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Research Article

Well-Posedness of Reset Control Systems as State-Dependent Impulsive Dynamical Systems

Alfonso Baños[1] and Juan I. Mulero[2]

[1] *Departamento de Informática y Sistemas, Universidad DE Murcia, 30071 Murcia, Spain*
[2] *Departamento de Ingenieria Sistemas y Automatica, Universidad Politécnica de Cartagena, Cartagena, Spain*

Correspondence should be addressed to Alfonso Baños, abanos@um.es

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Reset control systems are a special type of state-dependent impulsive dynamic systems, in which the time evolution depends both on continuous dynamics between resets and the discrete dynamics corresponding to the resetting times. This work is devoted to investigate well-posedness of reset control systems, taking as starting point the classical definition of Clegg and Horowitz. Well-posedness is related to the existence and uniqueness of solutions, and in particular to the resetting times to be well defined and distinct. A sufficient condition is developed for a reset system to have well-posed resetting times, which is also a sufficient condition for avoiding Zeno solutions and, thus, for a reset control system to be well-posed.

1. Introduction

Reset control systems [1–3] are a type of impulsive hybrid systems, in which the system state (or part of it) is reset at the instants it crosses some reset set. Impulsive hybrid systems are an active area of systems theory that has been developed in the last years. Two classical monographs are [4, 5], where reset control systems without external inputs are a particular case of autonomous system with impulse effects.

In this work, reset control systems will be formulated as a particular type of impulsive dynamical systems (IDSs), more specifically as state-dependent IDS, following the impulsive/hybrid dynamic system framework developed in [6]. In this framework, existence and uniqueness of solutions over a forward time interval is based on the well-posedness of resetting times.

The main goal of this work is to investigate well-posedness of reset control systems taking as starting point the classical definition of Clegg and Horowitz. This formulation has been also followed in several recent works, for example [7, 8] and references therein, and also [9–12]. As it is well known, reset control systems and IDS in general can exhibit behaviors that can be pathological from a control point of view. As it has been shown in [6], definition of IDS solutions has to deal with the problem of beating, deadlock, and Zenoness. In general, a reset control system will be considered well-posed if the resetting times are well-posed (they are well defined and are distinct), meaning that a number of beating or pulse phenomena are avoided, and in addition Zeno solutions do not exist.

As it will be shown, simple geometric conditions will be derived for avoiding the presence of these pathological behaviours. In Section 2, preliminary material and basic results are given. Section 3 develops a result for reset control systems to have well-posed resetting times. Finally, Section 4 tackles with the problem of existence of Zeno solutions.

Notation. In this work, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $(\mathbf{x}, \mathbf{y})$ denotes the column vector $\binom{\mathbf{x}}{\mathbf{y}}$. In addition, $\mathbf{e}_i \in \mathbb{R}^n$ stands for the unit vector $(0 \;\; \cdots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0)^T$ in which the ith-component is 1. For a set $\mathcal{M} \subset \mathbb{R}^n$, $\overline{\mathcal{M}}$ is the closure of $\mathcal{M}$. On the other hand, for a linear and time-invariant system with state space matrices (A, C), the subspace of unobservable states is given by the null space of the observability matrix O, where

$$O = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix}. \tag{1.1}$$

2. Preliminaries and Problem Setup

This work deals with a special class of hybrid systems called impulsive dynamical systems (IDSs, [6]). In particular, with state-dependent IDS having the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t), \quad \mathbf{x}(t) \notin \mathcal{M}, \\ \mathbf{x}(t^+) &= A_R\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{M}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $t \geq 0$, is the system state at the instant t, $\mathcal{M} \subset \mathbb{R}^n$ is the reset set, and A and A_R are matrices of dimension $n \times n$. The following material, including definition of IDS solutions and well-posedness of resetting times, is strongly based on [6]. The first equation in (2.1) will be referred to as the *continuous-time dynamics* or simply *base system dynamics*, while the second equation in (2.1) will be referred to as the *resetting law*. When at some *resetting* time $t \geq 0$, $\mathbf{x}(t) \in \mathcal{M}$ is true (the reset condition is active, and a *crossing* is performed), the state $\mathbf{x}(t)$ jumps to $\mathbf{x}(t^+) = A_R\mathbf{x}(t) \in \mathcal{M}_{\mathcal{R}}$; it will be assumed that resetting times are *well-posed*, that is, they are well defined and distinct for any initial condition. Otherwise, the state $\mathbf{x}(t)$ evolves with the base system dynamics. The set $\mathcal{M}_{\mathcal{R}}$ will be referred to as the after-reset set.

A function $\mathbf{x} : I_{\mathbf{x}_0} \to \mathbb{R}^n$ is a *solution* of the IDS system (2.1) on the interval $I_{\mathbf{x}_0} \subseteq \mathbb{R}$, with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, if $\mathbf{x}(\cdot)$ is left-continuous, and $\mathbf{x}(t)$ satisfies (2.1) for all $t \in I_{\mathbf{x}_0}$. For further discussion on solutions to impulsive differential equations and IDS solutions, see [4–6]. In general, there exists a unique solution $\psi(t) = e^{At}\psi_0$ of the (continuous) base system with initial condition $\psi(0) = \psi_0$ on $[0, \infty)$, for any $\psi_0 \in \mathbb{R}^n$. Informally speaking, the solution $\mathbf{x}$ of the IDS (2.1) from the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ for $0 < t \leq t_1$, where t_1 is the first resetting time satisfying $\mathbf{x}(t_1) \in \mathcal{M}$. Then, the state is instantaneously transferred to $A_R\mathbf{x}(t_1)$ according to the resetting law. The solution $\mathbf{x}(t)$, $t_1 < t \leq t_2$ (being t_2 the second resetting time given by $e^{A(t_2-t_1)}\mathbf{x}(t_1) \in \mathcal{M}$) is given by $\mathbf{x}(t) = e^{A(t-t_1)}\mathbf{x}(t_1) = e^{A(t-t_1)}A_R e^{At_1}\mathbf{x}_0$, and so forth. Note that the solution $\mathbf{x}$ of (2.1) is left-continuous, that is, it is continuous everywhere except at the resetting times t_k, and

$$\begin{aligned} \mathbf{x}(t_k) &= \lim_{\epsilon \to 0^+} \mathbf{x}(t_k - \epsilon), \\ \mathbf{x}\left(t_k^+\right) &= \lim_{\epsilon \to 0^+} \mathbf{x}(t_k + \epsilon) = A_R\mathbf{x}(t_k). \end{aligned} \tag{2.2}$$

2.1. Well-Posed Resetting Times and Zeno Solutions

Two standard assumptions for well-posedness of resetting times of state-dependent IDS [6], that will be used in this work, are

(A1) $\mathbf{x}(t) \in \overline{\mathcal{M}} \setminus \mathcal{M} \Rightarrow$ there exists $\epsilon > 0$ such that $\mathbf{x}(t+\delta) \notin \mathcal{M}$, for all $\delta \in (0, \epsilon)$.

