Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 975981, 32 pages
doi:10.1155/2012/975981

Research Article

Some Modified Extragradient Methods for Solving Split Feasibility and Fixed Point Problems

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Received 17 October 2012; Accepted 14 November 2012

Academic Editor: Jen-Chih Yao

We consider and study the modified extragradient methods for finding a common element of the solution set Γ of a split feasibility problem (SFP) and the fixed point set $\mathrm{Fix}(S)$ of a strictly pseudocontractive mapping S in the setting of infinite-dimensional Hilbert spaces. We propose an extragradient algorithm for finding an element of $\mathrm{Fix}(S)\cap\Gamma$ where S is strictly pseudocontractive. It is proven that the sequences generated by the proposed algorithm converge weakly to an element of $\mathrm{Fix}(S)\cap\Gamma$. We also propose another extragradient-like algorithm for finding an element of $\mathrm{Fix}(S)\cap\Gamma$ where $S : C \to C$ is nonexpansive. It is shown that the sequences generated by the proposed algorithm converge strongly to an element of $\mathrm{Fix}(S)\cap\Gamma$.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of $\mathcal{H}$ and let P_C be the metric projection from $\mathcal{H}$ onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers.

A mapping $A : C \to \mathcal{H}$ is called α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

For a given mapping $A : C \to \mathcal{H}$, we consider the following variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

The solution set of the VI (1.2) is denoted by $\mathrm{VI}(C, A)$. The variational inequality was first discussed by Lions [1] and now is well known. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems; see, for example, [2–4].

A mapping $S : C \to C$ is called k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C; \tag{1.3}$$

see [5]. We denote by $\mathrm{Fix}(S)$ the fixed point set of S; that is, $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$. In particular, if $k = 0$, then S is called a nonexpansive mapping. In 2003, for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ when $C \subset \mathcal{H}$ is nonempty, closed and convex, $S : C \to C$ is nonexpansive and $A : C \to \mathcal{H}$ is α-inverse strongly monotone, Takahashi and Toyoda [6] introduced the following Mann's type iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \geq 0, \tag{1.4}$$

where $x_0 \in C$ chosen arbitrarily, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that if $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Further, motivated by the idea of Korpelevich's extragradient method [7], Nadezhkina and Takahashi [8] introduced an iterative algorithm for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. They obtained a weak convergence theorem for two sequences generated by the proposed algorithm. Here the so-called extragradient method was first introduced by Korpelevich [7]. In 1976, She applied this method for finding a solution of a saddle point problem and proved the convergence of the proposed algorithm to a solution of this saddle point problem. Very recently, Jung [9] introduced a new composite iterative scheme by the viscosity approximation method and proved the strong convergence of the proposed scheme to a common element of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality for an inverse-strongly monotone mapping in a Hilbert space.

On the other hand, let C and Q be nonempty closed convex subsets of infinite-dimensional real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is to find a point x^* with the following property:

$$x^* \in C, \qquad Ax^* \in Q, \tag{1.5}$$

where $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $B(\mathcal{H}_1, \mathcal{H}_2)$ denote the family of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

In 1994, the SFP was first introduced by Censor and Elfving [10], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction problems can be formulated as the SFP; see, for example, [11] and the references therein. Recently, it is found that the SFP can also be applied to study intensity-modulated radiation therapy (IMRT) [12–14]. In the recent past, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP; see, for example, [11, 13, 15–19] and the references therein (see also [20] for relevant projection methods for solving image recovery problems). A special case of the SFP is the following convex constrained linear inverse problem [21] of finding an element x such that

$$x \in C, \qquad Ax = b. \tag{1.6}$$

It has been extensively investigated in the literature using the projected Landweber iterative method [22]. Comparatively, the SFP has received much less attention so far, due to the complexity resulting from the set Q. Therefore, whether various versions of the projected Landweber iterative method [23] can be extended to solve the SFP remains an interesting open topics. The original algorithm given in [10] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus, did not become popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [11, 15] which is found to be a gradient-projection method (GPM) in convex minimization. It is also a special case of the proximal forward-backward splitting method [24]. The CQ algorithm only involves the computation of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions; for example, C and Q are closed balls or half-spaces. However, it remains a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions, though theoretically, we can prove the (weak) convergence of the algorithm.

In 2010, Xu [25] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. He derived a weak convergence result, which shows that for suitable choices of iterative parameters (including the regularization), the sequence of iterative solutions can converge weakly to an exact solution of the SFP.

Very recently, Ceng et al. [26] introduced and studied an extragradient method with regularization for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional Hilbert spaces. By combining the regularization method and extragradient method due to Nadezhkina and Takahashi [8], the authors proposed an iterative algorithm for finding an element of $\mathrm{Fix}(S) \cap \Gamma$. The authors proved that the sequences generated by the proposed algorithm converge weakly to an element $z \in \mathrm{Fix}(S) \cap \Gamma$.

The purpose of this paper is to investigate modified extragradient methods for finding a common element of the solution set Γ of the SFP and the fixed point set $\mathrm{Fix}(S)$ of a strictly pseudocontractive mapping S in the setting of infinite-dimensional Hilbert spaces. Assume that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. By combining the regularization method and Nadezhkina and Takahashi's extragradient method [8], we propose an extragradient algorithm for finding an element of $\mathrm{Fix}(S) \cap \Gamma$. It is proven that the sequences generated by the proposed algorithm converge weakly to an element of $\mathrm{Fix}(S) \cap \Gamma$. This result represents the supplementation, improvement, and extension of the corresponding results in [25, 26]; for example, [25, Theorem 5.7] and

[26, Theorem 3.1]. On the other hand, by combining the regularization method and Jung's composite viscosity approximation method [9], we also propose another extragradient-like algorithm for finding an element of $\mathrm{Fix}(S) \cap \Gamma$ where $S : C \to C$ is nonexpansive. It is shown that the sequences generated by the proposed algorithm converge strongly to an element of $\mathrm{Fix}(S) \cap \Gamma$. Such a result substantially develops and improves the corresponding results in [9, 25, 26]; for example, [25, Theorem 5.7], [26, Theorem 3.1], and [9, Theorem 3.1]. It is worth pointing out that our results are new and novel in the Hilbert spaces setting. Essentially new approaches for finding the fixed points of strictly pseudocontractive mappings (including nonexpansive mappings) and solutions of the SFP are provided.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of $\mathscr{H}$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \tag{2.1}$$

Recall that the metric (or nearest point) projection from $\mathscr{H}$ onto K is the mapping $P_K : \mathscr{H} \to K$ which assigns to each point $x \in \mathscr{H}$ the unique point $P_K x \in K$ satisfying the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K). \tag{2.2}$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in \mathscr{H}$ and $z \in K$,*

(i) $z = P_K x \Leftrightarrow \langle x - z, y - z \rangle \le 0$, *for all $y \in K$;*

(ii) $z = P_K x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, *for all $y \in K$;*

(iii) $\langle P_K x - P_K y, x - y \rangle \ge \|P_K x - P_K y\|^2$, *for all $y \in \mathscr{H}$, which hence implies that P_K is nonexpansive and monotone.*

Definition 2.2. A mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in \mathscr{H}; \tag{2.3}$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in \mathscr{H}; \tag{2.4}$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{2.5}$$

where $S : \mathscr{H} \to \mathscr{H}$ is nonexpansive, projections are firmly nonexpansive.

Definition 2.3. Let T be a nonlinear operator with domain $D(T) \subseteq \mathscr{H}$ and range $R(T) \subseteq \mathscr{H}$, and let $\beta > 0$ and $\nu > 0$ be given constants. The operator T is called:

(a) monotone if

$$\langle x - y, Tx - Ty \rangle \geq 0, \quad \forall x, y \in D(T). \tag{2.6}$$

(b) β-strongly monotone if

$$\langle x - y, Tx - Ty \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in D(T). \tag{2.7}$$

(c) ν-inverse strongly monotone (ν-ism) if

$$\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in D(T). \tag{2.8}$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_K is 1-ism.

