$, then there exists $\delta > 0$ such that $P(f)(z) \neq \psi(z)$ on Δ_δ. By condition of Theorem, for every $g \in \mathcal{F}$, we know $P(g)(z) \neq \psi(z)$ on Δ_δ. By theorem D, $\mathcal{F}$ is normal on Δ_δ, so $\mathcal{F}$ is normal on $z = 0$.

Now, we consider $P(f)(0) = \psi(0)$. Suppose $P(f)(z) \not\equiv \psi(z)$ on the neighborhood $|z| < \delta$ (where δ is a small positive number) (otherwise, $P(f)(z) \equiv \psi(z)$ on the neighborhood $|z| < \delta$, by condition of theorem, for every $g \in \mathcal{F}$, we also obtain $P(g)(z) \equiv \psi(z)$. So $P(g)(z) \neq \psi(z) + 1$. By Theorem D, $\mathcal{F}$ is normal at $z = 0$. So Theorem 1.4 is proved), there exists $\delta > 0$ such that $P(f)(z) \neq \psi(z)$ on $(z \in \Delta'_\delta)$. So, for every $g \in \mathcal{F}$, we obtain

$$P(g)(z) \neq \psi(z) \quad (z \in \Delta'_\delta). \tag{3.2}$$

By Theorem D, $\mathcal{F}$ is normal on Δ'.

Next, we will prove $\mathcal{F}$ is normal at $z = 0$. Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z = 0 \in \Delta$, then there exists a sequence functions (we also denote $\mathcal{F}$) that has no any normal subsequence on $z = 0$.

Consider the family $\mathfrak{I} = \{g(z) = (f(z)/\varphi(z)) : f \in \mathcal{F}, z \in \Delta\}$. Since $f \neq 0$ for $f \in \mathcal{F}$, we have that $g(0) = \infty$ for each $g \in \mathfrak{I}$.

We first prove that $\mathfrak{I}$ is normal in Δ. Suppose, on the contrary, that $\mathfrak{I}$ is not normal at $z_0 \in \Delta$. By Lemma 2.1, there exist a sequence of functions $g_n \in \mathfrak{I}$, a sequence of complex numbers $z_n \to z_0$, and a sequence of positive numbers $\rho_n \to 0$, such that

$$G_n(\xi) = \frac{g_n(z_n + \rho_n\xi)}{\rho_n^k} \longrightarrow G(\xi) \tag{3.3}$$

converges spherically uniformly on compact subsets of $\mathbb{C}$ where $G(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, and $G(\xi) \neq 0$.

We distinguish two cases.

Case 1 ($z_n/\rho_n \to \infty$). By a simple calculation, for $0 \leq i \leq k$, we have

$$\begin{aligned} g_n^{(i)}(z) &= \frac{f_n^{(i)}(z)}{\varphi(z)} - \sum_{j=1}^{i} C_i^j g_n^{(i-j)}(z) \frac{\varphi^{(j)}(z)}{\varphi(z)} \\ &= \frac{f_n^{(i)}(z)}{\varphi(z)} - \sum_{j=1}^{i} \left[C_i^j g_n^{(i-j)}(z) \sum_{t=0}^{j} A_{jt} \frac{1}{z^{j-t}} \frac{\varphi^{(t)}(z)}{\varphi(z)} \right], \end{aligned} \tag{3.4}$$

where $A_{jt} = l(l-1)\cdots(l-j+t+1)C_j^t$ if $l < j$, for $t = 0, 1, \ldots, j-1$ and $A_{jj} = 1$.

Thus, from (3.4), we have

$$\begin{aligned} \rho_n^{k-i} G_n^{(i)}(\xi) &= g_n^{(i)}(z_n + \rho_n\xi) \\ &= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} - \sum_{j=1}^{i} \left[C_i^j g_n^{(i-j)}(z_n + \rho_n\xi) \sum_{t=0}^{j} A_{jt} \frac{1}{(z_n + \rho_n\xi)^{j-t}} \frac{\varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} \right] \\ &= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} - \sum_{j=1}^{i} \left[C_i^j \frac{g_n^{(i-j)}}{\rho_n^j}(z_n + \rho_n\xi) \sum_{t=0}^{j} A_{jt} \frac{1}{(z_n + \rho_n\xi)^{j-t}} \frac{\rho_n^t \varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} \right]. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{(z_n/\rho_n) + \xi} &= 0, \\ \lim_{n\to\infty} \frac{\rho_n^t \varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} &= 0, \end{aligned} \tag{3.6}$$

for $t \geq 1$. Noting that $g_n^{(i-j)}(z_n + \rho_n\xi)/\rho_n^j$ is locally bounded on $\mathbb{C}$ minus the set of poles of $G(\xi)$ since $g_n(z_n + \rho_n\xi)/\rho_n^k \to G(\xi)$. Therefore, on every subset of $\mathbb{C}$ which contains no poles of $G(\xi)$, we have

