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Research Article

Remarks on Confidence Intervals for Self-Similarity Parameter of a Subfractional Brownian Motion

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We first present two convergence results about the second-order quadratic variations of the subfractional Brownian motion: the first is a deterministic asymptotic expansion; the second is a central limit theorem. Next we combine these results and concentration inequalities to build confidence intervals for the self-similarity parameter associated with one-dimensional subfractional Brownian motion.

1. Introduction

A fundamental assumption in many statistical and stochastic models is that of independent observations. Moreover, many models that do not make the assumption have the convenient Markov property, according to which the future of the system is not affected by its previous states but only by the current one.

The *long-range dependence* property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing, and finance. The best known and most widely used process that exhibits the long-range dependence property is *fractional Brownian motion* (fBm in short). The fBm is a suitable generalization of the standard Brownian motion. The reader is referred, for example, to Alòs et al. [1] and Nualart [2] for a comprehensive introduction to fractional Brownian motion. On the other hand, many authors have proposed to use more general self-similar

Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes.

As a generalization of Brownian motion, recently, Bojdecki et al. [3, 4] introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fractional Brownian motion. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition, which is called the *subfractional Brownian motion*. The so-called subfractional Brownian motion (sub-fBm in short) with index $H \in (0,1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \geq 0\}$ with $S_0^H = 0$ and the covariance

$$R(t,s) \equiv E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right], \tag{1.1}$$

for all $s, t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion. S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths) and, for $s \leq t$, satisfies the following estimates:

$$\left[\left(2-2^{2H-1}\right) \wedge 1\right](t-s)^{2H} \leq E\left[\left(S_t^H - S_s^H\right)^2\right] \leq \left[\left(2-2^{2H-1}\right) \vee 1\right](t-s)^{2H}. \tag{1.2}$$

Thus, Kolmogorov continuity criterion implies that the subfractional Brownian motion is Hölder continuous of order ν for any $\nu < H$. But its increments are not stationary. More works for sub-fBm can be found in Bojdecki et al. [3, 4], Liu and Yan [5, 6], Liu [7], Tudor [8–12], Yan and Shen [13, 14], and others.

The problem of the statistical estimation of the self-similarity parameter is of great importance. The self-similarity parameter characterizes all of the important properties of the self-similar processes and consequently describes the behavior of the underlying physical system. Therefore, properly estimating them is of the utmost importance. Several statistics have been introduced to this end, such as wavelets, k-variations, variograms, maximum likelihood estimators, and spectral methods. This issue has generated a vast literature. See Chronopoulou et al. [15, 16], Liu [7], Tudor and Viens [17, 18], and references therein for more details. Recently, Breton et al. [19] firstly obtained the *nonasymptotic* construction of confidence intervals for the Hurst parameter H of fractional Brownian motion. Observe that the knowledge of explicit nonasymptotic confidence intervals may be of great practical value, for instance in order to evaluate the accuracy of a given estimation of H when only a fixed number of observations are available.

Motivated by all these results, in the present note, we will construct the confidence intervals for the self-similarity parameter associated with the so-called subfractional Brownian motion. It is well known that, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons are the complexity of dependence structures and the nonavailability of convenient stochastic integral representations for self-similar Gaussian processes which do not have stationary increments. As we know, in comparison with fractional Brownian motion, the subfractional Brownian motion has nonstationary increments, and the increments over nonoverlapping intervals are more weakly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki et al. [3] it is called subfractional Brownian motion). The above-mentioned properties make subfractional Brownian motion a possible candidate for models which involve long-range dependence, self-similarity, and nonstationary increments. Therefore, it seems interesting to construct the confidence intervals of self-similar parameter of subfractional Brownian motion. And we need more precise estimates to prove our results because of the nonstationary increments.

The first aim of this note is to prove a deterministic asymptotic expansion and a central limit theorem of the so-called second-order quadratic variation $V_n(S^H)$ which is defined by

$$V_n\left(S^H\right) = \sum_{k=0}^{n-1}\left(S^H_{(k+2)/n} + S^H_{k/n} - 2S^H_{(k+1)/n}\right)^2, \quad n \geq 1, \tag{1.3}$$

because the standard quadratic variation does not satisfy a central limit theorem in general. The second aim is to exploit the concentration inequality proved by Nourdin and Viens [20] in order to derive an exact (i.e., nonasymptotic) confidence interval for the self-similar parameter of subfractional Brownian motion S^H. Our formula hinges on the class of statistics $V_n(S^H)$ and $Z_n = n^{2H-1/2}V_n - \sqrt{n}(4 - 2^{2H})$.

