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Research Article

Principal Functions of Nonselfadjoint Discrete Dirac Equations with Spectral Parameter in Boundary Conditions

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Let L denote the operator generated in $\ell_2(\mathbb{N},\mathbb{C}^2)$ by $a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)}$, $a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}$, $n \in \mathbb{N}$, and the boundary condition $(\gamma_0 + \gamma_1\lambda)y_1^{(2)} + (\beta_0 + \beta_1\lambda)y_0^{(1)} = \mathbf{0}$, where $(a_n), (b_n), (p_n)$, and (q_n), $n \in \mathbb{N}$ are complex sequences, $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$, and λ is an eigenparameter. In this paper we investigated the principal functions corresponding to the eigenvalues and the spectral singularities of L.

1. Introduction

Consider the boundary value problem (BVP)

$$
\begin{gathered}
-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty, \\
y(0) = 0,
\end{gathered}
\tag{1.1}
$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter. The spectral theory of the above BVP with continuous and point spectrum was investigated by Naĭmark [1]. He showed the existence of the spectral singularities in the continuous spectrum of (1.1). Note that the eigen and associated functions corresponding to the spectral singularities are not the elements of $L^2(\mathbb{R}_+)$.

In [2, 3] the effect of the spectral singularities in the spectral expansion in terms of the principal vectors was considered. Some problems related to the spectral analysis of difference equations with spectral singularities were discussed in [4–7]. The spectral analysis of eigenparameter dependent nonselfadjoint difference equation was studied in [8, 9].

Let us consider the nonselfadjoint BVP for the discrete Dirac equations

$$\begin{aligned} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} &= \lambda y_n^{(1)}, \quad n \in \mathbb{N}, \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} &= \lambda y_n^{(2)}, \quad n \in \mathbb{N}, \end{aligned} \tag{1.2}$$

$$(\gamma_0 + \gamma_1\lambda)y_1^{(2)} + (\beta_0 + \beta_1\lambda)y_0^{(1)} = 0, \quad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq a_0^{-1}\beta_0, \tag{1.3}$$

where $\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $n \in \mathbb{N}$ are vector sequences, $a_n \neq 0$, $b_n \neq 0$ for all $n \in \mathbb{N}$, $\gamma_i, \beta_i \in \mathbb{C}$, and $i = 0, 1$ and, λ is a spectral parameter.

In [10] the authors proved that eigenvalues and spectral singularities of (1.2)-(1.3) have a finite number with finite multiplicities, if the condition,

$$\sum_{n=1}^{\infty} \exp\left(\varepsilon n^{\delta}\right)\left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty \tag{1.4}$$

holds, for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$.

In this paper, we aim to investigate the principal functions corresponding to the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3).

2. Discrete Spectrum of (1.2)-(1.3)

Let for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$,

$$\sum_{n=1}^{\infty} \exp\left(\varepsilon n^{\delta}\right)\left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty, \tag{2.1}$$

be satisfied. It has been shown that [10] under the condition (2.1), (1.2) has the solution

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n\left(I + \sum_{m=1}^{\infty} A_{nm}e^{imz}\right)\begin{pmatrix} e^{iz/2} \\ -i \end{pmatrix}e^{inz}, \quad n = 1, 2, \ldots, \tag{2.2}$$

$$f_0^{(1)}(z) = \alpha_0^{11}\left\{e^{iz/2}\left[1 + \sum_{m=1}^{\infty} A_{0m}^{11}e^{imz}\right] - i\sum_{m=1}^{\infty} A_{0m}^{12}e^{imz}\right\}, \tag{2.3}$$

for $\lambda = 2\sin(z/2)$ and $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}. \tag{2.4}$$

Note that α_n^{ij} and A_{nm}^{ij} $(i,j = 1,2)$ are uniquely expressed in terms of (a_n), (b_n), (p_n), and $(q_n), n \in \mathbb{N}$ as follows