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi),$$
$$\frac{f_n^{(i)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow 0, \tag{3.7}$$

for $i=0,1,\ldots,k-1$, and thus

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)+\Sigma_{i=0}^{k-1}a_i(z_n+\rho_n\xi)f_n^{(i)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi),$$
$$\frac{f_n^{(k)}(z_n+\rho_n\xi)+\Sigma_{i=0}^{k-1}a_i(z_n+\rho_n\xi)f_n^{(i)}(z_n+\rho_n\xi)-\psi(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi)-1, \tag{3.8}$$

since $a_0,\ldots,a_{k-1}$ are analytic in D.

By $G(\xi)\neq 0$, we know $G^{(k)}(\xi)\not\equiv 1$. In fact, if $G^{(k)}(\xi_0)=1$, by Hurwitz's theorem, then exists $\xi_n \to \xi_0$, for n sufficiently large,

$$P(f)(z_n+\rho_n\xi_n)=\psi(z_n+\rho_n\xi_n). \tag{3.9}$$

By the condition of theorem, for every positive number m, we obtain $P(f_m)(z_n+\rho_n\xi_n)=\psi(z_n+\rho_n\xi_n)$. We know $z_n+\rho_n\xi_n \to z_0\in\Delta_\delta$, and, for sufficiently large n, $z_n+\rho_n\xi_n\in\Delta_\delta$. However, $z_n+\rho_n\xi_n\neq 0$ (otherwise, $z_n+\rho_n\xi_n=0$, so $\xi_n=-(z_n/\rho_n)\to\infty$, a contradiction), so for sufficiently large n, $z_n+\rho_n\xi_n\in\Delta'_\delta$. This contradicts with (3.2).

So $G(\xi)\neq 0$ and $G^{(k)}(\xi)\neq 1$, by Lemma 2.8, we obtain G is a constant, a contradiction.

Case 2. $z_n/\rho_n \to \alpha$ is a finite complex number. Then,

$$\frac{g_n(\rho_n\xi)}{\rho_n^k}=\frac{g_n(z_n+\rho_n(\xi-(z_n/\rho_n)))}{\rho_n^k}=G_n\left(\xi-\frac{z_n}{\rho_n}\right)\longrightarrow G(\xi-\alpha)=\mathbb{G}(\xi). \tag{3.10}$$

Obviously, $\mathbb{G}(\xi)\neq 0$, and $\xi=0$ is a pole of $\mathbb{G}$ with order at least l.

Set

$$H_n(\xi)=\frac{f_n(\rho_n\xi)}{\rho_n^{k+l}}. \tag{3.11}$$

Then,

$$H_n(\xi)=\frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{f_n(\rho_n\xi)}{\rho_n^k\psi(\rho_n\xi)}=\frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{g_n(\rho_n\xi)}{\rho_n^k}. \tag{3.12}$$

Noting that $\psi(\rho_n\xi)/\rho_n^l \to \xi^l$, thus

$$H_n(\xi) \longrightarrow \xi^l \mathbb{G}(\xi) = H(\xi), \tag{3.13}$$

uniformly on compact subsets of $\mathbb{C}$. Since $\mathbb{G}$ has a pole of order at least at $\xi = 0$, we have $H(0) \neq 0$, so that $H(\xi) \neq 0$.

From (3.11), we get

$$H_n^{(i)} = \frac{f_n^{(i)}(\rho_n\xi)}{\rho_n^{k+l-i}} \longrightarrow H^{(i)}(\xi), \tag{3.14}$$

spherically uniformly on compact subsets of $\mathbb{C}$ minus the set of poles of $\mathbb{G}(\xi)$. As the above, on every compact subset of $\mathbb{C}$ minus the set of poles of $G(\xi)$, we have

$$\frac{f_n^{(k)}(\rho_n\xi) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi) f_n^{(i)}(\rho_n\xi)}{\rho_n^l} \longrightarrow H^{(k)}(\xi), \tag{3.15}$$

$$\frac{f_n^{(k)}(\rho_n\xi) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi) f_n^{(i)}(\rho_n\xi) - \psi(\rho_n\xi)}{\rho_n^l} \longrightarrow H^{(k)}(\xi) - \xi^l, \tag{3.16}$$

locally uniformly on $\mathbb{C}$.

By the assumption of Theorem and (3.16), Hurwitz's theorem implies $H^{(k)}(\xi) \neq 0$.

Next, we proof that if $\xi \in \mathbb{C}/\{0\}$, then $H^{(k)}(\xi) \neq \xi^l$.