This note is organized as follows. In Section 2 we present some preliminaries for concentration inequality and two convergence results about the quadratic variations of some Gaussian processes. In Section 3 we prove the asymptotic expansion and central limit theorem for the second-order quadratic variations of subfractional Brownian motion with $H \in (0,1)$. In Section 4 we state and prove the main result of this note.

Notation. Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C or c, which may not be the same in each occurrence. Sometimes we will emphasize the dependence of these constants upon parameters.

2. Preliminaries

Consider a finite centered Gaussian family $X = \{X_k : k = 0, \ldots, M\}$ and write $r(k,l) = E(X_k X_l)$. In what follows, we will consider two quadratic forms associated with X and with some real coefficient c. The first is obtained by summing up the squares of the elements of X and by subtracting the corresponding variances

$$Q_1(c, X) = c\sum_{k=0}^{M}\left(X_k^2 - r(k,k)\right); \tag{2.1}$$

the second quadratic form is

$$Q_2(c, X) = 2c^2 \sum_{k,l=0}^{M} X_k X_l r(k,l). \tag{2.2}$$

The following result, whose proof relies on the Malliavin calculus techniques developed in Nourdin and Peccati [21], Nourdin and Viens [20], characterizes the tail behavior of $Q_1(c, X)$.

Theorem 2.1 (Theorem 2.1 in Breton et al. [19]). *If the above assumptions are satisfied, suppose that $Q_1(c, X)$ is not a.s. zero and fix $\alpha > 0$ and $\beta > 0$. Assume that $Q_2(c, X) \leq \alpha Q_1(c, X) + \beta$, a.s.-P. Then, for all $z > 0$, one has*

$$P(Q_1(c,X) \geq z) \leq \exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right), \qquad P(Q_1(c,X) \leq -z) \leq \exp\left(-\frac{z^2}{2\beta}\right), \tag{2.3}$$

in particular,

$$P(|Q_1(c,X)| \geq z) \leq 2\exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right). \tag{2.4}$$

On the other hand, to be sure that the second-order quadratic variation $V_n(S^H)$ converges almost surely to a deterministic limit, we need to normalize this quantity. A result of the form

$$\lim_{n\to\infty} n^{1-\theta} V_n\left(S^H\right) = \int_0^1 f(u)du, \text{ a.s.} \tag{2.5}$$

is expected, where θ is related to the regularity of the paths of the subfractional Brownian motion S^H and f is related to the nondifferentiability of r on the diagonal $\{s = t\}$ and is called the singularity function of the process. Begyn [22] considered a class of processes for which a more general normalization is needed. Moreover, he presented a better result about the asymptotic expansion of the left hand of (2.5) and proved a central limit theorem. Because Theorems 1 and 2 in Begyn [22] are crucial in the proofs of Theorems 3.1 and 3.2, it is useful to recall the results.

We define the second-order increments of the covariance function R of a Gaussian process X as follows:

$$\begin{aligned} \delta_1^h R(s,t) &= R(s+h,t) + R(s-h,t) - 2R(s,t), \\ \delta_2^h R(s,t) &= R(s,t+h) + R(s,t-h) - 2R(s,t). \end{aligned} \tag{2.6}$$

First, we recall the result of asymptotic expansion of $V_n(X)$ under some certain conditions on the covariance function.

Theorem 2.2 (Theorem 1 in Begyn [22]). *Assume that the Gaussian process X satisfies the following statements.*

(1) *$t \longrightarrow M_t = \mathbb{E}X_t$ has a bounded first derivative in $[0, 1]$.*

(2) *The covariance function R of X has the following properties: (a) R is continuous in $[0,1]^2$. (b) The derivative $\partial^4 R/\partial s^2 \partial t^2$ exists and is continuous in $(0,1]^2/\{s = t\}$. There exists a constant $C > 0$, a real $\gamma \in (0, 2)$ and a positive slowly varying function $L : (0, 1) \longrightarrow (0, +\infty)$ such that*

$$\forall t, s \in \frac{(0,1]^2}{\{s=t\}}, \quad \left|\frac{\partial^4 R}{\partial s^2 \partial t^2}(s,t)\right| \leq C\frac{L(|s-t|)}{|s-t|^{2+\gamma}}. \tag{2.7}$$