Inverse strongly monotone (also referred to as cocoercive) operators have been applied widely to solve practical problems in various fields, for instance, in traffic assignment problems; see, for example, [27, 28].

Definition 2.4. A mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S, \tag{2.9}$$

where $\alpha \in (0, 1)$ and $S : \mathscr{H} \to \mathscr{H}$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $(1/2)$-averaged maps.

Proposition 2.5 (see [15]). *Let $T : \mathscr{H} \to \mathscr{H}$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Proposition 2.6 (see [15, 29]). *Let $S, T, V : \mathscr{H} \to \mathscr{H}$ be given operators.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ T_2 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N). \tag{2.10}$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping T, that is,* $\mathrm{Fix}(T) = \{x \in \mathscr{H} : Tx = x\}$.

On the other hand, it is clear that, in a real Hilbert space $\mathscr{H}$, $S : C \to C$ is k-strictly pseudocontractive if and only if there holds the following inequality:

$$\langle Sx - Sy, x - y\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in C. \tag{2.11}$$

This immediately implies that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $((1-k)/2)$-inverse strongly monotone; for further detail, we refer to [5] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings. The so-called demiclosedness principle for strict pseudocontractive mappings in the following lemma will often be used.

Lemma 2.7 (see [5, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ and $S : C \to C$ be a mapping.*

(i) *If S is a k-strict pseudocontractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x-y\|, \quad \forall x, y \in C. \tag{2.12}$$

(ii) *If S is a k-strict pseudocontractive mapping, then the mapping $I - S$ is semiclosed at 0; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, then $(I-S)\tilde{x} = 0$.*

(iii) *If S is k-(quasi-)strict pseudo-contraction, then the fixed point set* $\mathrm{Fix}(S)$ *of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

The following elementary result on real sequences is quite well known.

Lemma 2.8 (see [30, page 80]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\sigma_n)a_n + b_n, \quad \forall n \geq 1. \tag{2.13}$$

If $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 2.9 (see [31, page 303]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1. \tag{2.14}$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

It is easy to see that the following lemma holds.

Lemma 2.10 (see [32]). *Let $\mathscr{H}$ be a real Hilbert space. Then, for all $x, y \in \mathscr{H}$ and $\lambda \in [0,1]$,*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.15}$$

The following lemma plays a key role in proving weak convergence of the sequences generated by our algorithm.

Lemma 2.11 (see [33]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)k \leq \gamma$. Then*

$$\|\gamma(x-y) + \delta(Sx - Sy)\| \leq (\gamma+\delta)\|x-y\|, \quad \forall x, y \in C. \tag{2.16}$$

The following result is useful when we prove the weak convergence of a sequence.

Lemma 2.12 (see [25, Proposition 2.6]). *Let K be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:*

(i) *every weak limit point of $\{x_n\}$ lies in K;*

(ii) $\lim_{n\to\infty}\|x_n - x\|$ *exists for every $x \in K$.*

Then $\{x_n\}$ converges weakly to a point in K.

Let K be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ and let $F : K \to \mathscr{H}$ be a monotone mapping. The variational inequality (VI) is to find $x \in K$ such that

$$\langle Fx, y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.17}$$

The solution set of the VIP is denoted by $\mathrm{VI}(K,F)$. It is well known that

$$x \in \mathrm{VI}(K,F) \Longleftrightarrow x = P_K(x - \lambda Fx), \quad \forall \lambda > 0. \tag{2.18}$$

A set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called monotone if for all $x, y \in \mathcal{H}$, $f \in Tx$ and $g \in Ty$ imply

$$\langle x - y, f - g \rangle \geq 0. \tag{2.19}$$

A monotone mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in \mathcal{H} \times \mathcal{H}$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $F : K \to \mathcal{H}$ be a monotone and L-Lipschitz continuous mapping and let $N_K v$ be the normal cone to K at $v \in K$, that is,

$$N_K v = \{ w \in \mathcal{H} : \langle v - y, w \rangle \geq 0, \ \forall y \in K \}. \tag{2.20}$$

Define

$$Tv = \begin{cases} Fv + N_K v, & \text{if } v \in K, \\ \emptyset, & \text{if } v \notin K. \end{cases} \tag{2.21}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(K, F)$; see [34] for more details.

3. Some Modified Extragradient Methods

Throughout the paper, we assume that the SFP is consistent; that is, the solution set Γ of the SFP is nonempty. Let $f : \mathcal{H}_1 \to \mathbf{R}$ be a continuous differentiable function. the minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2} \left\| Ax - P_Q Ax \right\|^2 \tag{3.1}$$

is ill posed. Therefore, Xu [25] considered the following Tikhonov regularized problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2} \left\| Ax - P_Q Ax \right\|^2 + \frac{1}{2} \alpha \|x\|^2, \tag{3.2}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^* (I - P_Q) A + \alpha I \tag{3.3}$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone.

We can use fixed point algorithms to solve the SFP on the basis of the following observation.

Let $\lambda > 0$ and assume that $x^* \in \Gamma$. Then $Ax^* \in Q$, which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda A^*(I-P_Q)Ax^* = 0$. Hence, we have the fixed point equation $(I-\lambda A^*(I-P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$P_C(I - \lambda \nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*. \tag{3.4}$$

It is proven in [25, Proposition 3.2] that the solutions of the fixed point equation (3.4) are exactly the solutions of the SFP; namely, for given $x^* \in \mathcal{H}_1, x^*$ solves the SFP if and only if x^* solves the fixed point equation (3.4).

Proposition 3.1 (see [26, proposition 3.1]). *Given $x^*\in\mathcal{H}_1$, the following statements are equivalent:*

(i) *x^* solves the SFP;*

(ii) *x^* solves the fixed point equation* (3.4)*;*

(iii) *x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.5}$$

Remark 3.2. It is clear from Proposition 3.1 that

$$\Gamma = \mathrm{Fix}(P_C(I - \lambda \nabla f)) = \mathrm{VI}(C, \nabla f) \tag{3.6}$$

for all $\lambda > 0$, where $\mathrm{Fix}(P_C(I-\lambda\nabla f))$ and $\mathrm{VI}(C, \nabla f)$ denote the set of fixed points of $P_C(I-\lambda\nabla f)$ and the solution set of VIP (3.4), respectively.

We are now in a position to propose a modified extragradient method for solving the SFP and the fixed point problem of a k-strictly pseudocontractive mapping $S : C \to C$ and prove that the sequences generated by the proposed method converge weakly to an element of $\mathrm{Fix}(S) \cap \Gamma$.

Theorem 3.3. *Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following modified extragradient algorithm:*

$$\begin{gathered} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n, \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{gathered} \tag{3.7}$$

where $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 1/\|A\|^2)$ and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$*;*

(ii) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1/\|A\|^2$*;*

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$*;*

(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty}\delta_n > 0$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\widehat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Proof. First, taking into account $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1/\|A\|^2$, without loss of generality, we may assume that $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/\|A\|^2)$.

We observe that $P_C(I-\lambda\nabla f_\alpha)$ is ζ-averaged for each $\lambda \in (0,2/(\alpha+\|A\|^2))$, where

$$\zeta = \frac{2+\lambda\left(\alpha+\|A\|^2\right)}{4} \in (0,1). \tag{3.8}$$

See, for example, [35] and from which it follows that $P_C(I-\lambda\nabla f_\alpha)$ and $P_C(I-\lambda_n\nabla f_{\alpha_n})$ are nonexpansive for all $n \ge 0$.