$$
\begin{aligned}
\alpha_n^{11} &= \left[\prod_{k=n+1}^{\infty}(-1)^{n-k} b_k a_{k-1}\right]^{-1}, \\
\alpha_n^{12} &= 0, \\
\alpha_n^{22} &= \left[b_n \prod_{k=n+1}^{\infty}(-1)^{n-k+1} b_k a_{k-1}\right]^{-1}, \\
\alpha_n^{21} &= \alpha_n^{22}\left[p_n + \sum_{k=n+1}^{\infty}(p_k + q_k)\right], \\
A_{n1}^{12} &= -\sum_{k=n+1}^{\infty}(p_k + q_k), \\
A_{n1}^{11} &= \sum_{k=n+1}^{\infty}\left[a_{k+1}a_k + b_k^2 - p_k q_k + (p_k + q_k)A_{k1}^{12} - 2\right], \\
A_{n1}^{22} &= -1 + a_{n+1}a_n + \left(A_{n1}^{12}\right)^2 + A_{n1}^{11}, \\
A_{n1}^{21} &= -\sum_{k=n}^{\infty}\Big\{\left(q_{k+1} + A_{k1}^{12}\right)\Big[a_{k+1}a_k + q_{k+1}(p_{k+1} + q_{k+1}) + q_{k+1}A_{k1}^{12} \\
&\qquad + b_{k+1}^2 + A_{k+1,1}^{11} - 1\Big] - A_{k1}^{12}\left(1 + A_{k1}^{11}\right)\Big\} + \sum_{k=n+1}^{\infty}\left(q_k A_{k1}^{22} - b_k^2 p_k\right), \\
A_{n2}^{12} &= -a_{n+1}a_n\left(q_{n+1} + A_{n1}^{12}\right) + A_{n1}^{12}A_{n1}^{11} + A_{n1}^{12} - A_{n1}^{21}, \\
A_{n2}^{11} &= \sum_{k=n+1}^{\infty}\Big\{\left(b_k^2 - 1\right)A_{k1}^{11} - a_{k+1}a_k\left[\left(q_{k+1} + A_{k1}^{12}\right)A_{k+1,1}^{12} - A_{k+1,1}^{22}\right] \\
&\qquad - \left(p_k - A_{k1}^{12}\right)\left[q_k A_{k1}^{11} + A_{k1}^{12} - A_{k2}^{12}\right] - q_k A_{k1}^{21} + A_{k1}^{12}A_{k2}^{12} - A_{k1}^{22}\Big\}, \\
A_{n2}^{22} &= -a_{n+1}a_n\left(q_{n+1} + A_{n1}^{12}\right)A_{n+1,1}^{12} + a_{n+1}a_n A_{n+1,1}^{22} + A_{n1}^{12}A_{n2}^{12} - A_{n1}^{11} + A_{n2}^{11}, \\
A_{n2}^{21} &= \sum_{k=n}^{\infty}\Big\{A_{k1}^{12}A_{k2}^{11} + A_{k2}^{21} - a_{k+1}a_k\Big[\left(q_{k+1} + A_{k1}^{12}\right)A_{k+1,1}^{11} - A_{k+1,1}^{21}\Big]\Big\} \\
&\quad - \sum_{k=n+1}^{\infty}\Big[\left(q_k + A_{k-1,1}^{12}\right)\left(q_k A_{k2}^{12} - A_{k1}^{11} + A_{k2}^{11}\right) + b_k^2 A_{k2}^{21} - p_k A_{k2}^{22} + A_{k1}^{21}\Big],
\end{aligned}
\tag{2.5}
$$

and for $m \geq 3$

$$\begin{aligned}
A_{nm}^{12} =& -a_{n+1}a_n\left[\left(q_{n+1}+A_{n1}^{12}\right)A_{n+1,m-2}^{11}+A_{n+1,m-2}^{21}\right] \\
&+A_{n1}^{12}A_{n,m-1}^{11}+A_{n,m-1}^{12}-A_{n,m-1}^{21}, \\
A_{nm}^{11} =& -\sum_{k=n+1}^{\infty} a_{k+1}a_k\left[\left(q_{k+1}+A_{k1}^{12}\right)A_{k+1,m-1}^{12}-A_{k+1,m-1}^{22}\right] \\
&-\sum_{k=n+1}^{\infty}\left(p_k-A_{k1}^{12}\right)\left(q_kA_{k,m-1}^{11}+A_{k,m-1}^{12}-A_{km}^{12}\right)+\sum_{k=n+1}^{\infty}\left(b_k^2-1\right)A_{k,m-1}^{11} \\
&-\sum_{k=n+1}^{\infty} q_kA_{k,m-1}^{21}+\sum_{k=n+1}^{\infty}A_{k1}^{12}A_{km}^{12}-\sum_{k=n+1}^{\infty}A_{k,m-1}^{22}, \\
A_{nm}^{22} =& -a_{n+1}a_n\left[\left(q_{n+1}+A_{n1}^{12}\right)A_{n+1,m-1}^{11}-A_{n+1,m-1}^{22}\right]+A_{n1}^{12}A_{nm}^{12}+A_{nm}^{11}-A_{n,m-1}^{11}, \\
A_{nm}^{21} =& -\sum_{k=n}^{\infty} a_{k+1}a_k\left[\left(q_{k+1}+A_{k1}^{12}\right)A_{k+1,m-1}^{11}-A_{k+1,m-1}^{21}\right] \\
&-\sum_{k=n+1}^{\infty}\left(q_k-A_{k-1,1}^{12}\right)\left(q_kA_{km}^{21}+A_{k,m-1}^{11}-A_{km}^{22}\right)-\sum_{k=n+1}^{\infty}\left(b_k^2-1\right)A_{km}^{12} \\
&+\sum_{k=n}^{\infty}A_{k1}^{12}A_{km}^{22}+\sum_{k=n+1}^{\infty}q_kA_{km}^{22}+\sum_{k=n}^{\infty}A_{km}^{12}-\sum_{k=n+1}^{\infty}A_{k,m-1}^{21}.
\end{aligned} \tag{2.6}$$