(c) There exist $q+1$ functions ($q \in \mathbb{N}$) $g_0, g_1, \dots, g_q$ from $(0,1)$ to $\mathbb{R}$, q real numbers $0 < \nu_1 < \cdots < \nu_q$ and a function $\phi : (0,1) \to (0,\infty)$ such that (i) if $q \geq 1$, then for all $0 \leq i \leq q-1$, g_i is Lipschitz on $(0,1)$; (ii) g_q is bounded on $(0,1)$; (iii) one has

$$\sup_{h \leq t \leq 1-h} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(t,t)}{h^{2-\gamma} L(h)} - g_0(t) - \sum_{i=1}^{q} g_i(t) \phi(h)^{\nu_i} \right| = o\left(\phi(h)^{\nu_q}\right), \quad \text{as } h \longrightarrow 0+, \tag{2.8}$$

where the symbol "$\circ$" denotes the composition of functions and if $q = 0$, then $\sum_{i=1}^{q} g_i(t)\phi(h)^{\nu_i} = 0$ and $\phi(h)^{\nu_q} = 1$; else if $q \neq 0$, then $\lim_{h \to 0+} \phi(h) = 0$.

(3) *If $q \neq 0$, we assume that*

$$\lim_{n \to +\infty} \frac{\log n}{n \phi(1/n)^{\nu_q}} = 0. \tag{2.9}$$

(4) *If X is not centered, we make the additional assumption*

$$\lim_{n \to +\infty} \frac{1}{n^{\gamma} L(1/n) \phi(1/n)^{\nu_q}} = 0, \tag{2.10}$$

where if $q = 0$, then $\phi(1/n)^{\nu_q} = 1$.

Then, for all $t \in [0,1]$, one has almost surely

$$\lim_{n \to +\infty} \frac{n^{1-\gamma}}{L(1/n)} V_n(X) = \int_0^1 g_0(x)\,dx + \sum_{i=1}^{q} \left(\int_0^1 g_i(x)\,dx \right) \phi\left(\frac{1}{n}\right)^{\nu_i} + o\left(\phi\left(\frac{1}{n}\right)^{\nu_q} \right). \tag{2.11}$$

Second, let us recall the result of central limit theorem.

Theorem 2.3 (Theorem 2 in Begyn [22]). *Assume that the Gaussian process X is centered and satisfies the following statements.*

(1) *The covariance function R of X is continuous in $[0,1]^2$.*

(2) *Let $T = \{0 \leq t \leq s \leq 1\}$. We assume that the derivative $\partial^4 R / \partial s^2 \partial t^2$ exists in $(0,1]^2 / \{s = t\}$ and that there exists a continuous function $C : T \to \mathbb{R}$, a real $\gamma \in (0,2)$ and a positive slowly varying function $L : (0,1) \to \mathbb{R}$ such that*

$$\forall t, s \in \overline{T}, \quad \frac{(s-t)^{2+\gamma}}{L(s-t)} \frac{\partial^4 R}{\partial s^2 \partial t^2}(s,t) = C(s,t), \tag{2.12}$$

where $\overline{T}$ denotes the interior of T (i.e., $\overline{T} = \{0 < s < t < 1\}$).

(3) *We assume that there exist $q+1$ functions ($q \in \mathbb{N}$) $g_0, g_1, \dots, g_q$ from $(0,1)$ to $\mathbb{R}$, q real numbers $0 < \nu_1 < \cdots < \nu_q$ and a function $\phi : (0,1) \to (0,\infty)$ such that (a) if $q \geq 1$, then*

for all $0 \le i \le q-1$, g_i *is Lipschitz on* $(0,1)$*; (b)* g_q *is* $(1/2+\alpha_q)$*-Hölderian on* $(0,1)$ *with* $0 < \alpha_q \le 1/2$*; (c) there exists* $t \in (0,1)$ *such that* $g_0(t) \neq 0$*; (d) one has*

$$\lim_{h\to 0+} \frac{1}{\sqrt{h}} \left(\sup_{h \le t \le 1-h} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(t,t)}{h^{2-\gamma} L(h)} - g_0(t) - \sum_{i=1}^{q} g_i(t) \phi(h)^{\nu_i} \right| \right) = 0, \tag{2.13}$$