Next, we show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in \mathrm{Fix}(S)\cap\Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I-\lambda\nabla f)p = p$ for $\lambda \in (0,2/\|A\|^2)$. For simplicity, we write $v_n = P_C(x_n - \lambda_n\nabla f_{\alpha_n}(y_n))$ for all $n \ge 0$. Then we get $x_{n+1} = \beta_n x_n + \gamma_n v_n + \delta_n S v_n$ for all $n \ge 0$. From (3.7), it follows that

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n - P_C(I-\lambda_n\nabla f)p\| \\
&\le \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n - P_C(I-\lambda_n\nabla f_{\alpha_n})p\| + \|P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p\| \\
&\le \|x_n - p\| + \|P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p\| \\
&\le \|x_n - p\| + \|(I-\lambda_n\nabla f_{\alpha_n})p - (I-\lambda_n\nabla f)p\| \\
&= \|x_n - p\| + \lambda_n\alpha_n\|p\|.
\end{aligned} \tag{3.9}$$

Also, by Proposition 2.1(ii), we have

$$\begin{aligned}
\|v_n - p\|^2 &\le \|x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - p\|^2 - \|x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - v_n\|^2 \\
&= \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2\lambda_n\langle\nabla f_{\alpha_n}(y_n), p - v_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - v_n\|^2 \\
&\quad + 2\lambda_n(\langle\nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(p), p - y_n\rangle \\
&\qquad + \langle\nabla f_{\alpha_n}(p), p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - v_n\rangle) \\
&\le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2\lambda_n(\langle\nabla f_{\alpha_n}(p), p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - v_n\rangle) \\
&= \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2\lambda_n[\langle(\alpha_n I + \nabla f)p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - v_n\rangle] \\
&\le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - v_n\rangle] \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - v_n\rangle] \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, v_n - y_n\rangle \\
&\quad + 2\lambda_n\alpha_n\langle p, p - y_n\rangle.
\end{aligned} \tag{3.10}$$

Further, by Proposition 2.1(i), we have

$$\begin{aligned}
&\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - y_n, v_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n \nabla f_{\alpha_n}(x_n) - y_n, v_n - y_n\rangle + \langle \lambda_n \nabla f_{\alpha_n}(x_n) - \lambda_n \nabla f_{\alpha_n}(y_n), v_n - y_n\rangle \\
&\quad \leq \langle \lambda_n \nabla f_{\alpha_n}(x_n) - \lambda_n \nabla f_{\alpha_n}(y_n), v_n - y_n\rangle \\
&\quad \leq \lambda_n \| \nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n)\| \|v_n - y_n\| \\
&\quad \leq \lambda_n \left(\alpha_n + \|A\|^2\right) \|x_n - y_n\| \|v_n - y_n\|.
\end{aligned} \tag{3.11}$$

So, we obtain

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - y_n, v_n - y_n\rangle \\
&\quad + 2\lambda_n \alpha_n \langle p, p - y_n\rangle \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \left(\alpha_n + \|A\|^2\right) \|x_n - y_n\| \|v_n - y_n\| \\
&\quad + 2\lambda_n \alpha_n \|p\| \|p - y_n\| \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + \lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 \|x_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n \alpha_n \|p\| \|p - y_n\| \\
&= \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - y_n\| + \left(\lambda_n^2 \left(\alpha_n + \|A\|^2\right)^2 - 1\right) \|x_n - y_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - y_n\|.
\end{aligned} \tag{3.12}$$

Since $(\gamma_n + \delta_n)k \leq \gamma_n$, utilizing Lemmas 2.10 and 2.11, from (3.9) and the last inequality, we conclude that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n x_n + \gamma_n v_n + \delta_n S v_n - p\|^2 \\
&\quad = \left\| \beta_n (x_n - p) + (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} [\gamma_n (v_n - p) + \delta_n (S v_n - p)] \right\|^2 \\
&\quad = \beta_n \|x_n - p\|^2 + (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} [\gamma_n (v_n - p) + \delta_n (S v_n - p)] \right\|^2 \\
&\qquad - \beta_n (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} [\gamma_n (v_n - x_n) + \delta_n (S v_n - x_n)] \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\Big[\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\qquad + \left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2\Big] \\
&\quad - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad + (1-\beta_n)\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\left(\lambda_n^2\|p\|^2 + \|p - y_n\|^2\right) \\
&\quad + (1-\beta_n)\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\left[\lambda_n^2\|p\|^2 + \left(\|x_n - p\| + \lambda_n\alpha_n\|p\|\right)^2\right] \\
&\quad + (1-\beta_n)\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\left[\lambda_n^2\|p\|^2 + 2\|x_n - p\|^2 + 2\lambda_n^2\alpha_n^2\|p\|^2\right] \\
&\quad + (1-\beta_n)\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&= (1+2\alpha_n)\|x_n - p\|^2 + \alpha_n\lambda_n^2\|p\|^2\left(1 + 2\alpha_n^2\right) \\
&\quad + (1-\beta_n)\left(\lambda_n^2\left(\alpha_n + \|A\|^2\right)^2 - 1\right)\|x_n - y_n\|^2 - \frac{\beta_n}{1-\beta_n}\|x_{n+1} - x_n\|^2 \\
&\leq (1+2\alpha_n)\|x_n - p\|^2 + \alpha_n\lambda_n^2\|p\|^2\left(1 + 2\alpha_n^2\right) \\
&= (1+\sigma_n)\|x_n - p\|^2 + b_n,
\end{aligned}
\tag{3.13}
$$

where $\sigma_n = 2\alpha_n$ and $b_n = \alpha_n\lambda_n^2\|p\|^2(1+2\alpha_n^2)$. Since $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, 1/\|A\|^2)$, we conclude that $\sum_{n=0}^{\infty}\sigma_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Therefore, by Lemma 2.8, we deduce that

$$
\lim_{n\to\infty}\|x_n - p\| \text{ exists for each } p \in \mathrm{Fix}(S) \cap \Gamma,
\tag{3.14}
$$

and the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{v_n\}$. From the last relations, we also obtain

$$\begin{aligned}(1-\beta_n)\left(1-\lambda_n^2\left(\alpha_n+\|A\|^2\right)^2\right)\left\|x_n-y_n\right\|^2+\frac{\beta_n}{1-\beta_n}\|x_{n+1}-x_n\|^2\\ \leq(1+2\alpha_n)\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+\alpha_n\lambda_n^2\|p\|^2\left(1+2\alpha_n^2\right).\end{aligned} \tag{3.15}$$

Since $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,1/\|A\|^2), 0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$ and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\left\|x_n-y_n\right\|=\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.16}$$

Furthermore, we obtain

$$\begin{aligned}\left\|y_n-v_n\right\|&=\left\|P_C(x_n-\lambda_n\nabla f_{\alpha_n}(x_n))-P_C(x_n-\lambda_n\nabla f_{\alpha_n}(y_n))\right\|\\&\leq\left\|(x_n-\lambda_n\nabla f_{\alpha_n}(x_n))-(x_n-\lambda_n\nabla f_{\alpha_n}(y_n))\right\|\\&=\lambda_n\left\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(y_n)\right\|\\&\leq\lambda_n\left(\alpha_n+\|A\|^2\right)\left\|x_n-y_n\right\|.\end{aligned} \tag{3.17}$$

This together with (3.16) implies that

$$\lim_{n\to\infty}\left\|y_n-v_n\right\|=0. \tag{3.18}$$

Note that

$$\begin{aligned}\|v_n-x_n\|&\leq\left\|v_n-y_n\right\|+\left\|y_n-x_n\right\|,\\ \|\delta_n(Sv_n-x_n)\|&=\left\|x_{n+1}-x_n-\gamma_n(v_n-x_n)\right\|\\&\leq\|x_{n+1}-x_n\|+\gamma_n\|v_n-x_n\|\\&\leq\|x_{n+1}-x_n\|+\|v_n-x_n\|.\end{aligned} \tag{3.19}$$