(A2) $\mathbf{x} \in \mathcal{M} \Rightarrow A_R\mathbf{x} \notin \mathcal{M}$.

Note that for a particular solution $\mathbf{x}(\cdot)$, the first resetting time t_1 is well defined since $\min\{t \in \mathbb{R}^+ : \psi(t, 0, \mathbf{x}_0) = e^{At}\mathbf{x}_0 \in \mathcal{M}\}$ exists (and thus, it is unique by uniqueness of solutions of the base system). Analogously, for $k = 2, 3, \ldots$, the resetting time t_k is well defined since again $\min\{t \in \mathbb{R}^+ : \psi(t, t_{k-1}, A_R\mathbf{x}(t_{k-1})) \in \mathcal{M}\}$ exists, and in addition $0 = t_0 < t_1 < t_2 < \cdots$, for any $\mathbf{x}_0 \in \mathbb{R}^n$. Here, $\psi(t, t_0, \psi_0)$ is a solution of the base system with initial condition $\psi(t_0) = \psi_0$, that is, $\psi(t, t_0, \psi_0) = e^{A(t-t_0)}\psi_0$. Therefore, if for any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ the resetting times are well defined, functions $\tau_k : \mathbb{R}^n \to [0, \infty)$ are defined for $k = 1, 2, \ldots$, where $t_k = \tau_k(\mathbf{x}_0)$ is the kth resetting time, and by definition $\tau_0(\mathbf{x}_0) = 0$. Note that for a particular solution, there may exist no crossings, a finite or a infinite number of crossings, and in a finite or infinite time interval $I_{\mathbf{x}_0}$, and that functions $\tau_k(\mathbf{x}_0)$ are single valued by uniqueness of the base system solutions.

Since by assumptions A1 and A2, the resetting times are well defined and distinct, and since for a given initial condition, the solution to the base system differential equation exists and is unique, it follows that the solution of the IDS (2.1) also exists and is unique over a forward time interval [6]. For the IDS (2.1) with well-posed resetting times, a *Zeno solution* $\mathbf{x}_{\mathcal{Z}}(\cdot)$ exists on the interval $I_{\mathbf{x}_0} = [0, T]$ for some initial condition $\mathbf{x}_z(0) = \mathbf{x}_0 \in \mathbb{R}^n$, if there exists an infinite sequence of resetting times $(\tau_k(\mathbf{x}_0))_{k=0}^{\infty}$, and a positive number T, such as $\tau_k(\mathbf{x}_0) \to T$ as $k \to \infty$. Note that the solution is not defined beyond the time T. If there does not exist Zeno solutions for any initial condition, then the solutions of the IDS (2.1) exists and are unique for any initial condition on $I_{\mathbf{x}_0} = [0, \infty)$.

Note that conditions A1 and A2 can be interpreted as: (i) states that belong to the closure of $\mathcal{M}$, and does not belong to $\mathcal{M}$, evolve with the continuous base dynamics for some finite time interval (A1); (ii) after-reset states are not elements of the reset set $\mathcal{M}$ (A2). In other words, for resetting times to be well-posed a IDS system solution can only reach $\mathcal{M}$ through a point belonging to both $\mathcal{M}$ and its boundary $\partial\mathcal{M}$; and if a solution reaches a point in $\mathcal{M}$ that is on its boundary, then it is instantaneously removed from $\mathcal{M}$. Roughly speaking, condition A1 avoids the presence of deadlock, while condition A2 avoids beating or livelock (using these terms in the sense given in [6]).

In the following, two examples of ill-posed (not well-posed) second-order IDS are shown to illustrate conditions A1 and A2. In both cases, the base system corresponds to some matrix $A \in \mathbb{R}^{2\times 2}$, making the equilibrium point $\mathbf{x} = \mathbf{0}$ a center, and the resetting law is $x_1(t^+) = x_1(t), x_2(t^+) = 0$.

(a) (Figure 1(a)), here the reset set $\mathcal{M}^a$ is the rectangle

$$\mathcal{M}^a = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -1 \le x_1 \le 1,\ 0.7 < x_2 \le 1 \right\}, \tag{2.3}$$

and the after reset set is

$$\mathcal{M}^a_{\mathcal{R}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -1 \le x_1 \le 1,\ x_2 = 0 \right\}, \tag{2.4}$$

that is, the interval $[-1, 1]$ in the x_1-axis. From an initial condition in the point **A,** the trajectory reaches the reset set $\mathcal{M}^a$ at some point belonging to both $\mathcal{M}^a$ and its boundary $\partial\mathcal{M}^a$. Thus, the first resetting time $\tau_1(\mathbf{A})$ is well defined, and then the trajectory jumps to the point **B**. From the point **B**, the system trajectory evolves as the base system until it reaches a point **C** that belongs to $\partial\mathcal{M}^a$ but not to $\mathcal{M}^a$. Thus, condition A1 is not satisfied since the trajectory enters into $\mathcal{M}$ for any arbitrarily small time after reaching the point **C** (the second resetting time $\tau_2(\mathbf{A})$ is undefined).

(b) (Figure 1(b)), $\mathcal{M}^b$ is the grey region (it contains its boundary), and the after reset set is $\mathcal{M}^b_{\mathcal{R}} = \mathcal{M}^a_{\mathcal{R}}$. Note that the $\mathcal{M}^b \cap \mathcal{M}^b_{\mathcal{R}} = \{\mathbf{C}, \mathbf{D}\}$. In this case, a trajectory starting from the point **A** reaches $\mathcal{M}^b$ at the point **B** which belongs both to $\mathcal{M}^b$ and its boundary (thus, A1 is satisfied, and the first resetting time $\tau_1(\mathbf{A})$ is well defined). After that, the trajectory jumps to the point **C** that belongs both to $\mathcal{M}^b_{\mathcal{R}}$ and $\mathcal{M}^b$ and then make an infinite number of resets (condition A2 is not satisfied).