where if $q=0$*, then* $\sum_{i=1}^{q} g_i(t)\phi(h)^{\nu_i} = 0$ *and where if* $q \neq 0$*, then* $\lim_{h\to 0+}\phi(h) = 0$*. (e) there exists a bounded function* $\tilde{g} : (0,1) \longrightarrow \mathbb{R}$ *such that*

$$\lim_{h\to 0+} \sup_{h \le t \le 1-2h} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(t+h,t)}{h^{2-\gamma} L(h)} - \tilde{g}(t) \right| = 0. \tag{2.14}$$

Then one has

$$\sqrt{n}\left(\frac{n^{1-\gamma}}{L(1/n)} V_n(X) - \int_0^1 g_0(x)\,dx - \sum_{i=1}^{q} \left(\int_0^1 g_i(x)\,dx \right) \phi\left(\frac{1}{n}\right)^{\nu_i} \right) \longrightarrow \mathcal{N}\left(0, \sigma^2\right) \tag{2.15}$$

in distribution as n *tends to infinity where*

$$\sigma^2 = 2\int_0^1 g_0(x)^2\,dx + 4\int_0^1 \tilde{g}(x)^2\,dx + 4\left\|\rho_\gamma\right\|^2 \int_0^1 C(x,x)^2\,dx \tag{2.16}$$

and $\|\rho_\gamma\|^2 = \sum_{l=2}^{+\infty} \rho_\gamma(l)^2$ *with, if* $\gamma \neq 1$*,*

$$\rho_\gamma(l) = \frac{|l-2|^{2-\gamma} - 4|l-1|^{2-\gamma} + 6|l|^{2-\gamma} - 4|l+1|^{2-\gamma} + |l+2|^{2-\gamma}}{(\gamma-2)(\gamma-1)\gamma(\gamma+1)}; \tag{2.17}$$

if $\gamma = 1$*,*

$$\rho_1(l) = \frac{1}{2}\left(|l-2|\log|l-2| - 4|l-1|\log|l-1| + 6|l|\log|l| - 4|l+1|\log|l+1| + 2|l+2|\log|l+2|\right). \tag{2.18}$$

3. Asymptotic Expansion and Central Limit Theorem

In the following theorem the almost sure convergence of the second-order quadratic variations $V_n(S^H)$ is proved.

Theorem 3.1. *For all* $t \in [0,1]$*, one has almost surely*

$$\lim_{n\to\infty} n^{2H-1} V_n\left(S^H\right) = 4 - 2^{2H}. \tag{3.1}$$

Proof. It is clear that the derivative $(\partial^4/\partial s^2\partial t^2)R(s,t)$ exists on $(0,1]^2/\{s=t\}$. Moreover we can check that, for all $s,t\in(0,1]^2/\{s=t\}$,

$$\frac{\partial^4}{\partial s^2\partial t^2}R(s,t)=H(2H-1)(2H-2)(2H-3)\left[-|s-t|^{2H-4}-|s+t|^{2H-4}\right]. \tag{3.2}$$

Therefore the assumption 2(b) in Theorem 2.2 is satisfied with $L(H)=1$ and $\gamma=2-2H$.

For the assumption 2(b) in Theorem 2.2, standard computations yield

$$\frac{\left(\delta_1^h\circ\delta_2^h R\right)(t,t)}{h^H}=4-2^{2H}+\frac{\lambda_t(h)}{h^{2H}}, \tag{3.3}$$

with

$$\lambda_t(h)=\frac{1}{h^{2H}}\left[-2^{2H-1}(t+h)^{2H}-3\cdot 2^{2H}t^{2H}-2^{2H-1}(t-h)^{2H}+2(2t+h)^{2H}+2(2t-h)^{2H}\right], \tag{3.4}$$

and we can check $\lambda_t(0)=\lambda_t'(0)=\lambda_t''(0)=\lambda_t^{(3)}(0)=0$. So that Taylor formula yields

$$\lambda_t(h)=\int_0^h\frac{(h-x)^3}{3!}\lambda_t^{(4)}(x)dx,\quad \forall h\le t\le 1-h. \tag{3.5}$$