Moreover

$$\left|A_{nm}^{ij}\right| \leq C\sum_{k=n+[|m/2|]}^{\infty}\left(|1-a_k|+|1+b_k|+\left|p_k\right|+\left|q_k\right|\right) \tag{2.7}$$

holds, where $[|m/2|]$ is the integer part of $m/2$ and $C>0$ is a constant. Therefore f_n is vector-valued analytic function with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$ [10]. The solution $f(z)=(f_n(z))=\begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z)\end{pmatrix}$ is called Jost solution of (1.2).

Let us define

$$F(z)=\left(\gamma_0+2\gamma_1\sin\frac{z}{2}\right)f_1^{(2)}(z)+\left(\beta_0+2\beta_1\sin\frac{z}{2}\right)f_0^{(1)}(z). \tag{2.8}$$

It follows (2.2) and (2.3) that the function F is analytic in $\mathbb{C}_+$, continuous up to the real axis, and

$$F(z+4\pi)=F(z). \tag{2.9}$$

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and spectral singularities we have [10]

$$\sigma_d = \left\{\lambda : \lambda = 2\sin\frac{z}{2},\ z \in P_0,\ F(z) = 0\right\},$$
$$\sigma_{ss} = \left\{\lambda : \lambda = 2\sin\frac{z}{2},\ z \in [0, 4\pi],\ F(z) = 0\right\}, \tag{2.10}$$

where $P_0 := \{z : z \in \mathbb{C},\ z = x + iy,\ 0 \le x \le 4\pi,\ y > 0\}$. The finiteness of the multiplicities of eigenvalues and spectral singularities has been proven in [10]. Using (2.2), (2.3), and (2.8) we obtain

$$\begin{aligned}
F(z) &= \left\{\gamma_0 + \gamma_1\left[(-i)\left(e^{i(z/2)} - e^{-i(z/2)}\right)\right]\right\} f_1^{(2)}(z) \\
&\quad + \left\{\beta_0 + \beta_1\left[(-i)\left(e^{i(z/2)} - e^{-i(z/2)}\right)\right]\right\} f_0^{(1)}(z) \\
&= i\alpha_0^{11}\beta_1 + \left(\gamma_1\alpha_1^{22} + \alpha_0^{11}\beta_0\right)e^{i(z/2)} + i\left(-\gamma_0\alpha_1^{22} + \gamma_1\alpha_1^{22} - \alpha_0^{11}\beta_1\right)e^{iz} \\
&\quad + \left(\gamma_0\alpha_1^{21} - \gamma_1\alpha_1^{22}\right)e^{i(3z/2)} - i\gamma_1\alpha_1^{21}e^{2iz} \\
&\quad + \sum_{m=1}^{\infty}\alpha_0^{11}\beta_1 A_{0m}^{12} e^{i(m-(1/2))z} + i\sum_{m=1}^{\infty}\left(-\alpha_0^{11}\beta_0 A_{0m}^{12} + \alpha_0^{11}\beta_1 A_{0m}^{11}\right)e^{imz} \\
&\quad + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1^{21}A_{1m}^{12} + \gamma_1\alpha_1^{22}A_{1m}^{22} + \alpha_0^{11}\beta_0 A_{0m}^{11} - \alpha_0^{11}\beta_1 A_{0m}^{12}\right)e^{i(m+(1/2))z} \\
&\quad + i\sum_{m=1}^{\infty}\left(-\gamma_0\alpha_1^{21}A_{1m}^{12} - \gamma_0\alpha_1^{22}A_{1m}^{22} + \gamma_1\alpha_1^{21}A_{1m}^{11} + \gamma_1\alpha_1^{22}A_{1m}^{21}\right. \\
&\qquad\qquad \left. -\alpha_0^{11}\beta_1 A_{0m}^{11}\right)e^{i(m+1)z} \\
&\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1^{21}A_{1m}^{11} + \gamma_0\alpha_1^{22}A_{1m}^{21} - \gamma_1\alpha_1^{21}A_{1m}^{12} - \gamma_1\alpha_1^{22}A_{1m}^{22}\right)e^{i(m+(3/2))z} \\
&\quad + i\sum_{m=1}^{\infty}\left(-\gamma_1\alpha_1^{21}A_{1m}^{11} - \gamma_1\alpha_1^{22}A_{1m}^{21}\right)e^{i(m+2)z}. \\
&\quad + i\sum_{m=1}^{\infty}\left(-\gamma_1\alpha_1^{21}A_{1m}^{11} - \gamma_1\alpha_1^{22}A_{1m}^{21}\right)e^{i(m+2)z}.
\end{aligned} \tag{2.11}$$

Definition 2.1. The multiplicity of a zero of F in $P := P_0 \cup [0, 4\pi]$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2), (1.3).