2.2. Reset Control Systems

In this work, reset control systems will be represented by the state-dependent IDS (2.1). Consider the feedback system given by Figure 2, where the (single input-single ouput) plant is described by the following:

$$P : \begin{cases} \dot{\mathbf{x}}_p(t) = A_p \mathbf{x}_p(t) + B_p u(t), \quad \mathbf{x}_p(0) = \mathbf{x}_{p0}, \\ y(t) = C_p \mathbf{x}_p(t), \end{cases} \tag{2.5}$$

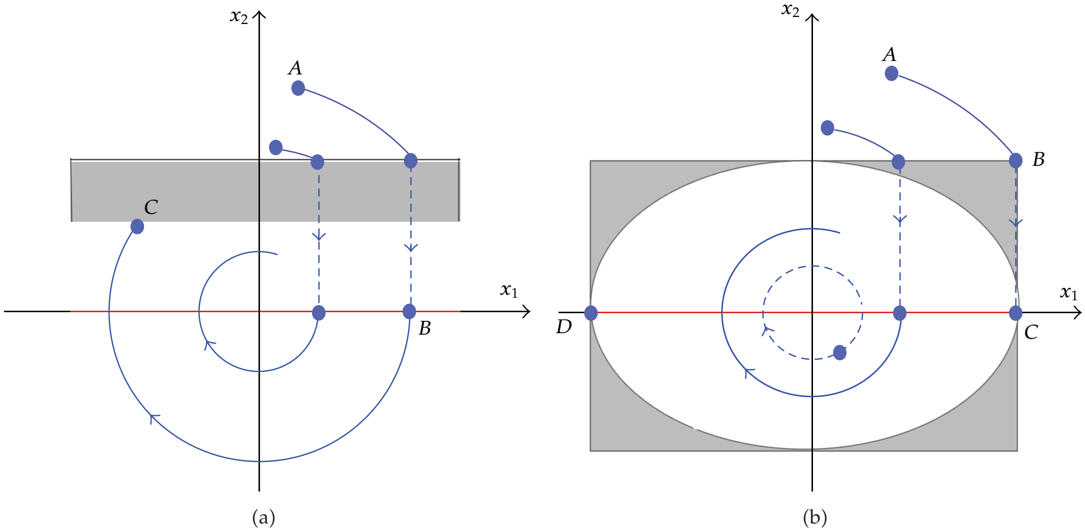

Figure 1: Examples of ill-posed IDSs.

and the (single input-single output) *reset* compensator is given by the impulsive differential equation:

$$C : \begin{cases} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r e(t), \quad \mathbf{x}_r(0) = \mathbf{x}_{r0},\ e(t) \neq 0, \\ \mathbf{x}_r(t+) = A_\rho \mathbf{x}_r(t), \quad e(t) = 0, \\ v(t) = C_r\mathbf{x}_r(t). \end{cases} \tag{2.6}$$

Here n_p is the dimension of the state $\mathbf{x}_p$, and n_r is the dimension of the state $\mathbf{x}_r$. A_ρ is a diagonal matrix with $(A_\rho)_{ii} = 0$ if the state $(\mathbf{x}_r)_i$ of the compensator is to be reset, and $(A_\rho)_{ii} = 1$ otherwise. In general, it is assumed that the last n_ρ compensator states are set to zero at the resetting times. In the case of a full-reset compensator, all the elements of A_ρ are 0. Consider the closed loop autonomous unforced system given by $e(t) = -y(t)$, $u(t) = v(t)$, and define the (closed loop) state $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_r)$ of dimension $n = n_p + n_r$, being $n_r = n_{\bar{\rho}} + n_\rho$. The result is that the reset control system is given by the state-dependent IDS

$$\begin{gathered} \dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(t) \notin \mathcal{M}, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathcal{M}, \\ \mathbf{x}(0) = \mathbf{x}_0, \\ y(t) = C\mathbf{x}(t), \end{gathered} \tag{2.7}$$

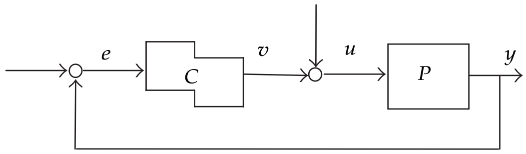

Figure 2: Reset control system.

where

$$A = \begin{pmatrix} A_p & B_p C_r \\ -B_r C_p & A_r \end{pmatrix}, \tag{2.8}$$

$$A_R = \operatorname{diag}\left(I_{n_p}, A_\rho\right) = \operatorname{diag}\left(I_{n_p}, \left(I_{n_{\bar\rho}}, O_{n_\rho}\right)\right), \tag{2.9}$$

$$C = \left[C_p, O\right], \tag{2.10}$$

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n : C\mathbf{x} = 0\}. \tag{2.11}$$

In control practice, it is required that reset control system solutions $\mathbf{x}(t)$ will be well-posed in the sense that they exist and are unique for $t \geq 0$. By definition, the reset control system (2.7)–(2.11) is *well-posed* if for any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, a solution $\mathbf{x}$ exists and is unique on $I_{\mathbf{x}_0} = [0, \infty)$. As a state-dependent IDS system, the reset control system is well-posed if resetting times are well-posed, and in addition, there do not exist Zeno solutions for any initial condition. In Section 3, the well-posedness of resetting times for reset control systems will be investigated. The existence of Zeno solutions is explored in Section 4.