Therefore, we have

$$\sup_{h\le t\le 1-h}\sup_{0\le x\le h}\left|\lambda_t^{(4)}(x)\right|=\mathcal{O}(1),\quad \text{as } h\longrightarrow 0+, \tag{3.6}$$

which yields

$$\sup_{h\le t\le 1-h}\left|\frac{\left(\delta_1^h\circ\delta_2^h R\right)(t,t)}{h^H}-\left(4-2^{2H}\right)\right|=\mathcal{O}\left(h^{4-2H}\right),\quad \text{as } h\longrightarrow 0+. \tag{3.7}$$

Therefore, the assumption 2(c) in Theorem 2.2 is fulfilled with

$$g_0(t)=4-2^{2H}. \tag{3.8}$$

Consequently, we can apply Theorem 2.2 to $V_n(S^H)$ and obtain the desired result. □

Next we study the weak convergence.

Theorem 3.2. *One has the following weak convergence*

$$\sqrt{n}\left(n^{2H-1}V_n\left(S^H\right)-\left(4-2^{2H}\right)\right)\longrightarrow_{(\mathcal{L})}N\left(0,\sigma_H^2\right),\quad \textit{as } n\longrightarrow\infty, \tag{3.9}$$

where

$$\sigma_H^2 = 2\left(4-2^{2H}\right)^2 + \left(2^{2H+2}-7-3^{2H}\right)^2 + [2H(2H-1)(2H-2)(2H-3)]^2\|\rho_{2-2H}\|^2,$$
$$\|\rho_{2-2H}\|^2 = \sum_{l=2}^{\infty}\left(\frac{-|l-2|^{2H}+4|l-1|^{2H}-6|l|^{2H}+4|l+1|^{2H}-|l+2|^{2H}}{2H(2H-1)(2H-2)(2H-3)}\right)^2. \tag{3.10}$$

Proof. We apply Theorem 2.3 to $V_n(S^H)$. As in the proof of Theorem 3.1, we need only to show that the assumptions 2 and 3 in Theorem 2.3 are satisfied.

For assumption 2, the previous computation showed that, for all $s, t \in (0,1]^2/\{s=t\}$,

$$\frac{\partial^4}{\partial s^2 \partial t^2}R(s,t) = H(2H-1)(2H-2)(2H-3)\left[-|s-t|^{2H-4} - |s+t|^{2H-4}\right]. \tag{3.11}$$

Therefore

$$(s-t)^{4-2H}\frac{\partial^4}{\partial s^2 \partial t^2}R(s,t) = -H(2H-1)(2H-2)\cdot(2H-3)\left[1+(s-t)^{4-2H}(s+t)^{2H-4}\right] :\equiv C(s,t). \tag{3.12}$$

This means that the assumption 2 in Theorem 2.3 is satisfied with $L(H)=1$, $\gamma = 2-2H$ and $C(s,t)$ defined by (3.12).

For assumption 3 in Theorem 2.3, the expression (3.7) shows that the assumption 3 in Theorem 2.3 is fulfilled with $q=0$, $g_0(t) = 4-2^{2H}$ and $\alpha_0 = 1/2$. Moreover, one can check that

$$\frac{\left(\delta_1^h \circ \delta_2^h R\right)(t,t+h)}{h^H} = \frac{1}{2}\left(2^{2H+2}-3^{2H}-7\right) + \frac{\xi_t(h)}{h^{2H}}. \tag{3.13}$$

Using the same arguments as those used for $\lambda_t(h)$ in the previous proof, we obtain

$$\sup_{h\le t\le 1-h} |\xi_t(h)| = \mathcal{O}\left(h^4\right), \quad \text{as } h \longrightarrow 0+. \tag{3.14}$$

This shows that the assumption 3(e) in Theorem 2.3 is satisfied with

$$\tilde{g}(t) = \frac{1}{2}\left(2^{2H+2}-3^{2H}-7\right). \tag{3.15}$$

Consequently, we can apply Theorem 2.3 to $V_n(S^H)$ to obtain the desired result. □

4. Confidence Intervals

Let S^H is a subfractional Brownian motion with unknown Hurst parameter $H \in (0, H_*]$, with $H_* < 1/2$ known. The following result is the main finding of the present note.

