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Research Article

Smooth Solutions of a Class of Iterative Functional Differential Equations

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By Faà di Bruno's formula, using the fixed-point theorems of Schauder and Banach, we study the existence and uniqueness of smooth solutions of an iterative functional differential equation $x'(t) = 1/(c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t))$.

1. Introduction

There has been a lot of monographs and research articles to discuss the kinds of solutions of functional differential equations since the publication of Jack Hale's paper [1]. Several papers discussed the iterative functional differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), \ldots, x^{[m]}(t)\right), \tag{1.1}$$

where $x^{[0]}(t) = t,\ x^{[1]}(t) = x(t),\ x^{[k]}(t) = x(x^{[k-1]}(t)),\ k = 2, \ldots, m$. More specifically, Eder [2] considered the functional differential equation

$$x'(t) = x^{[2]}(t) \tag{1.2}$$

and proved that every solution either vanishes identically or is strictly monotonic. Furthermore, Fečkan [3] and Wang [4] studied the equation

$$x'(t) = f\left(x^{[2]}(t)\right) \tag{1.3}$$

with different conditions. Staněk [5] considered the equation

$$x'(t) = x(t) + x^{[2]}(t) \tag{1.4}$$

and obtained every solution either vanishes identically or is strictly monotonic. Si and his coauthors [6, 7] studied the following equations:

$$\begin{aligned} x'(t) &= x^{[m]}(t), \\ x'(t) &= \frac{1}{x^{[m]}(t)}, \end{aligned} \tag{1.5}$$

$$x'(t) = \frac{1}{c_0 x^{[0]}(t) + c_1 x^{[1]}(t) + \cdots + c_m x^{[m]}(t)} \tag{1.6}$$

and established sufficient conditions for the existence of analytic solutions. Especially in [8, 9], the smooth solutions of the following equations:

$$\begin{aligned} x'(t) &= \sum_{j=1}^{m} a_j x^{[j]}(t) + F(t), \\ x'(t) &= \sum_{j=1}^{m} a_j(t) x^{[j]}(t) + F(t), \end{aligned} \tag{1.7}$$

have been studied by the fixed-point theorems of Schauder and Banach.

A smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. Let $x \in C^n$ if $x', \dots, x^{(n)}$ are continuous, $C^n(I, I)$ is the set in which $x \in C^n$ and maps a closed interval I into I. As in [9], we, using the same symbols, denote the norm

$$\|x\|_n = \sum_{k=0}^{n} \left\| x^{(k)} \right\|, \quad \|x\| = \max_{t \in I}\{|x(t)|\}, \tag{1.8}$$

then $C^n(I, R)$ with $\|\cdot\|_n$ is a Banach space, and $C^n(I, I)$ is a subset of $C^n(I, R)$. For given $M_i > 0$ $(i = 1, 2, \dots, n+1)$, let

$$\begin{aligned} \Omega(M_1, \dots, M_{n+1}; I) = \Big\{ x \in C^n(I, I) : \left| x^{(i)}(t) \right| \le M_i, i = 1, 2, \dots, n; \\ \left| x^{(n)}(t_1) - x^{(n)}(t_2) \right| \le M_{n+1} |t_1 - t_2|,\ t, t_1, t_2 \in I \Big\}. \end{aligned} \tag{1.9}$$

For convenience, we will make use of the notation

$$x_{ij}(t) = x^{(i)}\left(x^{[j]}(t)\right), \qquad x_{*jk}(t) = \left(x^{[j]}(t)\right)^{(k)}, \tag{1.10}$$

where i, j, and k are nonnegative integers. Let I be a closed interval in R. By induction, we may prove that

$$x_{*jk}(t) = P_{jk}\left(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t)\right), \tag{1.11}$$

$$\beta_{jk} = P_{jk}\left(\overbrace{x'(\xi), \dots, x'(\xi)}^{j \text{ terms}}; \dots; \overbrace{x^{(k)}(\xi), \dots, x^{(k)}(\xi)}^{j \text{ terms}}\right), \tag{1.12}$$

$$H_{jk} = P_{jk}\left(\overbrace{1, \dots, 1}^{j \text{ terms}}; \overbrace{M_2, \dots, M_2}^{j \text{ terms}}; \dots; \overbrace{M_k, \dots, M_k}^{j \text{ terms}}\right), \tag{1.13}$$

where P_{jk} is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can be found in [8].