3. Reset Control Systems with Well-Posed Resetting Times

The reset control system (2.7)–(2.11) is a particular case of the state-dependent IDS (2.1), with a reset set (it will be referred to as *reset surface*) $\mathcal{M} = \mathcal{N}(C)$, the null space of C, and with an after-reset set, or *after-reset surface*, $\mathcal{M}_\mathcal{R} = A_R(\mathcal{M})$. Since A_R is a projector, it results that in general $\mathcal{M}_\mathcal{R} \cap \mathcal{M} \neq \emptyset$, and thus, without any modification, the reset control system (2.7)–(2.11) does not have well-posed resetting times. This problem was detected in [7, 8], and a partial solution was given by redefining both sets as $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n \setminus \mathcal{M}_\mathcal{R} : C\mathbf{x} = 0\}$, and $\mathcal{M}_\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n : C\mathbf{x} = 0, (I - A_R)\mathbf{x} = 0\}$, where after-reset states are simply removed from $\mathcal{M}$ as given by (2.11). Since A_R is a projector, then the set $\{\mathbf{x} \in \mathbb{R}^n : (I - A_R)\mathbf{x} = 0\}$ is the column space of A_R, that is $\mathcal{R}(A_R)$. Thus, the above definitions are equivalent to

$$\mathcal{M}_\mathcal{R} = \mathcal{R}(A_R) \cap \mathcal{N}(C), \qquad \mathcal{M} = \mathcal{N}(C) \setminus \mathcal{M}_\mathcal{R}. \tag{3.1}$$

Proposition 3.1. *The reset system* (2.7)*, with* $\mathcal{M}$ *and* $\mathcal{M}_\mathcal{R}$ *given by* (3.1)*, has well-posed resetting times if*

$$\mathcal{M}_\mathcal{R} \cap \mathcal{N}(\mathcal{O}_{\text{base}}) = \{\mathbf{0}\}, \tag{3.2}$$

where $\mathcal{O}_{\text{base}}$ *is the base system observability matrix.*

Proof. If they do exist, by (3.1), resetting times are distinct since $\mathcal{M} \cap \mathcal{M}_{\mathcal{R}} = \emptyset$ is equivalent to A2, thus, the proof is centered in their existence. Note that by (3.1), $\overline{\mathcal{M}} \setminus \mathcal{M} = \mathcal{M}_{\mathcal{R}}$ in A1. By time invariance, A1 is equivalent to $\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}} \Rightarrow \mathbf{x}(t) \notin \mathcal{M}$ for $t \in (0, \epsilon)$ and some $\epsilon > 0$. Here ϵ depends on $\mathbf{x}_0$, but the dependence will not be explicitly shown by simplicity. In general, given $\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$, the first crossing with $\mathcal{M}$ is at the instant $t_1 = \tau_1(\mathbf{x}_0)$, and finally A1 is equivalent to the existence of a lower bound $\epsilon > 0$ for t_1, for any given $\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$. From (3.1), t_1 is simply given by $t_1 = \min\{t > 0 \mid Ce^{At}\mathbf{x}_0 = 0 \wedge e^{At}\mathbf{x}_0 \notin \mathcal{R}(A_R)\}$. If $\epsilon_1 > 0$ is by definition a lower bound of the set $\{t > 0 \mid Ce^{At}\mathbf{x}_0 = 0\}$, and $\epsilon_2 > 0$ is by definition a lower bound of the set $\{t > 0 \mid e^{At}\mathbf{x}_0 \notin \mathcal{R}(A_R)\}$ (both depending on $\mathbf{x}_0$), then $\epsilon = \max\{\epsilon_1, \epsilon_2\} \leq t_1$. By simplicity, consider in first instance that A has distinct eigenvalues; in this case (see the Appendix)

$$\left\{t > 0 \mid Ce^{At}\mathbf{x}_0 = 0\right\} = \left\{t > 0 \mid f_1(t) := e^{\Lambda t^T} U(\Lambda)\mathcal{O}_{\text{base}}\mathbf{x}_0 = 0\right\}. \tag{3.3}$$

Since $f_1(\cdot)$ is a sum of exponentials (in fact it is a Bohl function [13]), and thus, it is an analytical function, it is true that $f_1(t)$ is either zero for all $t \geq 0$ or has isolated zeros. As a result, two options are possible as follows $\epsilon_1 = 0$ if $x_0 \in \mathcal{N}(\mathcal{O}_{\text{base}})$ ($f_1(t) = 0$, for all $t \in [0, \infty)$), or there exist an interval $(0, \epsilon_1)$ in which $f_1(t) \neq 0$ for some $\epsilon_1 > 0$. Now, if condition (3.2) is satisfied then for any $\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$ only the second option is possible, and thus, $t_1 \geq \max\{\epsilon_1, \epsilon_2\} \geq \epsilon_1$, that is, A1 is satisfied (condition (3.2) is sufficient for A1). Finally, in the case in which the eigenvalues of A may be repeated, similar expressions may be found for f_1 (see the Appendix), and the above arguments are again applied. □

Remark 3.2. Note that, in particular, well-posedness of resetting times is obtained if the base system is observable, since in this case $\mathcal{N}(\mathcal{O}_{\text{base}}) = \{\mathbf{0}\}$. But some unobservable base linear systems can also produce reset systems with well-posed resetting times, as far as the after-reset surface does not contain unobservable states (different to $\mathbf{0}$).

Remark 3.3. Note that, in general, Proposition 3.1 may be applied to reset systems given by (2.7) with arbitrary values A, A_R, and C as far as the developed conditions apply (not necessarily reset control systems).

Remark 3.4. For the reset and after-reset sets given by (3.1), functions $\tau_k(\cdot)$, $k = 0, 1, 2, \ldots$ are homogenous (of degree 0), that is $\tau_k(\alpha\mathbf{x}_0) = \tau_k(\mathbf{x}_0)$ for any real number α, since $Ce^{At}(\alpha\mathbf{x}_0) = \alpha Ce^{At}\mathbf{x}_0 = 0$ at a resetting time t.

Example 3.5 (Ill-posed reset system). Consider a reset system (2.7), with the following system matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where the sets $\mathcal{M}_{\mathcal{R}}$ and $\mathcal{M}$ are defined according to (3.1) as $\mathcal{M}_{\mathcal{R}} = \mathcal{R}(A_R) \cap \mathcal{N}(C) = \text{span}\{(0,1,0)^T\}$, and $\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{M}_{\mathcal{R}} = \text{span}\{(0,1,0)^T, (0,0,1)^T\} \setminus \text{span}\{(0,1,0)^T\}$. This is due to the fact that the after-reset surface $\mathcal{M}_{\mathcal{R}}$ is a subset of the unobservable subspace of the linear base system, which is given in this case by

$$\mathcal{N}\left(\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}\right) = \mathcal{N}\left(\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \mathcal{N}(C) \supset \mathcal{M}_{\mathcal{R}}. \tag{3.5}$$

As a result, from any initial condition $\mathbf{x}_0 = (0, a, 0)^T \in \mathcal{M}_{\mathcal{R}}$, the condition A1 is not satisfied. Note that the origin is a stable focus in the plane $x_2 - x_3$, and that the first resetting time $\tau_1((0, a, 0)^T)$ is not well defined; in fact, the reset system is not well-posed.