Theorem 4.1. *For $V_n(S^H)$ defined in (1.3), fix $n \geq 0$ and a real number a such that $0 < a < (4 - 2^{2H_*})\sqrt{n}$. For $x \in (0,1)$, set $g_n(x) = x - \log(4 - 2^{2x})/2\log n$. Then, with probability at least*

$$\varphi(a) = \left[1 - 2\exp\left(-\frac{a^2}{4C_{H_*}(a/\sqrt{n} + 3 + C_{H_*}/n)}\right)\right]_+, \tag{4.1}$$

where C_{H_*} is a positive constant depending only on H_* and $[\cdot]_+$ stands for the positive part function; the unknown quantity $g_n(H)$ belongs to the following confidence intervals

$$\begin{aligned} I_n &= [I_l(n), I_r(n)] \\ &= \left[\frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log(1 - a/\sqrt{n}(4 - 2^{2H_*}))}{2\log n}, \frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log(1 + a/\sqrt{n}(4 - 2^{2H_*}))}{2\log n}\right]. \end{aligned} \tag{4.2}$$

Proof. The idea used here is essentially due to Breton et al. [19]. Define $X_n = \{X_{n,k};\ k = 0, 1, \ldots, n-1\}$, where

$$X_{n,k} = S^H_{(k+2)/n} + S^H_{k/n} - 2S^H_{(k+1)/n}. \tag{4.3}$$

One can prove by standard computations that the covariance structure of Gaussian family $X_{n,k}$ is described by the relation

$$E(X_{n,k}X_{n,l}) = \frac{1}{n^{2H}}\rho_H(k,l), \tag{4.4}$$

where

$$\begin{aligned} \rho_H(k,l) = &-\frac{1}{2}(k+l+4)^{2H} + 2(k+l+3)^{2H} - 3(k+l+2)^{2H} + 2(k+l+1)^{2H} - \frac{1}{2}(k+l)^{2H} \\ &- 3(k-l)^{2H} + 2(k-l+1)^{2H} - \frac{1}{2}(k-l+2)^{2H} + 2(k-l-1)^{2H} - \frac{1}{2}(k-l-2)^{2H}. \end{aligned} \tag{4.5}$$

Now let $Z_n = n^{2H-1/2}V_n - \sqrt{n}(4 - 2^{2H})$, where V_n is defined in (1.3). It is easy to see that

$$Z_n = Q_1\left(n^{2H-1/2}, X_n\right) + \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} a_{k,H}, \tag{4.6}$$

where

$$a_{k,H} = -\frac{1}{2}(2k+4)^{2H} + 2(2k+3)^{2H} - 3(2k+2)^{2H} + 2(2k+1)^{2H} - \frac{1}{2}(2k)^{2H}. \tag{4.7}$$

On the other hand

$$\begin{aligned}
Q_2\left(n^{2H-1/2}, X_n\right) &= 2n^{4H-1}\sum_{k,l=0}^{n-1} X_{n,k}X_{n,l}\rho_H(k,l) \\
&\le 2n^{2H-1}\sum_{k,l=0}^{n-1}\left|X_{n,k}\|X_{n,l}\|\rho_H(k,l)\right| \\
&\le n^{2H-1}\sum_{k,l=0}^{n-1}\left(|X_{n,k}|^2+|X_{n,l}|^2\right)\left|\rho_H(k,l)\right| \\
&= 2n^{2H-1}\sum_{k,l=0}^{n-1}|X_{n,k}|^2\left|\rho_H(k,l)\right| \\
&\le 2n^{2H-1}\sum_{k=0}^{n-1}|X_{n,k}|^2\left(\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\right) \\
&\le \frac{2}{\sqrt{n}}\left(\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\right)\left(Z_n+\left(4-2^{2H}\right)\sqrt{n}\right) \\
&\le \frac{2}{\sqrt{n}}\left(\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\right)\left(Z_n+3\sqrt{n}\right) \\
&= \frac{2}{\sqrt{n}}\left(\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\right)\left(Q_1\left(n^{2H-1/2}, X_n\right)+\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}a_{k,H}+3\sqrt{n}\right) \\
&\le \frac{2}{\sqrt{n}}\left(\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\right)\left(Q_1\left(n^{2H-1/2}, X_n\right)+\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}|a_{k,H}|+3\sqrt{n}\right) \\
&= \alpha_n Q_1\left(n^{2H-1/2}, X_n\right)+\beta_n,
\end{aligned} \tag{4.8}$$

with

$$\alpha_n = \frac{2}{\sqrt{n}}\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|, \qquad \beta = 2\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\left(3+\frac{1}{n}\sum_{k=0}^{n-1}|a_{k,H}|\right). \tag{4.9}$$