3. Principal Functions

In this section we also assume that (2.1) holds.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\nu$ denote the zeros of F in P_0 and $[0, 4\pi]$ with multiplicities $m_1, m_2, \ldots, m_k$ and $m_{k+1}, m_{k+2}, \ldots, m_\nu$, respectively.

Let us define $\ell := \binom{\tilde{\ell}}{\hat{\ell}}$ where

$$\begin{aligned}\left(\tilde{\ell}y\right)_n &= a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)}, \quad n \in \mathbb{N},\\ \left(\hat{\ell}y\right)_n &= a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)}, \quad n \in \mathbb{N}.\end{aligned} \tag{3.1}$$

Definition 3.1. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors y_n, $d/(d\lambda y_n), d^2/d\lambda^2 y_n, \dots, d^\nu/d\lambda^\nu y_n$,

$$\frac{d^j}{d\lambda^j}y := \left\{\frac{d^j}{d\lambda^j}y_n\right\}_{n\in\mathbb{N}}, \quad (j = 0, 1, \dots, \nu;\ n \in \mathbb{N}), \tag{3.2}$$

satisfy the equations

$$\begin{gathered}(\ell y)_n - \lambda_0 y_n = 0,\\ \left(\ell\left(\frac{d^j}{d\lambda^j}y\right)\right)_n - \lambda_0\frac{d^j}{d\lambda^j}y_n - \frac{d^{j-1}}{d\lambda^{j-1}}y_n = 0, \quad j = 1, 2, \dots, \nu,\ n \in \mathbb{N},\end{gathered} \tag{3.3}$$

then the vector y_n is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $(d/d\lambda)y_n, (d^2/d\lambda^2)y_n, \dots, (d^\nu/d\lambda^\nu)y_n$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$. The principal vectors of the spectral singularities of L are defined similarly.

We define the vectors

$$\begin{gathered}\frac{d^j}{d\lambda^j}V_n(\lambda_i) = \begin{pmatrix}\dfrac{1}{j!}\left\{\dfrac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i}\\ \dfrac{1}{j!}\left\{\dfrac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i}\end{pmatrix},\\ n \in \mathbb{N}, \quad j = 0, 1, \dots, m_i - 1, \quad i = 1, 2, \dots, k, k+1, \dots, \nu,\end{gathered} \tag{3.4}$$

where $\lambda = 2\sin(z/2)$ and

$$E_n(\lambda) = \begin{pmatrix}E_n^{(1)}(\lambda)\\ E_n^{(2)}(\lambda)\end{pmatrix} := f_n\left(2\arcsin\frac{\lambda}{2}\right) = \begin{pmatrix}f_n^{(1)}\left(2\arcsin\dfrac{\lambda}{2}\right)\\ f_n^{(2)}\left(2\arcsin\dfrac{\lambda}{2}\right)\end{pmatrix}. \tag{3.5}$$

If

$$y(\lambda) = \{y_n(\lambda)\} := \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix}_{n\in\mathbb{N}} \tag{3.6}$$

is a solution of (1.2), then

$$\frac{d^j}{d\lambda^j} y(\lambda) = \left\{ \left(\frac{d^j}{d\lambda^j} \right) y_n(\lambda) \right\}_{n\in\mathbb{N}} := \begin{Bmatrix} \left(\dfrac{d^j}{d\lambda^j} \right) y_n^{(1)}(\lambda) \\ \left(\dfrac{d^j}{d\lambda^j} \right) y_n^{(2)}(\lambda) \end{Bmatrix} \tag{3.7}$$

satisfies

$$\begin{aligned} &\begin{pmatrix} a_{n-1} \dfrac{d^j}{d\lambda^j} y_{n+1}^{(2)}(\lambda) + b_n \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + p_n \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \\ a_{n-1} \dfrac{d^j}{d\lambda^j} y_{n-1}^{(1)}(\lambda) + b_n \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + q_n \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \end{pmatrix} \\ &\quad = \begin{pmatrix} \lambda \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + j \dfrac{d^{j-1}}{d\lambda^{j-1}} y_n^{(1)}(\lambda) \\ \lambda \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + j \dfrac{d^{j-1}}{d\lambda^{j-1}} y_n^{(2)}(\lambda) \end{pmatrix}. \end{aligned} \tag{3.8}$$