Example 3.6 (Unobservable base system and well-posed resetting times). This example, adapted from [8], shows how an unobservable base system may define a reset system with well-posed resetting times, as far as the unobservable subspace does not contain after-reset states. Consider a reset control system (2.7) with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -0.2 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \tag{3.6}$$

that has a unobservable mode corresponding to a stable pole-zero cancellation in the linear base system, where the plant has a transfer function $P(s) = (s+1)/(s(s+0.2))$, and the base compensator is $C(s) = 1/(s+1)$ (the reset compensator is a first-order reset element -FORE). In addition, the after-reset and reset surfaces are given by $\mathcal{M}_{\mathcal{R}} = \mathcal{R}(A_R) \cap \mathcal{N}(C) = \operatorname{span}\{(1,0,0)^T\}$ and $\mathcal{M} = \mathcal{N}(C) \backslash \mathcal{M}_{\mathcal{R}} = \operatorname{span}\{(1,0,0)^T, (0,0,1)^T\} \backslash \operatorname{span}\{(1,0,0)^T\}$, respectively. In this case, the set $\mathcal{M}_{\mathcal{R}}$ is not a subset of the linear base system unobservable subspace, given by

$$\mathcal{N}\left(\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}\right) = \mathcal{N}\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & -0.2 & 1 \\ -0.2 & -0.96 & -0.2 \end{pmatrix}\right) = \operatorname{span}\left\{(1, 0, -1)^T\right\}. \tag{3.7}$$

As a result, Proposition 3.1 may be used to ensure that the system has well-posed resetting times (see Figure 3 for system solutions corresponding to two initial conditions).

4. Zeno Solutions of Reset Control Systems

In this Section, the existence of Zeno solutions is investigated for reset control systems described by (2.7)–(2.10), and with reset and after-reset surfaces given by (3.1). In principle, as discussed in Section 2.2, the reset control system may exhibit Zeno solutions even in the case in which resetting times are well-posed, that is, they are well defined and are distinct. However, as it will be shown in the following, well-posedness of resetting times is sufficient to avoid the existence of Zeno solutions in reset control systems, and thus, for reset control systems to be well-posed.

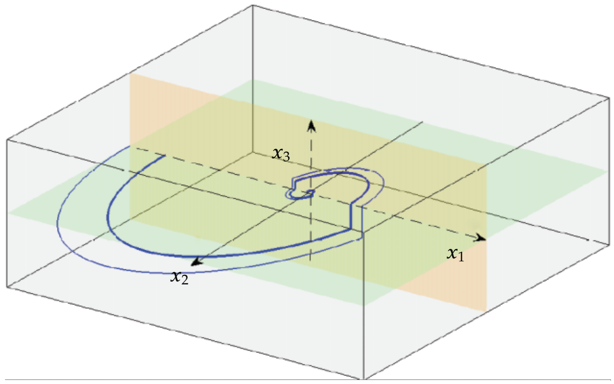

Figure 3: System solutions for the well-posed reset control system example.

Proposition 4.1. *The reset control system* (2.7)–(2.10)*, with reset and after-reset surfaces given by* (3.1)*, will not have Zeno solutions if it has well-posed resetting times.*

Proof. It will be assumed that $\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$, and that there exist an infinite number of crossings for $\mathbf{x}_0$ (otherwise no Zeno solution may exist), then reset intervals are given by $\Delta_k(\mathbf{x}_0) := \tau_k(\mathbf{x}_0) - \tau_{k-1}(\mathbf{x}_0) \neq 0$, $k = 1, 2, \ldots$. In the following, the notation $t_k = \tau_k(\mathbf{x}_0)$ is used, and the argument $\mathbf{x}_0$ is dropped by simplicity. Note that reset intervals Δ_k are well defined and $\Delta_k \neq 0$, $k = 1, 2, \ldots$, by well-posedness of the resetting times. The proof will be based on the fact that in general there can only exist a finite sequence of reset intervals $(\Delta_k)_{k=1}^{m-1}$ such as $\Delta_{m-1} < \Delta_{m-2} < \cdots < \Delta_1 = \epsilon$, for some $\epsilon > 0$ arbitrarily small but fixed, and some finite positive integer m, being m the dimension of the after reset surface $\mathcal{M}_{\mathcal{R}}$. Thus, the sequence of resetting times $(t_k = t_{k-1} + \Delta_k)_{k=1}^{\infty}$, with $t_0 = 0$, will not be a Cauchy sequence, and thus, $t_k \to \infty$, as $k \to \infty$. As a result, Zeno solutions does not exist. Assume that the plant state equations (2.5) are given in observer form (note that the plant has not to be necessarily observable), that is,

$$A_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_p-1} \end{pmatrix}, \qquad B_p = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_p-1} \end{pmatrix}, \tag{4.1}$$
$$C_p = \begin{pmatrix} 0 & 0 & \cdots & 1 \end{pmatrix},$$

then $C = (0, 0, \ldots, 1, 0, \ldots, 0)$, and $\mathcal{O}_{\text{base}}$ is

$$\mathcal{O}_{\text{base}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & -a_{n_p-1} & \star & \cdots & \star \\ 0 & 0 & \cdots & 1 & -a_{n_p-1} & \star & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -a_{n_p-1} & \cdots & \star & \star & \star & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & \end{pmatrix}, \tag{4.2}$$

where $\star$ stands for a non(necessarily) zero term. By simplicity, the case of full reset is approached in first instance. An after-reset state $\mathbf{x} \in \mathcal{M}_{\mathcal{R}}$ is given by

$$\mathbf{x} = \left(x_1, x_2, \ldots, x_{n_p-2}, x_{n_p-1}, 0, 0, \ldots, 0\right)^T, \tag{4.3}$$

for some values $x_1, \ldots, x_{n_p-1} \in \mathbb{R}$, being n_p the number of plant states. Thus, $m = n_p - 1$ in the case of full reset.