First, $H^{(k)}(\xi) \not\equiv \xi^l$, otherwise $H^{(k)}(\xi) \equiv \xi^l$, which contradicts with $H(\xi) \neq 0$. If there exists a $\xi_0 \neq 0$ such that $H^{(k)}(\xi_0) = \xi_0^l$, by Hurwitz's theorem and (3.16), there exists $\xi_n \to \xi_0$ such that $f_n^{(k)}(\rho_n\xi_n) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi_n) f_n^{(i)}(\rho_n\xi_n) = \psi(\rho_n\xi_n)$. By the assumption of Theorem 1.4, for every positive m such that $P(f_m)(\rho_n\xi_n) = \psi(\rho_n\xi_n)$. However, for n sufficiently large, $\rho_n\xi_n \in \Delta'_\delta$, all of these contradict with (3.2). So if $\xi \in \mathbb{C}/\{0\}$, then $H^{(k)}(\xi) \neq \xi^l$.

Noting $H(\xi) \neq 0$, By Lemma 2.9, we know H must be a rational function. If H is not a constant, By Lemma 2.4, we know $H^{(k)}(\xi) - \xi^l$ has at least two distinct zeros, a contradiction. So H must be a nonzero constant, also contradicts with $H^{(k)}(\xi) \neq 0$. Now, we have proved the $\mathfrak{I}$ is normal on Δ_δ.

It remains to show that $\mathcal{F}$ is normal at $z = 0$. Since $\mathfrak{I}$ is normal in Δ, then the family $\mathfrak{I}$ is equicontinuous on Δ with respect to the spherical distance. On the other hand, $g(0) = \infty$ for each $g \in \mathfrak{I}$, so there exists $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in \mathfrak{I}$ and each $z \in \Delta_\delta = \{z : |z| < \delta\}$. Suppose that $\mathcal{F}$ is not normal at $z = 0$. Since $\mathcal{F}$ is normal in $0 < |z| < 1$, the family $\mathcal{F}_1 = \{1/f : f \in \mathfrak{I}\}$ is normal in $\Delta = \{z : 0 < |z| < 1\}$, but it is not normal at $z = 0$. Then, there exists a sequence $\{1/f_n\} \subset \mathcal{F}_1$ which converges locally uniformly in Δ', but not in Δ. Noting that $f_n \neq 0$ in Δ, $1/f_n$ is holomorphic in Δ for each n. The maximum modulus principle implies that $1/f_n \to \infty$ in Δ'. Thus, $f_n \to 0$ converges locally uniformly in Δ', and hence so does $\{g_n\} \subset \mathfrak{I}$, where $g_n = f_n/\varphi$. But $|g_n(z)| \geq 1$ for each $z \in \Delta_\delta$, a contradiction. This finally completes the proof of Theorem 1.4. □

Acknowledgments

This paper is supported by Leading Academic Discipline Project 10XKJ01, by Key Development Project 12C102 of Shanghai Dianji University, by the National Natural Science Foundation of China (11171184), and by National Natural Science Youth Fund Project (51008190).

References

[1] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, UK, 1964.
[2] J. L. Schiff, *Normal Families*, Springer, New York, NY, USA, 1993.
[3] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.
[4] Y. X. Gu, "A normal criterion of mermorphic families," *Scientia Sinica*, vol. 1, pp. 267–274, 1979.
[5] L. Yang, "Normal families and fixed-points of meromorphic functions," *Indiana University Mathematics*, vol. 192, pp. 225–294, 2004.
[6] W. Schwick, "Exceptional functions and normality," *The Bulletin of the London Mathematical Society*, vol. 29, no. 4, pp. 425–432, 1997.
[7] Y. Xu, "On a result due to Yang and Schwick," *Science China Mathematics A*, vol. 40, no. 5, pp. 421–428, 2010 (Chinese).
[8] M. L. Fang and J. M. Chang, "Normal families and multiple values," *Archiv der Mathematik*, vol. 88, no. 6, pp. 560–568, 2007.
[9] J. Y. Xia and Y. Xu, "Normal families of meromorphic functions with multiple values," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 387–393, 2009.
[10] Y. T. Li and Y. X. Gu, "On normal families of meromorphic functions," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 2, pp. 421–425, 2009.
[11] X. Pang and L. Zalcman, "Normal families and shared values," *The Bulletin of the London Mathematical Society*, vol. 32, no. 3, pp. 325–331, 2000.
[12] L. Zalcman, "Normal families: new perspectives," *Bulletin of the American Mathematical Society*, vol. 35, no. 3, pp. 215–230, 1998.
[13] W. Bergweiler and X. Pang, "On the derivative of meromorphic functions with multiple zeros," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 285–292, 2003.
[14] Y. Xu, "Picard values and derivatives of meromorphic functions," *Kodai Mathematical Journal*, vol. 28, no. 1, pp. 99–105, 2005.