From (3.4) and (3.8) we get that

$$\begin{gathered} (\ell V(\lambda_i))_n - \lambda_0 V_n(\lambda_i) = 0, \\ \left(\ell \left(\frac{d^j}{d\lambda^j} V(\lambda_i) \right) \right)_n - \lambda_0 \frac{d^j}{d\lambda^j} V_n(\lambda_i) - \frac{d^{j-1}}{d\lambda^{j-1}} V_n(\lambda_i) = 0, \\ n \in \mathbb{N}, \quad j = 1, 2, \ldots, m_i - 1, \quad i = 1, 2, \ldots, \nu. \end{gathered} \tag{3.9}$$

The vectors $d^j/d\lambda^j V_n(\lambda_i)$, $j = 0, 1, 2, \ldots, m_i - 1$, $i = 1, 2, \ldots, k$ and $d^j/d\lambda^j V_n(\lambda_i)$, $j = 0, 1, 2, \ldots, m_i - 1$, $i = k+1, k+2, \ldots, \nu$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.2.

$$\begin{gathered} \frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1, \quad i = 1, 2, \ldots, k, \\ \frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1, \quad i = k+1, k+2, \ldots, \nu. \end{gathered} \tag{3.10}$$

Proof. Using (3.5) we get that

$$\left\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i}=\sum_{t=0}^{j}C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\right\}_{z=z_i},\quad n\in\mathbb{N},$$
$$\left\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i}=\sum_{t=0}^{j}D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\right\}_{z=z_i},\quad n\in\mathbb{N},\tag{3.11}$$

where $\lambda_i=2\sin z_i/2$, $z_i\in P=P_0\cup[0,4]$, $i=1,2,\ldots,k$ and C_t, D_t are constant depending on λ. From (2.2) we obtain that

$$\begin{aligned}\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\right\}_{z=z_i}&=\alpha_n^{11}i^t\left(n+\frac{1}{2}\right)^t e^{iz_i(n+(1/2))}\\&\quad+\sum_{m=1}^{\infty}\alpha_n^{11}\left\{A_{nm}^{11}i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\&\quad\left.-A_{nm}^{12}i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\},\end{aligned}\tag{3.12}$$

$$\begin{aligned}\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\right\}_{z=z_i}&=\alpha_n^{21}i^t\left(n+\frac{1}{2}\right)^t e^{iz_i(n+(1/2))}-i(in)^t\alpha_n^{22}e^{inz_i}\\&\quad+\sum_{m=1}^{\infty}\alpha_n^{21}\left\{A_{nm}^{11}i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\&\quad\left.-A_{nm}^{12}i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\}\\&\quad+\sum_{m=1}^{\infty}\alpha_n^{22}\left\{A_{nm}^{21}i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\&\quad\left.-A_{nm}^{22}i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\}.\end{aligned}\tag{3.13}$$

For the principal vectors $(d^j/d\lambda^j)V_n(\lambda_i)=\{(d^j/d\lambda^j)V_n(\lambda_i)\}_{n\in\mathbb{N}}$, $j=0,1,\ldots,m_i-1$, $i=1,2,\ldots,k$ corresponding to the eigenvalues of L we get

$$\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i}=\frac{1^j}{j!}\sum_{t=0}^{j}C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\},$$
$$j=0,1,\ldots,m_i-1,\quad i=1,2,\ldots,k,$$
$$\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i}=\frac{1^j}{j!}\sum_{t=0}^{j}D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\},$$
$$j=0,1,\ldots,m_i-1,\quad i=1,2,\ldots,k.\tag{3.14}$$

Since $\operatorname{Im}\lambda_i > 0$ for $i = 1, 2, \ldots, k$ from (3.14) we obtain that

$$\begin{aligned}
\left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 &= \sum_{n=1}^{\infty} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right) \\
&= \left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left(\left| \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right) \\
&\leq \left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left\{ \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 \right) + \left(\sum_{t=0}^{j} \left| D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right) \right\} \\
&\leq \left(\frac{1}{j!} \right)^2 \left(\sum_{n=1}^{\infty} \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \left(\left| \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| + \left| \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right| \right) \right)^2,
\end{aligned} \tag{3.15}$$