Let us start with the case $m = 1$. In this case, for any $\mathbf{x}_0 = (x_1, 0, 0, \ldots 0)^T$, it is clear that $\tau_1(\mathbf{x}_0) = \Delta$ for some constant $\Delta > 0$, since $\tau_1(\cdot)$ is homogenous (see Remark 3.4). In addition, $t_2 = \Delta + \tau_1(A_R e^{At_1}\mathbf{x}_0) = 2\Delta$ since $A_R e^{At_1}\mathbf{x}_0 = \alpha \mathbf{x}_0$ for some real number α. As a result $\Delta_k = \Delta$, $k = 0, 1, \ldots$, that is, resetting times are periodic with period Δ, and no Zeno solution may exist.

The case $m = 2$ is analyzed in the following. Consider an initial condition $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$, that is, $\mathbf{x}_0 = (x_1, x_2, 0, 0, \ldots 0)^T$. If the solution $\mathbf{x}(t, 0, \mathbf{x}_0)$ crosses the reset surface $\mathcal{M}_{\mathcal{R}}$ at time $t_1 = \epsilon_1$ for some $\epsilon_1 > 0$ arbitrarily small, and thus, $\Delta_1 = \epsilon_1$, then

$$0 = Ce^{A\epsilon_1}\mathbf{x}^1 = C\mathbf{x}^1 + \epsilon_1 CA\mathbf{x}^1 + \frac{\epsilon_1^2}{2}CA^2\mathbf{x}^1 + \cdots. \tag{4.4}$$

Now, since the control system (2.7)–(2.10) has well-posed resetting times, then the right hand of (4.4) is not identically zero for any $\mathbf{x}_0 \in \mathcal{M}_R$. Now, using the special structure given in (4.2), it is obtained that

$$0 = x_2 + \frac{\epsilon_1}{2}x_1 + O\left(\epsilon_1^2\right), \tag{4.5}$$

for arbitrarily small $\epsilon_1 > 0$. In addition, the following after-reset state is $\mathbf{x}_1 = A_R\mathbf{x}(t_1, 0, \mathbf{x}_0) = (x_1 + O(\epsilon_1^2), x_2 + \epsilon_1 x_1 + O(\epsilon_1^2), 0, 0, \ldots, 0)^T$. Repeating the above argument, the solution $\mathbf{x}(t, t_1, \mathbf{x}_1)$ will cross again $\mathcal{M}$ at the instant $t_2 = t_1 + \Delta_2$. If $\Delta_2 = \epsilon_2 \le \epsilon_1$ for some $\epsilon_2 > 0$, then

$$0 = x_2 + \epsilon_1 x_1 + \frac{\epsilon_2}{2}x_1 + O\left(\epsilon_1^2\right), \tag{4.6}$$

where the properties $O(\epsilon_2^2) = O(\epsilon_1^2)$ for $\epsilon_2 \le \epsilon_1$ and $O(k\epsilon) = O(\epsilon)$, for a real constant k, have been used. Now, using (4.5) and (4.6), the result is that given some $\epsilon_1 > 0$ arbitrarily small, then $\epsilon_2 = -\epsilon_1 + O(\epsilon_1^2) < 0$, which is absurd. Thus, by contradiction, it is true that $\epsilon_2 > \epsilon_1$, and thus any initial condition in the set $\mathcal{M}_{\mathcal{R}}$ that produces a first reset interval $\epsilon_1 > 0$ arbitrarily small, gives a larger second reset interval $\epsilon_2 > 0$. Thus, Zeno solutions does not exist in this case either.

In the rest of the proof, the terms $O(\epsilon_1^m)$ are directly neglected by simplicity. For the general case in which the dimension of $\mathcal{M}_{\mathcal{R}}$ is m, with initial state $\mathbf{x}_0 = (x_1, x_2, \dots, x_m, 0, 0, \dots 0)^T$, a similar reasoning results in the set of equations

$$\begin{gathered} \sum_{k=1}^{m} \frac{\epsilon_1^{m-1}}{(m+1-k)!} x_k = 0, \\ \sum_{i=1}^{m} \sum_{k=1}^{i} \frac{\epsilon_2^{m-i} \epsilon_1^{i-k}}{(m+1-i)!(i-k)!} x_k = 0, \\ \cdots \\ \sum_{i=1}^{m} \sum_{k=1}^{i} \frac{\epsilon_m^{m-i} (\epsilon_1 + \cdots + \epsilon_{m-1})^{i-k}}{(m+1-i)!(i-k)!} x_k = 0, \end{gathered} \tag{4.7}$$

which leads to an algebraic equation of order m in ϵ_m, being its solutions $\epsilon_m = -\epsilon_{m-1}$, $\epsilon_m = -(\epsilon_{m-1} + \epsilon_{m-2}), \dots, \epsilon_m = -(\epsilon_{m-1} + \epsilon_{m-2} + \cdots + \epsilon_1)$. And again, it can not exist a sequence of reset intervals $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$, with $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_m > 0$ and ϵ_1 arbitrarily small, showing that a Zeno solution is not possible in the full-reset case.

The case of partial reset can be conveniently transformed into the full-reset form by a change of coordinates, by a simple resorting of coordinates so that the bijectivity is guaranteed. We will consider the system structure decomposition by writing the states as $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_{\overline{\rho}}, \mathbf{x}_\rho)$, where $\mathbf{x}_p \in \mathbb{R}^{n_p}$ stands for the states of the plant, $\mathbf{x}_{\overline{\rho}} \in \mathbb{R}^{n_{\overline{\rho}}}$ for the nonresetting compensator states and $\mathbf{x}_\rho \in \mathbb{R}^{n_\rho}$ for the resetting compensator states. Define the linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$T\mathbf{x} = T(\mathbf{x}_p, \mathbf{x}_{\overline{\rho}}, \mathbf{x}_\rho) = (\mathbf{x}_{\overline{\rho}}, \mathbf{x}_p, \mathbf{x}_\rho) = \mathbf{z}, \tag{4.8}$$

that is,

$$T = \begin{pmatrix} 0_{n_{\overline{\rho}} \times n_p} & I_{n_{\overline{\rho}} \times n_{\overline{\rho}}} & 0_{n_{\overline{\rho}} \times n_\rho} \\ I_{n_p \times n_p} & 0_{n_p \times n_{\overline{\rho}}} & 0_{n_p \times n_\rho} \\ 0_{n_\rho \times n_p} & 0_{n_\rho \times n_{\overline{\rho}}} & I_{n_\rho \times n_\rho} \end{pmatrix}. \tag{4.9}$$