This together with (3.16), (3.18), and $\liminf\delta_n>0$ implies that

$$\lim_{n\to\infty}\|v_n-x_n\|=\lim_{n\to\infty}\|Sv_n-x_n\|=0. \tag{3.20}$$

So, we derive

$$\lim_{n\to\infty}\|Sv_n-v_n\|=0. \tag{3.21}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, from (3.18), we have

$$\lim_{n\to\infty} \left\| \nabla f(y_n) - \nabla f(v_n) \right\| = 0. \tag{3.22}$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $\widehat{x}$. We obtain that $\widehat{x} \in \mathrm{Fix}(S) \cap \Gamma$. First, we show that $\widehat{x} \in \Gamma$. Since $\|x_n - v_n\| \to 0$ and $\|y_n - v_n\| \to 0$, it is known that $v_{n_i} \rightharpoonup \widehat{x}$ and $y_{n_i} \rightharpoonup \widehat{x}$. Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{3.23}$$

where $N_C v = \{w \in \mathscr{H}_1 : \langle v - u, w \rangle \geq 0, \text{for all } u \in C\}$. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [34] for more details. Let $(v, w) \in G(T)$. Then, we have

$$w \in Tv = \nabla f(v) + N_C v \tag{3.24}$$

and hence,

$$w - \nabla f(v) \in N_C v. \tag{3.25}$$

So, we have

$$\left\langle v - u, w - \nabla f(v) \right\rangle \geq 0, \quad \forall u \in C. \tag{3.26}$$

On the other hand, from

$$y_n = P_C\left(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)\right), \qquad v \in C, \tag{3.27}$$

we have

$$\langle x_n - \lambda_n \nabla f_{\alpha_n}(x_n) - y_n, y_n - v \rangle \geq 0, \tag{3.28}$$

and hence,

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + \nabla f_{\alpha_n}(x_n) \right\rangle \geq 0. \tag{3.29}$$

Therefore, from

$$w - \nabla f(v) \in N_C v, \qquad y_{n_i} \in C, \tag{3.30}$$

we have

$$\begin{aligned}
\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, \nabla f(v)\rangle \\
&\geq \langle v - y_{n_i}, \nabla f(v)\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f_{\alpha_{n_i}}(x_{n_i})\right\rangle \\
&= \langle v - y_{n_i}, \nabla f(v)\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f(x_{n_i})\right\rangle - \alpha_{n_i}\langle v - y_{n_i}, x_{n_i}\rangle \\
&= \langle v - y_{n_i}, \nabla f(v) - \nabla f(y_{n_i})\rangle + \langle v - y_{n_i}, \nabla f(y_{n_i}) - \nabla f(x_{n_i})\rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle - \alpha_{n_i}\langle v - y_{n_i}, x_{n_i}\rangle \\
&\geq \langle v - y_{n_i}, \nabla f(y_{n_i}) - \nabla f(x_{n_i})\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle - \alpha_{n_i}\langle v - y_{n_i}, x_{n_i}\rangle.
\end{aligned} \tag{3.31}$$

Hence, we obtain

$$\langle v - \hat{x}, w\rangle \geq 0, \qquad \text{as } i \longrightarrow \infty. \tag{3.32}$$

Since T is maximal monotone, we have $\hat{x} \in T^{-1}0$, and hence, $\hat{x} \in \mathrm{VI}(C, \nabla f)$. Thus it is clear that $\hat{x} \in \Gamma$.

We show that $\hat{x} \in \mathrm{Fix}(S)$. Indeed, since $v_{n_i} \rightharpoonup \hat{x}$ and $\|v_{n_i} - Sv_{n_i}\| \to 0$ by (3.21), by Lemma 2.7(ii), we get $\hat{x} \in \mathrm{Fix}(S)$. Therefore, we have $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$. This shows that $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma$, where

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \tag{3.33}$$

Since the limit $\lim_{n\to\infty}\|x_n - p\|$ exists for every $p \in \mathrm{Fix}(S) \cap \Gamma$, by Lemma 2.12, we know that

$$x_n \rightharpoonup \hat{x} \in \mathrm{Fix}\,(S) \cap \Gamma. \tag{3.34}$$

Further, from $\|x_n - y_n\| \to 0$, it follows that $y_n \rightharpoonup \hat{x}$. This shows that both sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$. □

Remark 3.4. It is worth emphasizing that the modified extragradient algorithm in Theorem 3.3 is essentially the predictor-corrector algorithm. Indeed, the first iterative step $y_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n$ is the predictor one, and the second iterative step $x_{n+1} = \beta_n x_n + \gamma_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n))$ is actually the corrector one. In addition, Theorem 3.3 extends the extragradient method due to Nadezhkina and Takahashi [8, Theorem 3.1].

Corollary 3.5. *Let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{gathered} x_0 = x \in C \ \textit{chosen arbitrarily}, \\ y_n = P_C(I - \lambda_n \nabla f_{\alpha_n}) x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{gathered} \tag{3.35}$$

where $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 1/\|A\|^2)$, and $\{\beta_n\} \subset [0, 1]$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/\|A\|^2$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Proof. In Theorem 3.3, putting $\gamma_n = 0$ for every $n \geq 0$, we obtain that $\beta_n + \delta_n = \beta_n + \gamma_n + \delta_n = 1$ and

$$\begin{aligned} x_0 &= x \in C \ \text{chosen arbitrarily}, \\ y_n &= P_C(I - \lambda_n \nabla f_{\alpha_n}) x_n, \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) \\ &= \beta_n x_n + \delta_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0. \end{aligned} \tag{3.36}$$

Since $S : C \to C$ is a nonexpansive mapping, $S : C \to C$ must be a k-strictly pseudocontractive mapping with coefficient $k = 0$. It is clear that $(\gamma_n + \delta_n)k \leq \gamma_n$ for every $n \geq 0$ and $\liminf_{n\to\infty} \delta_n = 1 - \limsup_{n\to\infty} \beta_n > 0$. In this case, all conditions in Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, we derive the desired result. □

Remark 3.6. Compared with [26, Theorem 3.1], Corollary 3.5 is essentially coincident with [26, Theorem 3.1]. Hence our Theorem 3.3 includes [26, Theorem 3.1] as a special case. Utilizing [8, Theorem 3.1], Ceng et al. gave the following weak convergence result [26, Theorem3.2].

Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following Nadezhkina and Takahashi extragradient algorithm

$$\begin{gathered} x_0 = x \in C \ \text{chosen arbitrarily}, \\ y_n = P_C(I - \lambda_n \nabla f) x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S P_C(x_n - \lambda_n \nabla f(y_n)), \quad \forall n \geq 0, \end{gathered} \tag{3.37}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/\|A\|^2)$ and $\{\beta_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Obviously, there is no doubt that [26, Theorem 3.2] is a weak convergence result for $\{\alpha_n\}$ satisfying $\alpha_n = 0$, for all $n \geq 0$. However, Corollary 3.5 is another weak convergence one for the sequence of regularization parameters $\{\alpha_n\} \subset (0, \infty)$.

Remark 3.7. Theorem 3.3 improves, extends, and develops [25, theorem 5.7] and [26, Theorem 3.1] in the following aspects.

(a) The corresponding iterative algorithms in [25, Theorem 5.7] and [26, Theorem 3.1] are extended for developing our modified extragradient algorithm with regularization in Theorem 3.3.

(b) The technique of proving weak convergence in Theorem 3.3 is different from those in [25, Theorem 5.7] and [26, Theorem 3.1] because our technique depends on the properties of maximal monotone mappings and strictly pseudocontractive mappings (e.g., Lemma 2.11) and the demiclosedness principle for strictly pseudocontractive mappings (e.g., Lemma 2.7) in Hilbert spaces.

(c) The problem of finding an element of $\mathrm{Fix}(S) \cap \Gamma$ with $S : C \to C$ being strictly pseudocontractive is more general than the problem of finding a solution of the SFP in [25, Theorem 5.7] and the problem of finding an element of $\mathrm{Fix}(S) \cap \Gamma$ with $S : C \to C$ being nonexpansive in [26, Theorem 3.1].

(d) The second iterative step $x_{n+1} = \beta_n x_n + \gamma_n P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n))$ in our algorithm reduces to the the second iterative one $x_{n+1} = \beta_n x_n + (1-\beta_n) S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n))$ in the algorithm of [26, Theorem 3.1] whenever $\gamma_n = 0$ for all $n \geq 0$.