In order to seek a solution $x(t)$ of (1.6), in $C^n(I, I)$ such that ξ is a fixed point of the function $x(t)$, that is, $x(\xi) = \xi$, it is natural to seek an interval I of the form $[\xi - \delta, \xi + \delta]$ with $\delta > 0$.

Let us define

$$\begin{aligned} &X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I) \\ &\quad = \left\{x \in \Omega(1, M_2, \dots, M_{n+1}; I) : x(\xi) = \xi_0 = \xi, x^{(i)}(\xi) = \xi_i,\ i = 1, 2, \dots, n\right\}. \end{aligned} \tag{1.14}$$

2. Smooth Solutions of (1.6)

In this section, we will prove the existence theorem of smooth solutions for (1.6). First of all, we have the inequalities in the following for all $x(t), y(t) \in X$:

$$\left|x^{[j]}(t_1) - x^{[j]}(t_2)\right| \le |t_1 - t_2|, \quad t_1, t_2 \in I,\ j = 0, 1, \dots, m, \tag{2.1}$$

$$\left\|x^{[j]} - x^{[j]}\right\| \le j\|x - y\|, \quad j = 1, \dots, m, \tag{2.2}$$

$$\|x - y\| \le \delta^n \left\|x^{(n)} - y^{(n)}\right\|, \tag{2.3}$$

and the proof can be found in [9].

Theorem 2.1. *Let $I = [\xi - \delta, \xi + \delta]$, where ξ and δ satisfy*

$$\xi \ge \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \qquad 0 < \delta \le \xi - \frac{1}{|c_0| - \sum_{i=1}^m |c_i|}, \tag{2.4}$$

where $|c_0| > \sum_{i=0}^m |c_i|$, then (1.6) *has a solution in*

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \tag{2.5}$$

provided the following conditions hold:

(i)

$$\xi_1 = \xi^{-1}\left(\sum_{i=0}^{m} c_i\right)^{-1}, \tag{2.6}$$

$$\begin{aligned}\xi_k = \sum \frac{(-1)^s\ (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!}\left(\xi\sum_{i=0}^{m} c_i\right)^{-s-1}\left(\frac{\sum_{i=0}^{m} c_i\beta_{i1}}{1!}\right)^{s_1} \\ \times\left(\frac{\sum_{i=0}^{m} c_i\beta_{i2}}{2!}\right)^{s_2}\cdots\left(\frac{\sum_{i=0}^{m} c_i\beta_{ik-1}}{(k-1)!}\right)^{s_{k-1}},\end{aligned} \tag{2.7}$$

where $k = 2,\dots,n$, *and the sum is over all nonnegative integer solutions of the Diophantine equation* $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ *and* $s = s_1 + s_2 + \cdots + s_{k-1}$,

(ii)

$$\begin{aligned}\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!}(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\frac{\sum_{i=0}^{m}|c_i|H_{i1}}{1!}\right)^{s_1} \\ \times\left(\frac{\sum_{i=0}^{m}|c_i|H_{i2}}{2!}\right)^{s_2}\cdots\left(\frac{\sum_{i=0}^{m}|c_i|H_{ik-1}}{(k-1)!}\right)^{s_{k-1}} \le M_k, \quad k = 2,\dots,n,\end{aligned} \tag{2.8}$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ *and* $s = s_1 + s_2 + \cdots + s_{k-1}$,