or

$$\begin{aligned}
&\left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 \\
&\leq \left(\frac{1}{j!} \right)^2 \Bigg\{ \sum_{n=1}^{\infty} \Bigg[\sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \\
&\qquad \times \left\{ \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left(\left| n + \frac{1}{2} \right|^t e^{-(n+(1/2)) \operatorname{Im} z_i} \right) + \left|\alpha_n^{22}\right| |n|^t e^{-n \operatorname{Im} z_i} \right\} \\
&\qquad + \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \Bigg\{ \sum_{m=1}^{\infty} \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left(\left|A_{nm}^{11}\right| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+(1/2)) \operatorname{Im} z_i} \right) \\
&\qquad\qquad + \left|A_{nm}^{12}\right| |m+n|^t e^{-(m+n) \operatorname{Im} z_i} \Bigg\} \\
&\qquad + \sum_{t=0}^{j} |D_t| \Bigg\{ \sum_{m=1}^{\infty} \left|\alpha_n^{22}\right| \Bigg(\left|A_{nm}^{21}\right| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+(1/2)) \operatorname{Im} z_i} + \left|A_{nm}^{22}\right| |m+n|^t \\
&\qquad\qquad \times e^{-(m+n) \operatorname{Im} z_i} \Bigg) \Bigg\} \Bigg] \Bigg\}^2.
\end{aligned} \tag{3.16}$$

From (3.16),

$$\begin{aligned}
&\left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \left\{ \begin{array}{c} \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left(\left| n + \frac{1}{2} \right|^t e^{-(n+(1/2)) \operatorname{Im} z_i} \right) \\ + \left|\alpha_n^{22}\right| |n|^t e^{-n \operatorname{Im} z_i} \end{array} \right\} \\
&\leq \frac{A}{(j!)^2} \sum_{n=1}^{\infty} \left(1 + \left(n + \frac{1}{2} \right) + \left(n + \frac{1}{2} \right)^2 + \cdots + \left(n + \frac{1}{2} \right)^j \right) e^{-(n+(1/2)) \operatorname{Im} z_i}
\end{aligned}$$

$$
\begin{aligned}
&+\left(1+n+n^2+\cdots+n^j\right)e^{-n\,\mathrm{Im}\,z_i} \\
&\leq \frac{A(j+1)^2}{(j!)^2}\sum_{n=1}^{\infty}\left[\left(n+\frac{1}{2}\right)^j e^{-(n+(1/2))\,\mathrm{Im}\,z_i}+n^j e^{-n\,\mathrm{Im}\,z_i}\right]<\infty.
\end{aligned} \tag{3.17}
$$

Holds, where

$$
A=\max\{|C_t|,|D_t|\}\max\left\{\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right),\left|\alpha_n^{22}\right|\right\}. \tag{3.18}
$$

Now we define the function

$$
\begin{aligned}
g_n(z)=\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}&\left\{\sum_{m=1}^{\infty}\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left(\left|A_{nm}^{11}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\,\mathrm{Im}\,z_i}\right.\right.\\
&\qquad\left.\left.+\left|A_{nm}^{12}\right||m+n|^t e^{-(m+n)\,\mathrm{Im}\,z_i}\right)\right\}\\
&+\sum_{t=0}^{j}|D_t|\left\{\sum_{m=1}^{\infty}\left|\alpha_n^{22}\right|\left(\left|A_{nm}^{21}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\,\mathrm{Im}\,z_i}\right.\right.\\
&\qquad\left.\left.+\left|A_{nm}^{22}\right||m+n|^t e^{-(m+n)\,\mathrm{Im}\,z_i}\right)\right\}.
\end{aligned} \tag{3.19}
$$

So we get

$$
\begin{aligned}
&\left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\left\{\sum_{m=1}^{\infty}\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left(\left|A_{nm}^{11}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\,\mathrm{Im}\,z_i}\right.\right.\right.\\
&\qquad\left.\left.+\left|A_{nm}^{12}\right||m+n|^t e^{-(m+n)\,\mathrm{Im}\,z_i}\right)\right\}\\
&\quad+\sum_{t=0}^{j}|D_t|\left\{\sum_{m=1}^{\infty}\left|\alpha_n^{22}\right|\left(\left|A_{nm}^{21}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\,\mathrm{Im}\,z_i}\right.\right.\\
&\qquad\left.\left.\left.+\left|A_{nm}^{22}\right||m+n|^t e^{-(m+n)\,\mathrm{Im}\,z_i}\right)\right\}\right]\\
&=\left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty}g_n(z).
\end{aligned} \tag{3.20}
$$

Using the boundness of A_{nm}^{ij} and α_n^{ij}, $i,j=1,2$ we obtain that

$$g_n(z) \le \max\{|C_t|,|D_t|\} M \sum_{t=0}^{j}\sum_{m=1}^{\infty}\left\{\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\operatorname{Im} z_i} + |m+n|^t e^{-(m+n)\operatorname{Im} z_i}\right\}, \tag{3.21}$$

where

$$\begin{aligned} M = \max\Big\{&\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left|A_{nm}^{11}\right|, \left|\alpha_n^{22}\right|\left|A_{nm}^{21}\right|, \\ &\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left|A_{nm}^{12}\right|, \left|\alpha_n^{22}\right|\left|A_{nm}^{22}\right|\Big\}. \end{aligned} \tag{3.22}$$