Utilizing Theorem 3.3, we have the following two new results in the setting of real Hilbert spaces.

Corollary 3.8. *Let $S : \mathscr{H}_1 \to \mathscr{H}_1$ be a k-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap (\nabla f)^{-1} 0 \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by*

$$\begin{aligned} x_0 &= x \in C \ \textit{chosen arbitrarily}, \\ y_n &= (I - \lambda_n \nabla f_{\alpha_n}) x_n, \\ x_{n+1} &= \beta_n x_n + \gamma_n (x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n S(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{aligned} \tag{3.38}$$

where $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 1/\|A\|^2)$ and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/\|A\|^2$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n) k \leq \gamma_n$ *for all* $n \geq 0$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap (\nabla f)^{-1} 0$.

Proof. In Theorem 3.3, putting $C = \mathscr{H}_1$, we have

$$(\nabla f)^{-1}0 = \mathrm{VI}(\mathscr{H}_1, \nabla f) = \Gamma \tag{3.39}$$

and $P_{\mathscr{H}_1} = I$ the identity mapping. By Theorem 3.3, we obtain the desired result. □

Remark 3.9. In Corollary 3.8, putting $\gamma_n = 0$ for every $n \geq 0$ and letting $S : \mathscr{H}_1 \to \mathscr{H}_1$ be a nonexpansive mapping, Corollary 3.8 essentially reduces to [26, Corollary 3.2]. Hence, Corollary 3.8 includes [26, Corollary 3.2] as a special case.

Corollary 3.10. *Let $B : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ be a maximal monotone mapping such that $B^{-1}0 \cap (\nabla f)^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{gathered} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = (I - \lambda_n \nabla f_{\alpha_n}) x_n, \\ x_{n+1} = \beta_n x_n + \gamma_n (x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) + \delta_n J_r^B (x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{gathered} \tag{3.40}$$

where $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 1/\|A\|^2)$ and $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/\|A\|^2$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in B^{-1}0 \cap (\nabla f)^{-1}0$.

Proof. In Theorem 3.3, putting $C = \mathscr{H}_1$ and $S = J_r^B$ the resolvent of B, we know that $P_{\mathscr{H}_1} = I$ the identity mapping and S is nonexpansive. In this case, we get $\mathrm{Fix}(S) = \mathrm{Fix}(J_r^B) = B^{-1}0$ and

$$(\nabla f)^{-1}0 = \mathrm{VI}(\mathscr{H}_1, \nabla f) = \Gamma. \tag{3.41}$$

By Theorem 3.3, we obtain the desired result. □

Remark 3.11. In Corollary 3.10, putting $\gamma_n = 0$ for every $n \geq 0$, Corollary 3.10 essentially reduces to [26, Corollary 3.3]. Hence, Corollary 3.10 includes [26, Corollary 3.3] as a special case.

On the other hand, by combining the regularization method and Jung's composite viscosity approximation method [9], we introduce another new composite iterative scheme for finding an element of $\mathrm{Fix}(S) \cap \Gamma$, where $S : C \to C$ is nonexpansive, and prove strong convergence of this scheme. To attain this object, we need to use the following lemmas.

Lemma 3.12 (see [36]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - s_n) a_n + s_n t_n + r_n, \quad \forall n \geq 0, \tag{3.42}$$

where $\{s_n\} \subset (0,1]$ *and* $\{t_n\}$ *are such that*

(i) $\sum_{n=0}^{\infty} s_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} t_n \le 0$ *or* $\sum_{n=0}^{\infty} |s_n t_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} r_n < \infty$ *where* $r_n \ge 0$, *for all* $n \ge 0$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 3.13. *In a real Hilbert space* $\mathcal{H}$, *there holds the following inequality:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathcal{H}. \tag{3.43}$$

Theorem 3.14. *Let* $Q : C \to C$ *be a contractive mapping with coefficient* $\rho \in [0,1)$ *and* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \ne \emptyset$. *Assume that* $0 < \lambda < 2/\|A\|^2$, *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences in* C *generated by the following composite extragradient-like algorithm:*

$$\begin{gathered} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = \beta_n Q x_n + (1 - \beta_n) S P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = (1 - \gamma_n) y_n + \gamma_n S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \quad \forall n \ge 0, \end{gathered} \tag{3.44}$$

where the sequences of parameters $\{\alpha_n\} \subset (0,\infty)$ *and* $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, both the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in \mathrm{Fix}(S) \cap \Gamma$, *which is a unique solution of the following variational inequality:*

$$\langle (I - Q)q, q - p\rangle \le 0, \quad \forall p \in \mathrm{Fix}(S) \cap \Gamma. \tag{3.45}$$

Proof. Repeating the same argument as in the proof of Theorem 3.3, we obtain that for each $\lambda \in (0, 2/(\alpha + \|A\|^2))$, $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged with

$$\zeta = \frac{1}{2} + \frac{\lambda\left(\alpha + \|A\|^2\right)}{2} - \frac{1}{2} \cdot \frac{\lambda\left(\alpha + \|A\|^2\right)}{2} = \frac{2 + \lambda\left(\alpha + \|A\|^2\right)}{4} \in (0,1). \tag{3.46}$$

This shows that $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\lambda \in (0, 2/\|A\|^2)$, utilizing the fact that $\lim_{n\to\infty}(2/(\alpha_n + \|A\|^2)) = 2/\|A\|^2$ we may assume that

$$0 < \lambda < \frac{2}{\alpha_n + \|A\|^2}, \quad \forall n \ge 0. \tag{3.47}$$

Consequently, it follows that for each integer $n \geq 0$, $P_C(I - \lambda\nabla f_{\alpha_n})$ is ζ_n-averaged with

$$\zeta_n = \frac{1}{2} + \frac{\lambda\left(\alpha_n + \|A\|^2\right)}{2} - \frac{1}{2} \cdot \frac{\lambda\left(\alpha_n + \|A\|^2\right)}{2} = \frac{2 + \lambda\left(\alpha_n + \|A\|^2\right)}{4} \in (0,1). \tag{3.48}$$

This immediately implies that $P_C(I - \lambda\nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$. Next, we divide the remainder of the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, put $u_n = P_C(x_n - \lambda\nabla f_{\alpha_n}(x_n))$ and $v_n = P_C(y_n - \lambda\nabla f_{\alpha_n}(y_n))$ for every $n \geq 0$. Take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I - \lambda\nabla f)p = p$ for $\lambda \in (0, 2/\|A\|^2)$. Hence, we have

$$\begin{aligned}
\|u_n - p\| &= \|P_C(I - \lambda\nabla f_{\alpha_n})x_n - P_C(I - \lambda\nabla f)p\| \\
&\leq \|P_C(I - \lambda\nabla f_{\alpha_n})x_n - P_C(I - \lambda\nabla f_{\alpha_n})p\| + \|P_C(I - \lambda\nabla f_{\alpha_n})p - P_C(I - \lambda\nabla f)p\| \\
&\leq \|x_n - p\| + \|P_C(I - \lambda\nabla f_{\alpha_n})p - P_C(I - \lambda\nabla f)p\| \\
&\leq \|x_n - p\| + \lambda\alpha_n\|p\|.
\end{aligned} \tag{3.49}$$

Similarly we get $\|v_n - p\| \leq \|y_n - p\| + \lambda\alpha_n\|p\|$. Thus, from (3.44), we have