(iii)

$$\begin{aligned}&\sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\ &\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1+1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2} \\ &\quad\times\cdots\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}} + s_1(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1} \\ &\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\left(\sum_{i=0}^{m}|c_i|H_{i3}\right)^{s_3}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}} \\ &\quad+\cdots+s_{n-1}(\xi-\delta)^{-s-1}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-2}\right)^{s_{n-2}} \\ &\quad\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in}\right)\Bigg] \\ &\le M_{n+1},\end{aligned} \tag{2.9}$$

where $s_1 + 2s_2 + \cdots + (n-1)s_{n-1} = n-1$ *and* $s = s_1 + s_2 + \cdots + s_{n-1}$, *Proof.* Define an operator T from X into $C^n(I, I)$ by

$$(Tx)(t) = \xi + \int_{\xi}^{t} \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)} ds. \tag{2.10}$$

We will prove that for any $x \in X$, $Tx \in X$,

$$|(Tx)(t) - \xi| = \left| \int_{\xi}^{t} \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(s)} ds \right| \le \left((\xi - \delta) \left(|c_0| - \sum_{i=1}^{m} |c_i| \right) \right)^{-1} |t - \xi| \le \delta, \tag{2.11}$$

where the second inequality is from (2.4) and $x(I) \subseteq I$. Thus, $(Tx)(I) \subseteq I$.

It is easy to see that

$$(Tx)'(t) = \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)}, \tag{2.12}$$

and by Faà di Bruno's formula, for $k = 2, \ldots, n$, we have

$$\begin{aligned} (Tx)^{(k)}(t) &= \left(\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)} \right)^{(k-1)} = \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{s+1}} \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)'}{1!} \right)^{s_1} \\ &\quad \times \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)''}{2!} \right)^{s_2} \cdots \left(\frac{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{(k-1)}}{(k-1)!} \right)^{s_{k-1}} \\ &= \sum \frac{(-1)^s (k-1)! s!}{s_1! s_2! \cdots s_{k-1}!} \frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i x_{*i1}(t)}{1!} \right)^{s_1} \\ &\quad \times \left(\frac{\sum_{i=0}^{m} c_i x_{*i2}(t)}{2!} \right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i x_{*ik-1}(t)}{(k-1)!} \right)^{s_{k-1}}, \end{aligned} \tag{2.13}$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k - 1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Furthermore, note $(Tx)(\xi) = \xi$, by (2.6) and (2.7),

$$
\begin{aligned}
(Tx)'(\xi) &= \frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(\xi)} = \frac{1}{\xi \sum_{i=0}^{m} c_i} = \xi_1,\\
(Tx)^{(k)}(\xi) &= \sum \frac{(-1)^s (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left(\xi \sum_{i=0}^{m} c_i\right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i x_{*i1}(\xi)}{1!}\right)^{s_1}\\
&\quad \times \left(\frac{\sum_{i=0}^{m} c_i x_{*i2}(\xi)}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i x_{*ik-1}(\xi)}{(k-1)!}\right)^{s_{k-1}}\\
&= \sum \frac{(-1)^s (k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left(\xi \sum_{i=0}^{m} c_i\right)^{s+1}} \left(\frac{\sum_{i=0}^{m} c_i \beta_{i1}}{1!}\right)^{s_1}\\
&\quad \times \left(\frac{\sum_{i=0}^{m} c_i \beta_{i2}}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m} c_i \beta_{ik-1}}{(k-1)!}\right)^{s_{k-1}}\\
&= \xi_k, \qquad k = 2, \ldots, n,
\end{aligned}
\tag{2.14}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. Thus, $(Tx)^{(k)}(\xi) = \xi_k$ for $k = 0, 1, \ldots, n$,

$$
\left|(Tx)'(t)\right| = \left|\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)}\right| \le \left((\xi - \delta)\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)\right)^{-1} \le 1 = M_1,
\tag{2.15}
$$