If we take $\max\{|C_t|,|D_t|\}M = N$, we can write

$$\begin{aligned} g_n(z) &\le N\sum_{t=0}^{j} e^{-n\operatorname{Im} z_i}\sum_{m=1}^{\infty}\left\{\left(m+n+\frac{1}{2}\right)^t e^{-m\operatorname{Im} z_i} + (m+n)^t e^{-m\operatorname{Im} z_i}\right\} \\ &= Ne^{-n\operatorname{Im} z_i}\left\{\sum_{m=1}^{\infty} 2e^{-m\operatorname{Im} z_i} + \sum_{m=1}^{\infty} e^{-m\operatorname{Im} z_i}\left(\left(m+n+\frac{1}{2}\right)+(m+n)\right)+\cdots \right.\\ &\qquad \left. + \sum_{m=1}^{\infty} e^{-m\operatorname{Im} z_i}\left(\left(m+n+\frac{1}{2}\right)^j + (m+n)^j\right)\right\} \\ &\le Ne^{-n\operatorname{Im} z_i}\sum_{m=1}^{\infty}\sum_{t=0}^{j} e^{-m\operatorname{Im} z_i}\left(\left(m+n+\frac{1}{2}\right)^t + (m+n)^t\right) \\ &\le Be^{-n\operatorname{Im} z_i}, \end{aligned} \tag{3.23}$$

where

$$B = A\sum_{m=1}^{\infty}\sum_{t=0}^{j} e^{-m\operatorname{Im} z_i}\left(\left(m+n+\frac{1}{2}\right)^t + (m+n)^t\right). \tag{3.24}$$

Therefore we have

$$\left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty} g_n(z) \le \left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty} Be^{-n\operatorname{Im} z_i} < \infty. \tag{3.25}$$

From (3.17) and (3.25) $d^j/d\lambda^j V_n(\lambda_i) \in \ell_2(\mathbb{N},\mathbb{C}^2)$, $j=0,1,\ldots,m_i-1$, $i=1,2,\ldots,k$.

On the other hand, since $\operatorname{Im} z_i = 0$ for $j = 0, 1, \dots, m_i - 1$; $i = k+1, k+2, \dots, \nu$ using (3.12) we find that

$$\sum_{n=1}^{\infty} \left| \alpha_n^{11} i^t \left(n + \frac{1}{2} \right)^t e^{iz_i(n+(1/2))} \right|^2 = \infty, \tag{3.26}$$

but the other terms in (3.12) belongs $\ell_2(\mathbb{N}, \mathbb{C}^2)$, so $d^j/(d\lambda^j) E_n^{(1)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$. Similarly from (3.13) we get $d^j/(d\lambda^j) E_n^{(2)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$, then we obtain that $d^j/(d\lambda^j) V_n(\lambda_i) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$, $j = 0, 1, \dots, m_i - 1; i = k+1, k+2, \dots, \nu$. □

Let us introduce Hilbert space $j = 0, 1, 2, \dots$

$$H_{-j}(\mathbb{N}) = \left\{ y = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} : \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left(\left| y_n^{(1)} \right|^2 + \left| y_n^{(2)} \right|^2 \right) < \infty \right\}, \tag{3.27}$$

with

$$\|y\|_{-j}^2 = \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left(\left| y_n^{(1)} \right|^2 + \left| y_n^{(2)} \right|^2 \right). \tag{3.28}$$

Theorem 3.3. *$d^j/(d\lambda^j) V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$, $j = 0, 1, 2, \dots, m_i - 1$, $i = k+1, k+2, \dots, \nu$.*

Proof. Using (3.4), (3.14) we have

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda = \lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda = \lambda_i} \right|^2 \right) \\
&\quad = \sum_{n \in \mathbb{N}} \frac{(1 + |n|)^{-2(j+1)}}{(j!)^2} \left\{ \left| \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right\} \\
&\quad \le \frac{1}{(j!)^2} \sum_{n=1}^{\infty} (1 + |n|)^{-2(j+1)} \left\{ \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 + \left(\sum_{t=0}^{j} \left| D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right| \right)^2 \right\},
\end{aligned} \tag{3.29}$$

for $j = 0, 1, 2, \dots, m_i - 1$, $i = k+1, k+2, \nu$. Since $\operatorname{Im} z_i = 0$, using (3.29) we obtain