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n(Qx_n - p) + (1 - \beta_n)(Su_n - p)\| \\
&\leq \beta_n\|Qx_n - p\| + (1 - \beta_n)\|u_n - p\| \\
&\leq \beta_n\|Qx_n - Qp\| + \beta_n\|Qp - p\| + (1 - \beta_n)(\|x_n - p\| + \lambda\alpha_n\|p\|) \\
&\leq \beta_n\rho\|x_n - p\| + \beta_n\|Qp - p\| + (1 - \beta_n)(\|x_n - p\| + \lambda\alpha_n\|p\|) \\
&= (1 - \beta_n(1 - \rho))\|x_n - p\| + \beta_n\|Qp - p\| + (1 - \beta_n)\lambda\alpha_n\|p\| \\
&= (1 - \beta_n(1 - \rho))\|x_n - p\| + \beta_n(1 - \rho)\frac{\|Qp - p\|}{1 - \rho} + (1 - \beta_n)\lambda\alpha_n\|p\| \\
&\leq \max\left\{\|x_n - p\|, \frac{\|Qp - p\|}{1 - \rho}\right\} + \lambda\alpha_n\|p\|,
\end{aligned} \tag{3.50}$$

and hence,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \gamma_n)(y_n - p) + \gamma_n(Sv_n - p)\| \\
&\leq (1 - \gamma_n)\|y_n - p\| + \gamma_n\|v_n - p\| \\
&\leq (1 - \gamma_n)\|y_n - p\| + \gamma_n(\|y_n - p\| + \lambda\alpha_n\|p\|) \\
&\leq \|y_n - p\| + \lambda\alpha_n\|p\| \\
&\leq \max\left\{\|x_n - p\|, \frac{\|Qp - p\|}{1 - \rho}\right\} + \lambda\alpha_n\|p\| + \lambda\alpha_n\|p\| \\
&= \max\left\{\|x_n - p\|, \frac{\|Qp - p\|}{1 - \rho}\right\} + 2\lambda\alpha_n\|p\|.
\end{aligned} \tag{3.51}$$

By induction, we get

$$\|x_{n+1} - p\| \leq \max\left\{ \|x_0 - p\|, \frac{\|Qp - p\|}{1 - \rho} \right\} + 2\lambda\|p\| \cdot \sum_{i=0}^{n} \alpha_i, \quad \forall n \geq 0. \tag{3.52}$$

This implies that $\{x_n\}$ is bounded and so are $\{y_n\}, \{u_n\}, \{v_n\}$. It is clear that both $\{Su_n\}$ and $\{Sv_n\}$ are also bounded. By condition (ii), we also obtain

$$\|y_n - Su_n\| = \beta_n \|Qx_n - Su_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.53}$$

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Indeed, from (3.44), we have

$$\begin{aligned} y_n &= \beta_n Qx_n + (1 - \beta_n) Su_n, \\ y_{n-1} &= \beta_{n-1} Qx_{n-1} + (1 - \beta_{n-1}) Su_{n-1}, \quad \forall n \geq 1. \end{aligned} \tag{3.54}$$

Simple calculations show that

$$y_n - y_{n-1} = (1 - \beta_n)(Su_n - Su_{n-1}) + (\beta_n - \beta_{n-1})(Qx_{n-1} - Su_{n-1}) + \beta_n(Qx_n - Qx_{n-1}). \tag{3.55}$$

Since

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|P_C(I - \lambda\nabla f_{\alpha_n})x_n - P_C(I - \lambda\nabla f_{\alpha_n})x_{n-1}\| \\ &\quad + \|P_C(I - \lambda\nabla f_{\alpha_n})x_{n-1} - P_C(I - \lambda\nabla f_{\alpha_{n-1}})x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + \|(I - \lambda\nabla f_{\alpha_n})x_{n-1} - (I - \lambda\nabla f_{\alpha_{n-1}})x_{n-1}\| \\ &= \|x_n - x_{n-1}\| + \|\lambda\nabla f_{\alpha_n}(x_{n-1}) - \lambda\nabla f_{\alpha_{n-1}}(x_{n-1})\| \\ &= \|x_n - x_{n-1}\| + \lambda|\alpha_n - \alpha_{n-1}|\|x_{n-1}\| \end{aligned} \tag{3.56}$$

for every $n \geq 1$, we have

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &\quad\leq (1 - \beta_n)\|Su_n - Su_{n-1}\| + |\beta_n - \beta_{n-1}|\|Qx_{n-1} - Su_{n-1}\| + \beta_n\|Qx_n - Qx_{n-1}\| \\ &\quad\leq (1 - \beta_n)\|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|\|Qx_{n-1} - Su_{n-1}\| + \beta_n\rho\|x_n - x_{n-1}\| \\ &\quad\leq (1 - \beta_n)[\|x_n - x_{n-1}\| + \lambda|\alpha_n - \alpha_{n-1}|\|x_{n-1}\|] + |\beta_n - \beta_{n-1}|\|Qx_{n-1} - Su_{n-1}\| \\ &\qquad + \beta_n\rho\|x_n - x_{n-1}\| \\ &\quad\leq (1 - \beta_n(1 - \rho))\|x_n - x_{n-1}\| + \lambda|\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Qx_{n-1} - Su_{n-1}\| \\ &\quad\leq (1 - \beta_n(1 - \rho))\|x_n - x_{n-1}\| + M_1[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|] \end{aligned} \tag{3.57}$$

for every $n \geq 1$, where $M_1 = \sup\{\lambda\|x_{n-1}\| + \|Qx_{n-1} - Su_{n-1}\| : n \geq 1\}$.

On the other hand, from (3.44), we have

$$x_{n+1} = (1-\gamma_n)y_n + \gamma_n S v_n,$$
$$x_n = (1-\gamma_{n-1})y_{n-1} + \gamma_{n-1} S v_{n-1}. \tag{3.58}$$

Also, simple calculations show that

$$x_{n+1} - x_n = (1-\gamma_n)(y_n - y_{n-1}) + \gamma_n(Sv_n - Sv_{n-1}) + (\gamma_n - \gamma_{n-1})(Sv_{n-1} - y_{n-1}). \tag{3.59}$$

Since

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le \left\|P_C(I-\lambda\nabla f_{\alpha_n})y_n - P_C(I-\lambda\nabla f_{\alpha_n})y_{n-1}\right\| \\
&\quad + \left\|P_C(I-\lambda\nabla f_{\alpha_n})y_{n-1} - P_C(I-\lambda\nabla f_{\alpha_{n-1}})y_{n-1}\right\| \\
&\le \left\|y_n - y_{n-1}\right\| + \left\|(I-\lambda\nabla f_{\alpha_n})y_{n-1} - (I-\lambda\nabla f_{\alpha_{n-1}})y_{n-1}\right\| \\
&= \left\|y_n - y_{n-1}\right\| + \left\|\lambda\nabla f_{\alpha_n}(y_{n-1}) - \lambda\nabla f_{\alpha_{n-1}}(y_{n-1})\right\| \\
&= \left\|y_n - y_{n-1}\right\| + \lambda|\alpha_n - \alpha_{n-1}|\left\|y_{n-1}\right\|
\end{aligned} \tag{3.60}$$

for every $n \ge 1$, it follows from (3.57) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le (1-\gamma_n)\left\|y_n - y_{n-1}\right\| + \gamma_n\|Sv_n - Sv_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right|\left\|Sv_{n-1} - y_{n-1}\right\| \\
&\quad \le (1-\gamma_n)\left\|y_n - y_{n-1}\right\| + \gamma_n\|v_n - v_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right|\left\|Sv_{n-1} - y_{n-1}\right\| \\
&\quad \le (1-\gamma_n)\left\|y_n - y_{n-1}\right\| + \gamma_n\left[\left\|y_n - y_{n-1}\right\| + \lambda|\alpha_n - \alpha_{n-1}|\left\|y_{n-1}\right\|\right] \\
&\qquad + \left|\gamma_n - \gamma_{n-1}\right|\left\|Sv_{n-1} - y_{n-1}\right\| \\
&\quad \le \left\|y_n - y_{n-1}\right\| + \lambda|\alpha_n - \alpha_{n-1}|\left\|y_{n-1}\right\| + \left|\gamma_n - \gamma_{n-1}\right|\left\|Sv_{n-1} - y_{n-1}\right\| \\
&\quad \le \left\|y_n - y_{n-1}\right\| + M_2\left[|\alpha_n - \alpha_{n-1}| + \left|\gamma_n - \gamma_{n-1}\right|\right] \\
&\quad \le (1-\beta_n(1-\rho))\|x_n - x_{n-1}\| + M_1\left[|\alpha_n - \alpha_{n-1}| + \left|\beta_n - \beta_{n-1}\right|\right] \\
&\qquad + M_2\left[|\alpha_n - \alpha_{n-1}| + \left|\gamma_n - \gamma_{n-1}\right|\right] \\
&\quad = (1-\beta_n(1-\rho))\|x_n - x_{n-1}\| + (M_1 + M_2)|\alpha_n - \alpha_{n-1}| + M_1\left|\beta_n - \beta_{n-1}\right| + M_2\left|\gamma_n - \gamma_{n-1}\right|
\end{aligned} \tag{3.61}$$