By (2.8), we have

$$
\begin{aligned}
\left|(Tx)^{(k)}(t)\right| &\le \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} \frac{1}{\left|\sum_{i=0}^{m} c_i x^{[i]}(t)\right|^{s+1}} \left(\frac{\sum_{i=0}^{m}|c_i||x_{*i1}(t)|}{1!}\right)^{s_1}\\
&\quad \times \left(\frac{\sum_{i=0}^{m}|c_i||x_{*i2}(t)|}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|}{(k-1)!}\right)^{s_{k-1}}\\
&\le \sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!} (\xi - \delta)^{-s-1} \left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-s-1} \left(\frac{\sum_{i=0}^{m}|c_i|H_{i1}}{1!}\right)^{s_1}\\
&\quad \times \left(\frac{\sum_{i=0}^{m}|c_i|H_{i2}}{2!}\right)^{s_2} \cdots \left(\frac{\sum_{i=0}^{m}|c_i|H_{ik-1}}{(k-1)!}\right)^{s_{k-1}}\\
&\le M_k, \quad k = 2, \ldots, n,
\end{aligned}
\tag{2.16}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$.

Finally,

$$
\begin{aligned}
&\left|(Tx)^{(n)}(t_1)-(Tx)^{(n)}(t_2)\right| \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_1)\right)^{-s-1}\left(\sum_{i=0}^{m}c_i x_{*i1}(t_1)\right)^{s_1}\left(\sum_{i=0}^{m}c_i x_{*i2}(t_1)\right)^{s_2}\cdots\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_1)\right)^{s_{n-1}}\right. \\
&\qquad\quad\left.-\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\left(\sum_{i=0}^{m}c_i x_{*i1}(t_2)\right)^{s_1}\left(\sum_{i=0}^{m}c_i x_{*i2}(t_2)\right)^{s_2}\cdots\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_2)\right)^{s_{n-1}}\right| \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left[\left|\frac{1}{\left(\sum_{i=0}^{m}c_i x^{[i]}(t_1)\right)^{s+1}}-\frac{1}{\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{s+1}}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t_1)|\right)^{s_1}\right. \\
&\qquad\quad\times\cdots\left(\sum_{i=0}^{m}|c_i||x_{*in-1}(t_1)|\right)^{s_{n-1}}+\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\right| \\
&\quad\times\left|\left(\sum_{i=0}^{m}c_i x_{*i1}(t_1)\right)^{s_1}-\left(\sum_{i=0}^{m}c_i x_{*i1}(t_2)\right)^{s_1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i2}(t_1)|\right)^{s_2}\cdots \\
&\quad\times\left(\sum_{i=0}^{m}|c_i||x_{*in-1}(t_1)|\right)^{s_{n-1}}+\cdots+\left|\left(\sum_{i=0}^{m}c_i x^{[i]}(t_2)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t_2)|\right)^{s_1} \\
&\quad\times\left(\sum_{i=0}^{m}|c_i||x_{*i2}(t_2)|\right)^{s_2}\cdots\left(\sum_{i=0}^{m}|c_i||x_{*in-2}(t_2)|\right)^{s_{n-2}} \\
&\quad\left.\times\left|\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_1)\right)^{s_{n-1}}-\left(\sum_{i=0}^{m}c_i x_{*in-1}(t_2)\right)^{s_{n-1}}\right|\right] \\
&\quad\le \sum \frac{(n-1)!s!}{s_1!s_2!\cdots s_{n-1}!1!^{s_1}2!^{s_2}\cdots(n-1)!^{s_{n-1}}} \\
&\qquad\times\left[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1+1}\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2}\right. \\
&\qquad\quad\times\cdots\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}}+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1} \\
&\qquad\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\left(\sum_{i=0}^{m}|c_i|H_{i3}\right)^{s_3}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}} \\
&\qquad\quad+\cdots+s_{n-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{in-2}\right)^{s_{n-2}} \\
&\qquad\quad\left.\times\left(\sum_{i=0}^{m}|c_i|H_{in-1}\right)^{s_{n-1}-1}\left(\sum_{i=0}^{m}|c_i|H_{in}\right)\right]|t_1-t_2|,
\end{aligned}
\tag{2.17}
$$

where $s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k-1$ and $s = s_1 + s_2 + \cdots + s_{k-1}$. By (2.9), we see that

$$\left|(Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2)\right| \le M_{n+1}|t_1 - t_2|. \tag{2.18}$$

Now, we can say that T is an operator from X into itself.