Note that T is a square matrix, all of whose entries are 0 or 1, and in each row and column of T there is precisely one 1. This means that T is a permutation matrix. Clearly, such a matrix is unitary, hence orthogonal, so $T^T = T^{-1}$. The nonsingular matrix T allows to rewrite the dynamical system via a similarity transformation (congruence transformation):

$$\begin{gathered} \dot{\mathbf{z}}(t) = \overline{A}\mathbf{z}(t), \quad \mathbf{z}(t) \notin \widetilde{\mathcal{M}}, \\ \mathbf{z}(t^+) = \overline{A}_R \mathbf{z}(t), \quad \mathbf{z}(t) \in \widetilde{\mathcal{M}}, \\ y(t) = \overline{C}\mathbf{z}(t), \end{gathered} \tag{4.10}$$

where $\overline{A} = TAT^T$, $\overline{A}_R = TA_RT^T$, and $\overline{C} = CT^T$, and in addition, the reset surface is transformed into $\widetilde{\mathcal{M}} = \{\mathbf{z} \in \mathbb{R}^n : T^T z \in \mathcal{M}\}$. Note that $\overline{C} = CT^T = \mathbf{e}_{n_p+n_{\overline{\rho}}}$ so that the output is not changed by the transformation, that is, $y(t) = z_{n_p+n_{\overline{\rho}}}(t)$ as expected. Henceforth, (4.10) is in full-reset form. Since observability is invariant under similarity transformations, it is clear that (2.8) is well-posed if and only if (4.10) is well-posed. Finally, to complete the proof it is necessary to show that the observability matrix has the structure given in (4.2) (using state transformations if needed). This is simply done by considering the substate $\mathbf{z}_1 = (\mathbf{x}_{\overline{\rho}}, \mathbf{x}_p)$. In general, there always exists a state transformation of $\mathbf{z} = (\mathbf{z}_1, \mathbf{x}_\rho)$ to $\mathbf{w} = (\mathbf{w}_1, \mathbf{x}_\rho)$, such that the state submatrix corresponding to $\mathbf{z}_1$ is in the observer form, and thus, the observability matrix has the structure given in (4.2) once unobservable states are eliminated. □

4.1. Example: Well-Posed Reset Control System with Partial Reset

Consider a reset control system, where the plant, with state $\mathbf{x}_p = (x_1 \ \ x_2)^T$, is given by

$$A_p = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{4.11}$$

and the reset compensator, with state $\mathbf{x}_r = (x_3 \ \ x_4)^T$, by

$$A_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad C_r = \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad A_\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.12}$$

that is the reset control system has a partial reset compensator: it is a parallel connection of an integrator and a Clegg integrator, where only the state x_4 is set to zero at the resetting times. The closed-loop system is given by the matrices

$$A = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.13}$$

and the closed-loop state $\mathbf{x} = (x_1 \ \ x_2 \ \ x_3 \ \ x_4)^T$. This reset control system is well-posed, since

$$\mathcal{M}_R = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{O}_{\text{base}} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}, \tag{4.14}$$

and then, $\mathcal{M}_{\mathcal{R}} \cap \mathcal{N}(\mathcal{O}_{\text{base}}) = \{\mathbf{0}\}$. Following the reasoning given in the proof of Proposition 4.1, the closed-loop state $\mathbf{x}$ can be transformed into a state $\mathbf{z}$ in which the observability matrix has the form (4.2). In this case, this is obtained with $\mathbf{z} = (x_3 \ \ x_1 \ \ x_2 \ \ x_4)^T$. Thus, the initial

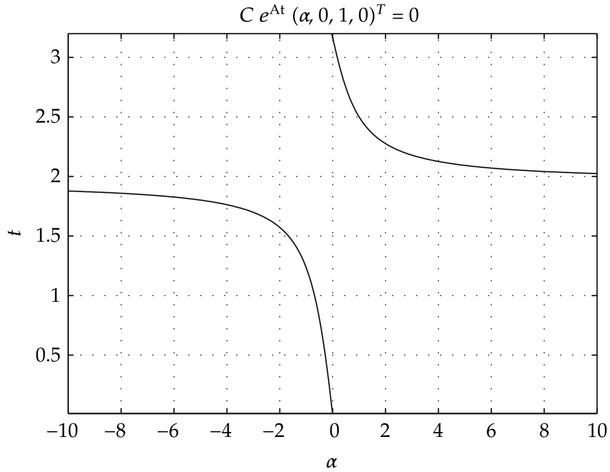

Figure 4: First resetting time as a function of α.

conditions that produces a crossing in a arbitrarily small time $\epsilon > 0$ are of the form $\mathbf{z}^1 = (1 \;\; -\epsilon/2 \;\; 0 \;\; 0)^T$ or equivalently

$$\mathbf{x}^1 = \left(-\frac{\epsilon}{2} \;\; 0 \;\; 1 \;\; 0\right)^T. \tag{4.15}$$

Now, the second after-reset state is given by

$$\mathbf{x}^2 = A_R e^{A\epsilon}\left(-\frac{\epsilon}{2} \;\; 0 \;\; 1 \;\; 0\right)^T = \left(\frac{\epsilon}{2} \;\; 0 \;\; 1 \;\; 0\right)^T, \tag{4.16}$$

and according to Proposition 4.1, $\mathbf{x}^2$ cannot produce a new crossing in a time less than or equal to ϵ. This fact can be verified by computing solutions to the implicit equation $0 = Ce^{At}(\alpha \;\; 0 \;\; 1 \;\; 0)^T$ for t, given $\alpha \in \mathbb{R}$. The solution is shown in Figure 3, where $t = \tau_1((\alpha \;\; 0 \;\; 1 \;\; 0)^T)$ is given.

Note that for t to be arbitrarily small, the initial condition $\mathbf{x}^1$ in the after-reset surface must be given by (4.15), that is, $\alpha = -\epsilon/2$. Then, as a result, the state after the first reset $\mathbf{x}^2$ is given by (4.16). And then the value of the second resetting time can be obtained from Figure 4 with $\alpha = \epsilon/2$. The result is that if the first resetting time is arbitrarily small, then the second resetting time is arbitrarily close to a number $t^* = 3.1698\cdots$ (it can be approximately computed by numerically solving the implicit equation).