for every $n \geq 1$, where $M_2 = \sup\{\lambda\|y_{n-1}\| + \|Sv_{n-1} - y_{n-1}\| : n \geq 1\}$. From conditions (i), (ii), (iii), it is easy to see that

$$\lim_{n\to\infty} \beta_n(1-\rho) = 0, \qquad \sum_{n=0}^{\infty} \beta_n(1-\rho) = \infty,$$
$$\sum_{n=0}^{\infty} \left[(M_1 + M_2)|\alpha_n - \alpha_{n-1}| + M_1\left|\beta_n - \beta_{n-1}\right| + M_2\left|\gamma_n - \gamma_{n-1}\right|\right] < \infty. \tag{3.62}$$

Applying Lemma 3.12 to (3.61), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.63}$$

From (3.57), we also have that $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Step 3. $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|x_n - Su_n\| = 0$.

Indeed, it follows that

$$\begin{aligned}
\left\|x_{n+1} - y_n\right\| &= \gamma_n\left\|Sv_n - y_n\right\| \\
&\leq \gamma_n\left(\|Sv_n - Su_n\| + \left\|Su_n - y_n\right\|\right) \\
&\leq \gamma_n\left(\|v_n - u_n\| + \left\|Su_n - y_n\right\|\right) \\
&= \gamma_n\left(\left\|P_C(I - \lambda\nabla f_{\alpha_n})y_n - P_C(I - \lambda\nabla f_{\alpha_n})x_n\right\| + \left\|Su_n - y_n\right\|\right) \\
&\leq \gamma_n\left(\left\|y_n - x_n\right\| + \left\|Su_n - y_n\right\|\right) \\
&\leq \gamma_n\left(\left\|y_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| + \left\|Su_n - y_n\right\|\right),
\end{aligned} \tag{3.64}$$

which implies that

$$\begin{aligned}
(1 - \gamma_n)\left\|y_n - x_{n+1}\right\| &\leq \gamma_n\left(\|x_{n+1} - x_n\| + \left\|Su_n - y_n\right\|\right) \\
&\leq \|x_{n+1} - x_n\| + \left\|Su_n - y_n\right\|.
\end{aligned} \tag{3.65}$$

Obviously, utilizing (3.53), $\|x_{n+1} - x_n\| \to 0$ and $\limsup_{n\to\infty}\gamma_n < 1$, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that

$$\left\|x_n - y_n\right\| \leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.66}$$

From (3.53) and (3.66), we also have

$$\|x_n - Su_n\| \leq \left\|x_n - y_n\right\| + \left\|y_n - Su_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.67}$$

Step 4. $\lim_{n\to\infty}\|x_n - u_n\| = 0$.

Indeed, take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Then, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n(Qx_n - p) + (1-\beta_n)(Su_n - p)\|^2 \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\|Su_n - p\|^2 \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\|^2 \\
&= \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\|(I-\lambda\nabla f)x_n - (I-\lambda\nabla f)p - \lambda\alpha_n x_n\|^2 \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\Big[\|(I-\lambda\nabla f)x_n - (I-\lambda\nabla f)p\|^2 \\
&\qquad\qquad -2\lambda\alpha_n\langle x_n, (I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\rangle\Big] \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\left[\|x_n - p\|^2 + 2\lambda\left(\lambda - \frac{2}{\|A\|^2}\right)\|\nabla f(x_n) - \nabla f(p)\|^2\right. \\
&\qquad\qquad \left. + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\|\right] \\
&\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\Big[\|x_n - p\|^2 + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\|\Big].
\end{aligned} \tag{3.68}$$

So we obtain

$$\begin{aligned}
&(1-\beta_n)2\lambda\left(\frac{2}{\|A\|^2} - \lambda\right)\|\nabla f(x_n) - \nabla f(p)\|^2 \\
&\quad\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \|y_n - p\|^2 \\
&\qquad + (1-\beta_n)2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\
&\quad\le \beta_n\|x_n - p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\
&\qquad + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\
&\quad\le \beta_n\|x_n - p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\
&\qquad + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\|.
\end{aligned} \tag{3.69}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, $\|x_n - y_n\| \to 0$, and $0 < \lambda < 2/\|A\|^2$, from the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows that $\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(p)\| = 0$, and hence

$$\lim_{n\to\infty}\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\| = 0. \tag{3.70}$$

Moreover, from the firm nonexpansiveness of P_C, we obtain

$$\begin{aligned}\|u_n - p\|^2 &= \|P_C(I-\lambda\nabla f_{\alpha_n})x_n - P_C(I-\lambda\nabla f)p\|^2 \\ &\le \langle (I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p, u_n - p\rangle \\ &= \frac{1}{2}\Big\{ \|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\|^2 + \|u_n - p\|^2 \\ &\quad - \|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p - (u_n - p)\|^2 \Big\} \\ &\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| + \|u_n - p\|^2 \\ &\quad - \|x_n - u_n\|^2 + 2\lambda\langle x_n - u_n, \nabla f_{\alpha_n}(x_n) - \nabla f(p)\rangle - \lambda^2\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\|^2 \Big\},\end{aligned} \tag{3.71}$$

and so

$$\begin{aligned}\|u_n - p\|^2 \le{}& \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\ &+ 2\lambda\langle x_n - u_n, \nabla f_{\alpha_n}(x_n) - \nabla f(p)\rangle - \lambda^2\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\|^2.\end{aligned} \tag{3.72}$$

Thus, we have

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n\|Qx_n - p\|^2 + (1-\beta_n)\ \|u_n - p\|^2 \\ &\le \beta_n\|Qx_n - p\|^2 + \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 \\ &\quad + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\ &\quad + 2(1-\beta_n)\lambda\langle x_n - u_n, \nabla f_{\alpha_n}(x_n) - \nabla f(p)\rangle - (1-\beta_n)\lambda^2\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\|^2,\end{aligned} \tag{3.73}$$

which implies that

$$\begin{aligned}&(1-\beta_n)\|x_n - u_n\|^2 \\ &\quad\le \beta_n\|x_n - p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\ &\qquad + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\ &\qquad + 2(1-\beta_n)\lambda\langle x_n - u_n, \nabla f_{\alpha_n}(x_n) - \nabla f(p)\rangle - (1-\beta_n)\lambda^2\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\|^2 \\ &\quad\le \beta_n\|x_n - p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - y_n\|) \\ &\qquad + 2\lambda\alpha_n\|x_n\|\|(I-\lambda\nabla f_{\alpha_n})x_n - (I-\lambda\nabla f)p\| \\ &\qquad + 2(1-\beta_n)\lambda\langle x_n - u_n, \nabla f_{\alpha_n}(x_n) - \nabla f(p)\rangle - (1-\beta_n)\lambda^2\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\|^2.\end{aligned} \tag{3.74}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, $\|x_n - y_n\| \to 0$, and $\|\nabla f_{\alpha_n}(x_n) - \nabla f(p)\| \to 0$, from the boundedness of $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$, it follows that $\lim_{n\to\infty}\|x_n - u_n\| = 0$ and hence