Next, we will show that T is continuous. Let $x, y \in X$, then

$$\begin{aligned}
\|Tx - Ty\|_n &= \|Tx - Ty\| + \left\|(Tx)' - (Ty)'\right\| + \sum_{k=2}^{n}\left\|(Tx)^{(k)} - (Ty)^{(k)}\right\| \\
&= \max_{t\in I}\left|\int_{\xi}^{t}\left(\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(s)} - \frac{1}{\sum_{i=0}^{m} c_i y^{[i]}(s)}\right)ds\right| \\
&\quad + \max_{t\in I}\left|\frac{1}{\sum_{i=0}^{m} c_i x^{[i]}(t)} - \frac{1}{\sum_{i=0}^{m} c_i y^{[i]}(t)}\right| \\
&\quad + \sum_{k=2}^{n}\max_{t\in I}\left|\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}}\right. \\
&\quad \times \left[\frac{1}{\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{s+1}}\left(\sum_{i=0}^{m} c_i x_{*i1}(t)\right)^{s_1}\left(\sum_{i=0}^{m} c_i x_{*i2}(t)\right)^{s_2}\cdots\right. \\
&\quad \times \left(\sum_{i=0}^{m} c_i x_{*ik-1}(t)\right)^{s_{k-1}} - \frac{1}{\left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{s+1}}\left(\sum_{i=0}^{m} c_i y_{*i1}(t)\right)^{s_1} \\
&\quad \left.\left.\times \left(\sum_{i=0}^{m} c_i y_{*i2}(t)\right)^{s_2}\cdots\left(\sum_{i=0}^{m} c_i y_{*ik-1}(t)\right)^{s_{k-1}}\right]\right| \\
&\le \delta(\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]} - y^{[i]}\right\|\right) \\
&\quad + (\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]} - y^{[i]}\right\|\right) \\
&\quad + \sum_{k=2}^{n}\max_{t\in I}\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \\
&\quad \times \left[\left|\left(\sum_{i=0}^{m} c_i x^{[i]}(t)\right)^{-s-1} - \left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i||x_{*i1}(t)|\right)^{s_1}\cdots\right. \\
&\qquad \times \left(\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|\right)^{s_{k-1}} + \left|\left(\sum_{i=0}^{m} c_i y^{[i]}(t)\right)^{-s-1}\right| \\
&\qquad \times \left|\left(\sum_{i=0}^{m} c_i x_{*i1}(t)\right)^{s_1} - \left(\sum_{i=0}^{m} c_i y_{*i1}(t)\right)^{s_1}\right|
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{i=0}^{m}|c_i||x_{*i2}(t)|\right)^{s_2}\cdots\left(\sum_{i=0}^{m}|c_i||x_{*ik-1}(t)|\right)^{s_{k-1}} \\
&+\cdots+\left|\left(\sum_{i=0}^{m}c_i y^{[i]}(t)\right)^{-s-1}\right|\left(\sum_{i=0}^{m}|c_i|\left|y_{*i1}(t)\right|\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|\left|y_{*ik-2}(t)\right|\right)^{s_{k-2}} \\
&\times\left|\left(\sum_{i=0}^{m}c_i x_{*ik-1}(t)\right)^{s_{k-1}}-\left(\sum_{i=0}^{m}c_i y_{*ik-1}(t)\right)^{s_{k-1}}\right|\Bigg] \\
\leq{}& (\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\| \\
&+\sum_{k=2}^{n}\sum\frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \\
&\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right) \\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}} \\
&\quad+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1} \\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right) \\
&\quad+\cdots+s_{k-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots \\
&\quad\times\left(\sum_{i=0}^{m}|c_i|H_{ik-2}\right)^{s_{k-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}-1}\left(\sum_{i=0}^{m}c_i H_{ik}\right) \\
&\quad\times\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\Bigg] \\
&+\delta^{n+1}(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\left\|x^{(n)}-y^{(n)}\right\|,
\end{aligned}
\tag{2.19}
$$