Since $Z_n \neq 0$, Theorem 2.1 yields

$$P(|Z_n| \ge a) \le 2\exp\left(-\frac{a^2}{4\sum_{i,j\in\mathbb{Z}}\left|\rho_H(i,j)\right|\left(a/\sqrt{n}+3+(1/n)\sum_{k=0}^{n-1}|a_{k,H}|\right)}\right). \tag{4.10}$$

Now let us find bounds on $\sum_{i,j\in\mathbb{Z}}|\rho_H(i,j)|$. Using

$$(1+x)^{\alpha}=1+\sum_{k=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k,\quad \text{for } -1<x<1. \tag{4.11}$$

We denote by

$$\rho_H(i,j)=\rho_{H,1}(i,j)+\rho_{H,2}(i,j), \tag{4.12}$$

where

$$\begin{aligned}\rho_{H,1}(i,j)&=-\frac{1}{2}(k+l+4)^{2H}+2(k+l+3)^{2H}-3(k+l+2)^{2H}+2(k+l+1)^{2H}-\frac{1}{2}(k+l)^{2H},\\ \rho_{H,2}(i,j)&=-3(k-l)^{2H}+2(k-l+1)^{2H}-\frac{1}{2}(k-l+2)^{2H}+2(k-l-1)^{2H}-\frac{1}{2}(k-l-2)^{2H}.\end{aligned} \tag{4.13}$$

The second term $\rho_{H,2}(i,j)$ has been bounded by Breton et al. [19]. They proved that

$$\sum_{i,j\in\mathbb{Z}}\rho_{H,2}(i,j)\le\frac{71}{4}. \tag{4.14}$$

Now let us bound the first term $\rho_{H,1}(i,j)$. We denote by

$$\rho_{H,1}(i,j)=\rho_{H,1}(r),\quad r=i+j. \tag{4.15}$$

We can write for any $r\ge 5$,

$$\begin{aligned}\rho_{H,1}(r)&=\frac{r^{2H}}{2}\left[-\left(1+\frac{4}{r}\right)^{2H}+4\left(1+\frac{3}{r}\right)^{2H}-6\left(1+\frac{2}{r}\right)^{2H}+4\left(1+\frac{1}{r}\right)^{2H}-1\right]\\ &=\frac{r^{2H}}{2}\left[\sum_{k=1}^{\infty}\frac{2H(2H-1)\cdots(2H-k+1)}{k!}\left(-4^k+4\cdot 3^k-6\cdot 2^k+4\right)r^{-k}\right].\end{aligned} \tag{4.16}$$

Note that the sign of $2H(2H-1)\cdots(2H-k+1)$ is the same as that of $(-1)^{k-1}$, and

$$|2H(2H-1)\cdots(2H-k+1)|=2H|(2H-1)|\cdots|2H-k+1|<2\cdot 1\cdot 2\cdots(k-1)=2(k-1)!. \tag{4.17}$$

Hence we can write, for any $r \geq 5$,

$$\begin{aligned}
\left|\rho_{H,1}(r)\right| &\leq \frac{r^{2H}}{2}\sum_{k=1}^{\infty}\left|\frac{1}{k}\left(-4^k+4\cdot 3^k-6\cdot 2^k+4\right)r^{-k}\right| \\
&= \frac{r^{2H}}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left|-\left(\frac{4}{r}\right)^k+4\cdot\left(\frac{3}{r}\right)^k-6\cdot\left(\frac{2}{r}\right)^k+4\left(\frac{1}{r}\right)^k\right| \\
&= \frac{r^{2H}}{2}\left|\log\left(1-\frac{4}{r}\right)-4\log\left(1-\frac{3}{r}\right)+6\log\left(1-\frac{2}{r}\right)-4\log\left(1-\frac{1}{r}\right)\right| \\
&\leq \frac{r^{2H}}{2}\left[\left|\log\left(1-\frac{4}{r}\right)-4\log\left(1-\frac{1}{r}\right)\right|+\left|-4\log\left(1-\frac{3}{r}\right)+6\log\left(1-\frac{2}{r}\right)\right|\right].
\end{aligned} \tag{4.18}$$