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \frac{1}{(j!)^2} \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 \\
&\quad = \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \sum_{t=0}^{j} (1 + |n|)^{-(j+1)} \left(n + \frac{1}{2} \right)^t \left| \alpha_n^{11} \right| |C_t| \right. \\
&\qquad \left. + \sum_{t=0}^{j} |C_t| \left| \alpha_n^{11} \right| (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left| A_{nm}^{11} \right| \left(m + n + \frac{1}{2} \right)^t + \left| A_{nm}^{12} \right| (m+n)^t \right\}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(j!)^2}\sum_{n=1}^{\infty}\Bigg\{\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\left(n+\frac{1}{2}\right)^{t}\left|\alpha_n^{11}\right||C_t|\right)^2 \\
&\qquad + 2(1+|n|)^{-2(j+1)}\left|\alpha_n^{11}\right|^2\left[\sum_{t=0}^{j}\left(n+\frac{1}{2}\right)^{t}|C_t|\right] \\
&\qquad \times\left[\sum_{t=0}^{j}|C_t|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^{t}+\left|A_{nm}^{12}\right|(m+n)^{t}\right] \\
&\qquad + \left(\sum_{t=0}^{j}|C_t|(1+|n|)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^{t}+\left|A_{nm}^{12}\right|(m+n)^{t}\right)^2\Bigg\}.
\end{aligned}
\tag{3.30}
$$

Using (3.30), (2.1), and (2.7) we first obtain that

$$
\begin{aligned}
&\left(\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(\left|A_{nm}^{11}\right|\left(m+\mathrm{n}+\frac{1}{2}\right)^{t}+\left|A_{nm}^{12}\right|(m+n)^{t}\right)\right)^2 \\
&\quad\le 4\Bigg(\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(m+n+\frac{1}{2}\right)^{t}C \\
&\qquad\quad \times\sum_{j=n+\lceil m/2\rceil}^{\infty}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)e^{-\varepsilon j^{\delta}}e^{\varepsilon j^{\delta}}\Bigg)^2 \\
&\quad\le 4\Bigg\{\sum_{t=0}^{j}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(m+n+\frac{1}{2}\right)^{t}C\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{\delta}\right) \\
&\qquad\quad \times\sum_{j=n+\lceil m/2\rceil}^{\infty}e^{\varepsilon j^{\delta}}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)\Bigg\}^2 \\
&\quad\le C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^{t}\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{\delta}\right)\right)^2 \\
&\quad\le C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^{t}\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{1/2}\right)\right)^2 \\
&\quad\le C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^{t}\exp\left(-\varepsilon\frac{\sqrt{2}}{4}\left(n^{\frac{1}{2}}+m^{\frac{1}{2}}\right)\right)\right)^2
\end{aligned}
$$

$$= C_1(1+|n|)^{-2(j+1)} \exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t \exp\left(-\varepsilon\frac{\sqrt{2}}{4}m^{1/2}\right)\right)^2$$
$$= G\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{\frac{1}{2}}\right)(1+|n|)^{-2(j+1)}, \tag{3.31}$$

where

$$C_1 = \left(2C\left|\alpha_n^{11}\right|\sum_{j=n+[m/2]}^{\infty} e^{\varepsilon j^{\delta}}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)\right)^2,$$
$$G = C_1\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t \exp\left(-\varepsilon\frac{\sqrt{2}}{4}m^{1/2}\right)\right)^2. \tag{3.32}$$

So we get from (3.31)

$$\sum_{n=1}^{\infty}\left(\sum_{t=0}^{j}|C_t|(1+|n|)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^t+\left|A_{nm}^{12}\right|(m+n)^t\right)^2$$
$$\leq G\sum_{n=1}^{\infty}\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)(1+|n|)^{-2(j+1)}<\infty. \tag{3.33}$$

Secondly, using (3.30) and (3.31) we obtain that

$$\sum_{n=1}^{\infty}2\left\{\left[\sum_{t=0}^{j}\left|\alpha_n^{11}\right||C_t|(1+|n|)^{-(j+1)}\left(n+\frac{1}{2}\right)^t\right]\right.$$
$$\left.\times\left[\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(\left(m+n+\frac{1}{2}\right)^t\left|A_{nm}^{11}\right|+(m+n)^t\left|A_{nm}^{12}\right|\right)\right]\right\} \tag{3.34}$$
$$\leq\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}(1+|n|)^{-2(j+1)}\left(n+\frac{1}{2}\right)^t\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)G^{1/2}\right]<\infty,$$

and also the first part of the (3.31) obviously convergent so, we get from (3.33) and (3.34)

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\}\right|\right)^2<\infty, \tag{3.35}$$

and similarly

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\}\right|\right)^2<\infty. \tag{3.36}$$

Finally $d^j/(d\lambda^j)V_n(\lambda_i)\in H_{-(j+1)}(\mathbb{N})$, $j=0,1,2,\ldots,m_i-1$, $i=k+1,k+2,\ldots,\nu$. □

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