where $s_1+2s_2+\cdots+(k-1)s_{k-1}=k-1$ and $s=s_1+s_2+\cdots+s_{k-1}$.

Moreover, we can find some constants P_k such that

$$\begin{aligned}
&\sum_{k=2}^{n}\sum \frac{(k-1)!s!}{s_1!s_2!\cdots s_{k-1}!1!^{s_1}2!^{s_2}\cdots(k-1)!^{s_{k-1}}} \\
&\quad\times\Bigg[(s+1)(\xi-\delta)^{-s-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-2}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\times \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}} \\
&\qquad+s_1(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1-1} \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{i2}\right)^{s_2+1}\cdots\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}}\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right) \\
&\qquad+\cdots+s_{k-1}(\xi-\delta)^{-s-1}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-s-1}\left(\sum_{i=0}^{m}|c_i|H_{i1}\right)^{s_1}\cdots \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|H_{ik-2}\right)^{s_{k-2}}\left(\sum_{i=0}^{m}|c_i|H_{ik-1}\right)^{s_{k-1}-1}\left(\sum_{i=0}^{m}c_iH_{ik}\right) \\
&\qquad\times\left(\sum_{i=0}^{m}|c_i|\left\|x^{[i]}-y^{[i]}\right\|\right)\Bigg] \\
&\le\sum_{k=1}^{n-1}P_k\left(\xi,\delta,c_i,H_{ij}\right)\left\|x^{(k)}-y^{(k)}\right\|,
\end{aligned} \tag{2.20}$$

where

$$P_k\left(\xi,\delta,c_i,H_{ij}\right)=P(\xi,\delta;c_1,\ldots,c_m;H_{11},\ldots,H_{1k+1};\ldots;H_{m1},\ldots,H_{mk+1};) \tag{2.21}$$

are the positive constants depend on ξ,δ,c_i, and $H_{ij},\ i=1,\ldots,m;\ j=1,\ldots,k+1$. Then

$$\begin{aligned}
\left\|Tx-Ty\right\|_n &\le(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\|x-y\| \\
&\quad+\sum_{k=1}^{n-1}P_k\left(\xi,\delta,c_i,H_{ij}\right)\left\|x^{(k)}-y^{(k)}\right\| \\
&\quad+\delta^{n+1}(\xi-\delta)^{-2}\left(|c_0|-\sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0|+\sum_{i=1}^{m}i|c_i|\right)\left\|x^{(n)}-y^{(n)}\right\| \\
&\le\Gamma\left\|x-y\right\|_n.
\end{aligned} \tag{2.22}$$

Here,

$$\Gamma = \max\left\{ (\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0| + \sum_{i=1}^{m} i|c_i|\right);\ \max_{1\le k\le n-1}\{P_k(\xi,\delta,c_i,H_{ij})\};\right.$$
$$\left. \delta^{n+1}(\xi-\delta)^{-2}\left(|c_0| - \sum_{i=1}^{m}|c_i|\right)^{-2}\left(|c_0| + \sum_{i=1}^{m} i|c_i|\right)\right\}, \tag{2.23}$$

$k = 1,\dots,n-1$. So we can say that T is continuous.