One can easily check that $|\log(1-4x)-4\log(1-x)| \leq (243/20)x^2$, if $0 \leq x \leq 1/5$. And moreover,

$$\left|-4\log(1-3x)+6\log(1-2x)\right| \leq \left|\log(1-4x)-4\log(1-x)\right|. \tag{4.19}$$

Then we have, for any $r \geq 5$,

$$\left|\rho_{H,1}(r)\right| \leq \frac{243}{20}r^{-3/2+H_*}. \tag{4.20}$$

Consequently, we get

$$\begin{aligned}
\sum_{r\in\mathbb{Z}}\left|\rho_{H,1}(r)\right| &\leq \left|\rho_{H,1}(0)\right|+\left|\rho_{H,1}(1)\right|+\left|\rho_{H,1}(2)\right|+\left|\rho_{H,1}(3)\right|+\left|\rho_{H,1}(4)\right|+\sum_{r\geq 5}\left|\rho_{H,1}(r)\right| \\
&= \frac{1}{2}\left|-4^{2H}+4\cdot 3^{2H}-6\cdot 2^{2H}+4\right|+\frac{1}{2}\left|-5^{2H}+4\cdot 4^{2H}-6\cdot 3^{2H}+4\cdot 2^{2H}-1\right| \\
&\quad+\frac{1}{2}\left|-6^{2H}+4\cdot 5^{2H}-6\cdot 4^{2H}+4\cdot 3^{2H}-2^{2H}\right| \\
&\quad+\frac{1}{2}\left|-7^{2H}+4\cdot 6^{2H}-6\cdot 5^{2H}+4\cdot 4^{2H}-3^{2H}\right| \\
&\quad+\frac{1}{2}\left|-8^{2H}+4\cdot 7^{2H}-6\cdot 6^{2H}+4\cdot 5^{2H}-4^{2H}\right|+\sum_{r\geq 5}\left|\rho_{H,1}(r)\right| \\
&\leq C_1+C_{H_*} :\equiv C_{H_*} < \infty,
\end{aligned} \tag{4.21}$$

and the positive constant C_{H_*} does not depend on the unknown parameter H. Putting this bound in (4.10) yields

$$P(|Z_n| \geq a) \leq 2\exp\left(-\frac{a^2}{4C_{H_*}\left(a/\sqrt{n}+3+C_{H_*}/n\right)}\right). \tag{4.22}$$

Now we can construct the confidence interval for $g_n(H) = H - \log(4-2^{2H})/2\log n$. First observe that $Z_n = n^{2H-1/2}V_n - \sqrt{n}(4-2^{2H})$. Using the assumption $H \leq H_* < 1/2$ on the one hand and (4.22) on the other hand, we get

$$\begin{aligned}
&P\left(\frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log\left(1 - a/\left(4-2^{2H_*}\right)\sqrt{n}\right)}{2\log n} \leq g_n(H) \leq \frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log\left(1 + a/\left(4-2^{2H_*}\right)\sqrt{n}\right)}{2\log n}\right) \\
&\quad \geq P\left(\frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log\left(1 - a/\left(4-2^{2H}\right)\sqrt{n}\right)}{2\log n} \leq H - \frac{\log\left(4-2^{2H}\right)}{2\log n}\right. \\
&\qquad\qquad \left. \leq \frac{1}{2} - \frac{\log V_n}{2\log n} + \frac{\log\left(1 + a/\left(4-2^{2H}\right)\sqrt{n}\right)}{2\log n}\right) \\
&\quad = P\left(\frac{1}{4} - \frac{\log V_n}{2\log n} + \frac{\log\left(\left(4-2^{2H}\right)\sqrt{n} - a\right)}{2\log n} \leq H \leq \frac{1}{4} - \frac{\log V_n}{2\log n} + \frac{\log\left(\left(4-2^{2H}\right)\sqrt{n} + a\right)}{2\log n}\right) \\
&\quad = P(|Z_n| \leq a) \geq 1 - 2\exp\left(-\frac{a^2}{4C_{H_*}\left(a/\sqrt{n} + 3 + C_{H_*}/n\right)}\right),
\end{aligned} \tag{4.23}$$

where $\sum_{i,j\in\mathbb{Z}}|\rho(i,j)| \leq C_{H_*}$ and the positive constant C_{H_*} does not depend on the unknown parameter H. This is the desired result. □

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