5. Conclusions

Well-posedness of reset control system has been investigated using a state-dependent impulsive dynamic system (IDS) representation. Reset systems have been shown to be well-posed, in the sense that resetting times of the IDS are well defined and are distinct for any initial condition, and in addition, no Zeno solutions do exist. A sufficient condition for the well-posedness of resetting times has been elaborated, based on the observability of after-reset states. In addition, it has been shown that reset control systems do no exhibit Zeno solutions if resetting times are well-posed. On the other hand, it has also been revealed several properties related with the structure of the resetting times: (i) an initial condition in the after-reset surface (having dimension m) will have sequences of decreasing reset intervals with length at most $m-1$; (ii) resetting times as a function of the initial condition, is in general a discontinuous mapping, which explains to certain extent the complexity in the analysis of reset control systems.

Appendix

In the following, it is shown that $Ce^{At}\mathbf{x}_0 = e^{\boldsymbol{\lambda} t_1}{}^T U(\boldsymbol{\lambda})\mathcal{O}_{\text{base}}\mathbf{x}_0$.

By simplicity, consider in first instance that the closed-loop state matrix A has distinct eigenvalues, then the matrix exponential may be computed by use the Caley-Hamilton method, that is,

$$e^{At_1} = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1}, \tag{A.1}$$

where α_i, $i = 0,\ldots,n-1$, are given by

$$\begin{aligned} e^{\lambda_1 t_1} &= \alpha_0 + \alpha_1\lambda_1 + \cdots + \alpha_{n-1}\lambda_1^{n-1}, \\ e^{\lambda_2 t_1} &= \alpha_0 + \alpha_1\lambda_2 + \cdots + \alpha_{n-1}\lambda_2^{n-1}, \\ \cdots \\ e^{\lambda_n t_1} &= \alpha_0 + \alpha_1\lambda_n + \cdots + \alpha_{n-1}\lambda_n^{n-1}. \end{aligned} \tag{A.2}$$

Using the notation $\boldsymbol{\lambda}^T = (\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_n)$, $\boldsymbol{\alpha}^T = (\alpha_0 \;\; \alpha_1 \;\; \cdots \;\; \alpha_{n-1})$ and $e^{\boldsymbol{\lambda} t_1} = \sum_{i=1}^{n} e^{\lambda_i t_i}\mathbf{e}_i$, where $\mathbf{e}_i$ stands for the unit $(0 \;\; \cdots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0)^T$ in which the i-component is 1, (A.2) can be compactly written as

$$e^{\boldsymbol{\lambda} t_1} = V(\boldsymbol{\lambda})^T\boldsymbol{\alpha}, \tag{A.3}$$

where $V(\boldsymbol{\lambda})$ is a (nonsingular) Vandermonde matrix. Now, by eliminating $\boldsymbol{\alpha}$ from (A.1) and (A.3), the equation $Ce^{At_1}\mathbf{x}_0$ is transformed into

$$\boldsymbol{\alpha}^T\begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1}\end{pmatrix}\mathbf{x}_0 = e^{\boldsymbol{\lambda} t_1}{}^T U(\boldsymbol{\lambda})\mathcal{O}_{\text{base}}\mathbf{x}_0 = e^{\boldsymbol{\lambda} t_1}{}^T U(\boldsymbol{\lambda})\mathcal{O}_{\text{base}}\mathbf{x}_0, \tag{A.4}$$

where $U(\boldsymbol{\lambda}) = V(\boldsymbol{\lambda})^{-1}$.

In the case in which the eigenvalues of A may be repeated, a similar argument may be applied. Note that e^{At} may be written as the infinite series $D(A) = \sum_{i=0}^{\infty} A^i t^i/i!$. Thus, the polynomial $D(\lambda) = \sum_{i=0}^{\infty} \lambda^i t^i / i!$ can be factorized by $D(\lambda) = Q(\lambda)P(\lambda) + R(\lambda)$, with $R(\lambda) = 0$ or degree (R) < degree $(P) = n$. In addition, R has degree no greater than $n-1$, and thus, $R(\lambda) = \sum_{j=0}^{n-1} \alpha_j \lambda^j$. Since the characteristic polynomial is zero for the eigenvalues of A, then $D(\lambda_k) = R(\lambda_k)$ for $k = 0, 1, \ldots, n-1$. And then $D(\lambda_k) = \sum_{i=0}^{\infty} \lambda_k^i t_1^i / i! = e^{\lambda_k t_1} = R(\lambda_k) = \sum_{j=0}^{n-1} \alpha_j \lambda_k^j$ para $k = 0, 1, \ldots, n-1$. This can be compactly expressed as $V^T(\boldsymbol{\lambda})\boldsymbol{\alpha} = e^{\boldsymbol{\lambda} t}$, and the expression (A.3) is obtained. Now, if A has r different eigenvalues with multiplicity order n_r, and as a consequence, the characteristic polynomial is $p(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{n_i}$. Again, there exists unique polynomials Q and R such as $D(\lambda) = Q(\lambda)P(\lambda) + R(\lambda)$, where $D(\lambda) = e^{\lambda t_1}$ y $R = 0$ or deg (R) < deg (P). Here, R can be expressed as $R(\lambda) = \sum_{i=0}^{n-1} \alpha_i \lambda^i$, where the coefficients are unique. Since p and its derivatives up to order n_r are zero at λ_i, then

$$\frac{d^j D(\lambda_i)}{d\lambda^j} = \frac{d^j R(\lambda_i)}{d\lambda^j} \quad \forall i = 1, 2, \ldots, r, \ \forall j = 0, 1, \ldots, n_i - 1. \tag{A.5}$$

This can be expressed by $\boldsymbol{\mu} = W\boldsymbol{\alpha}$, where

$$\begin{aligned} \boldsymbol{\mu} &= \sum_{i=1}^{r} \sum_{j=0}^{n_i - 1} \mathbf{e}_i e^{\lambda_i t_1} \otimes \mathbf{e}_j \lambda_i^j, \\ W &= \left(\sum_{i=1}^{r} \sum_{j=0}^{n_i - 1} \mathbf{e}_i \otimes \mathbf{e}_j \mathbf{e}_i^T \frac{\partial^j V(\boldsymbol{\lambda})}{\partial \lambda_i^j} \right). \end{aligned} \tag{A.6}$$

By using arguments based on the Lagrange-Hermite interpolation problem, it can be shown that in fact the matrix W is invertible. And then, an expression similar to (A.4) may be obtained.

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