It is easy to see that X is closed and convex. We now show that X is a relatively compact subset of $C^n(I,I)$. For any $x = x(t) \in X$,

$$\|x\|_n \le \|x\| + \sum_{k=1}^{n}\left\|x^{(k)}\right\| \le |\xi| + \delta + 1 + \sum_{k=2}^{n} M_k. \tag{2.24}$$

Next, for any t_1, t_2 in I, we have

$$|x(t_1) - x(t_2)| \le |t_1 - t_2|. \tag{2.25}$$

Hence, X is bounded in $C^n(I,I)$ and equicontinuous on I, and by the Arzela-Ascoli theorem, we know X is relatively compact in $C^n(I,I)$, since $C^n(I,I)$ is the subset of $C^n(I,R)$, and we can say that X is relatively compact in $C^n(I,R)$.

From Schauder's fixed-point theorem, we conclude that

$$x(t) = \xi + \int_{\xi}^{t} \frac{1}{c_0 x^{[0]}(s) + c_1 x^{[1]}(s) + \cdots + c_m x^{[m]}(s)}\, ds, \tag{2.26}$$

for some $x = x(t)$ in X. By differentiating both sides of the above equality, we see that x is the desired solution of (1.6). This completes the proof. □

Theorem 2.2. *Let* $I = [\xi - \delta, \xi + \delta]$*, where* ξ *and* δ *satisfy* (2.4)*, then* (1.6) *has a unique solution in*

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I), \tag{2.27}$$

provided the conditions (2.6)–(2.9) *hold and* $\Gamma < 1$ *in* (2.23).

Proof. Since $\Gamma < 1$, we see that T defined by (2.10) is contraction mapping on the close subset X of $C^n(I,I)$. Thus, the fixed point x in the proof of Theorem 2.1 must be unique. This completes the proof. □

Remark 2.3. By Theorem 2.1 or Theorem 2.2, the existence and uniqueness of smooth solutions of an iterative functional differential equation of the form (1.6) can be obtained. If $n \to +\infty$, we can also find that the solution is C^∞-smooth.

Now, we will show that the conditions in Theorem 2.1 do not self-contradict. Consider the following equation:

$$x'(t) = \frac{1}{t + (1/2)x(t) + (1/4)x(x(t))}, \tag{2.28}$$

where $c_0 = 1$, $c_1 = (1/2)$, $c_2 = (1/4)$, and $\xi \geq 4$. Moreover, we take $0 < \delta \leq \xi - 4$. Then, (2.4) is satisfied, and ξ, δ define the interval $I = [\xi - \delta, \xi + \delta]$. Now, take $\xi_0 = \xi$,

$$\begin{gathered} \xi_1 = \frac{4}{7}\xi^{-1}, \\ \xi_2 = -\frac{16}{49}\xi^{-2}\left(1 + \frac{1}{2}\xi_1 + \frac{1}{4}\xi_1^2\right), \\ \xi_3 = \frac{4}{343}\xi^{-3}\left(4 + 2\xi_1 + \xi_1^2\right)^2 - \frac{4}{49}\xi^{-2}\xi_2\left(2 + \xi_1 + \xi_1^2\right), \end{gathered} \tag{2.29}$$

then (2.6) and (2.7) are satisfied.

Finally, if we take

$$M_1 = 1, \quad M_2 = 28(\xi - \delta)^{-2}, \quad M_3 = 392(\xi - \delta)^{-3} + 16(\xi - \delta)^{-2} M_2 \tag{2.30}$$

as positive, and

$$M_4 = 8232(\xi - \delta)^{-4} + 576(\xi - \delta)^{-3} M_2 + 8(\xi - \delta)^{-2}\left(6M_2^2 + 5M_3\right), \tag{2.31}$$

then (2.8) and (2.9) are satisfied.

Thus, we have shown that when $\xi_0, \ldots, \xi_3$ and $M_1, \ldots, M_4$ are defined as above, then there will be a solution for (2.28) in $X(\xi; \xi_0, \ldots, \xi_3; 1, \ldots, M_4; I)$.

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