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.75}$$

Step 5. $\limsup_{n\to\infty}\langle Qq - q, y_n - q\rangle \le 0$ for $q \in \mathrm{Fix}(S)\cap\Gamma$, where q is a unique solution of the variational inequality

$$\big\langle (I - Q)q, q - p\big\rangle \le 0, \qquad \forall p \in \mathrm{Fix}(S)\cap\Gamma. \tag{3.76}$$

Indeed, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\big\langle Qq - q, Su_n - q\big\rangle = \lim_{i\to\infty}\big\langle Qq - q, Su_{n_i} - q\big\rangle. \tag{3.77}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $\overline{u}$. Without loss of generality we may assume that $u_{n_i} \rightharpoonup \overline{u}$. Then we can obtain $\overline{u} \in \mathrm{Fix}(S)\cap\Gamma$. Let us first show that $\overline{u} \in \Gamma$. Define

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{3.78}$$

where $N_C v = \{w \in \mathcal{H}_1 : \langle v - u, w\rangle \ge 0, \text{ for all } u \in C\}$. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [34] for more details. Let $(v, w) \in G(T)$. Then, we have

$$w \in Tv = \nabla f(v) + N_C v \tag{3.79}$$

and hence,

$$w - \nabla f(v) \in N_C v. \tag{3.80}$$

So, we have

$$\big\langle v - u, w - \nabla f(v)\big\rangle \ge 0, \quad \forall u \in C. \tag{3.81}$$

On the other hand, from

$$u_n = P_C\big(x_n - \lambda\nabla f_{\alpha_n}(x_n)\big), \qquad v \in C, \tag{3.82}$$

we have

$$\big\langle x_n - \lambda\nabla f_{\alpha_n}(x_n) - u_n, u_n - v\big\rangle \ge 0, \tag{3.83}$$

and hence,

$$\left\langle v - u_n, \frac{u_n - x_n}{\lambda} + \nabla f_{\alpha_n}(x_n) \right\rangle \geq 0. \tag{3.84}$$

Therefore, from

$$w - \nabla f(v) \in N_C v, \qquad u_{n_i} \in C, \tag{3.85}$$

we have

$$\begin{aligned}
\langle v - u_{n_i}, w \rangle &\geq \langle v - u_{n_i}, \nabla f(v) \rangle \\
&\geq \langle v - u_{n_i}, \nabla f(v) \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda} + \nabla f_{\alpha_{n_i}}(x_{n_i}) \right\rangle \\
&= \langle v - u_{n_i}, \nabla f(v) \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda} + \nabla f(x_{n_i}) \right\rangle - \alpha_{n_i} \langle v - u_{n_i}, x_{n_i} \rangle \\
&= \langle v - u_{n_i}, \nabla f(v) - \nabla f(u_{n_i}) \rangle + \langle v - u_{n_i}, \nabla f(u_{n_i}) - \nabla f(x_{n_i}) \rangle \\
&\quad - \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda} \right\rangle - \alpha_{n_i} \langle v - u_{n_i}, x_{n_i} \rangle \\
&\geq \langle v - u_{n_i}, \nabla f(u_{n_i}) - \nabla f(x_{n_i}) \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda} \right\rangle - \alpha_{n_i} \langle v - u_{n_i}, x_{n_i} \rangle.
\end{aligned} \tag{3.86}$$

Hence, we obtain

$$\langle v - \overline{u}, w \rangle \geq 0, \qquad \text{as } i \longrightarrow \infty. \tag{3.87}$$

Since T is maximal monotone, we have $\overline{u} \in T^{-1}0$, and hence, $\overline{u} \in \mathrm{VI}(C, \nabla f)$. Thus it is clear that $\overline{u} \in \Gamma$.

On the other hand, by Steps 3 and 4, $\|u_n - Su_n\| \leq \|u_n - x_n\| + \|x_n - Su_n\| \to 0$. So, by Lemma 2.7(ii), we derive $\overline{u} \in \mathrm{Fix}(S)$ and hence $\overline{u} \in \mathrm{Fix}(S) \cap \Gamma$. Then from (3.76) and (3.77), we have

$$\begin{aligned}
\limsup_{n \to \infty} \langle Qq - q, Su_n - q \rangle &= \lim_{i \to \infty} \langle Qq - q, Su_{n_i} - q \rangle = \langle Qq - q, \overline{u} - q \rangle \\
&= \langle (I - Q)q, q - \overline{u} \rangle \leq 0.
\end{aligned} \tag{3.88}$$

Thus, from (3.53), we obtain

$$\begin{aligned}
\limsup_{n \to \infty} \langle Qq - q, y_n - q \rangle &\leq \limsup_{n \to \infty} \langle Qq - q, y_n - Su_n \rangle + \limsup_{n \to \infty} \langle Qq - q, Su_n - q \rangle \\
&\leq \limsup_{n \to \infty} \|Qq - q\| \|y_n - Su_n\| + \limsup_{n \to \infty} \langle Qq - q, Su_n - q \rangle \\
&\leq 0.
\end{aligned} \tag{3.89}$$

Step 6. $\lim_{n\to\infty}\|x_n - q\| = 0$ for $q \in \mathrm{Fix}(S)\cap\Gamma$, where q is a unique solution of the variational inequality

$$\langle (I-Q)q, q-p\rangle \le 0, \qquad \forall p \in \mathrm{Fix}(S)\cap\Gamma. \tag{3.90}$$

Indeed, utilizing (3.49), (3.51), and Lemma 3.13, we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\quad\le \left(\|y_n-q\| + \lambda\alpha_n\|q\|\right)^2\\
&\quad= \|y_n-q\|^2 + \lambda\alpha_n\|q\|\left(2\|y_n-q\| + \lambda\alpha_n\|q\|\right)\\
&\quad\le \|y_n-q\|^2 + M_3\alpha_n = \|\beta_n(Qx_n-q) + (1-\beta_n)(Su_n-q)\|^2 + M_3\alpha_n\\
&\quad\le (1-\beta_n)^2\|Su_n-q\|^2 + 2\beta_n\langle Qx_n-q, y_n-q\rangle + M_3\alpha_n\\
&\quad\le (1-\beta_n)^2\|u_n-q\|^2 + 2\beta_n\langle Qx_n-Qq, y_n-q\rangle + 2\beta_n\langle Qq-q, y_n-q\rangle + M_3\alpha_n\\
&\quad\le (1-\beta_n)^2\left(\|x_n-q\| + \lambda\alpha_n\|q\|\right)^2 + 2\beta_n\rho\|x_n-q\|\|y_n-q\|\\
&\qquad+ 2\beta_n\langle Qq-q, y_n-q\rangle + M_3\alpha_n\\
&\quad\le (1-\beta_n)^2\Big(\|x_n-q\|^2 + M_3\alpha_n\Big) + 2\beta_n\rho\|x_n-q\|\left(\|y_n-x_n\| + \|x_n-q\|\right)\\
&\qquad+ 2\beta_n\langle Qq-q, y_n-q\rangle + M_3\alpha_n\\
&\quad\le (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n\rho\|x_n-q\|\left(\|y_n-x_n\| + \|x_n-q\|\right)\\
&\qquad+ 2\beta_n\langle Qq-q, y_n-q\rangle + 2M_3\alpha_n\\
&\quad= \left[1-2\beta_n(1-\rho)\right]\|x_n-q\|^2 + \beta_n\Big[2\rho\|x_n-q\|\|y_n-x_n\| + \beta_n\|x_n-q\|^2 + 2\langle Qq-q, y_n-q\rangle\Big]\\
&\qquad+ 2M_3\alpha_n
\end{aligned} \tag{3.91}$$

for every $n \ge 0$, where $M_3 = \sup\{\lambda\|q\|[2(\|x_n-q\| + \|y_n-q\|) + \lambda\alpha_n\|q\|] : n \ge 0\}$. Now, put $a_n = \|x_n-q\|^2$, $s_n = 2\beta_n(1-\rho)$, $t_n = (1/2(1-\rho))[2\rho\|x_n-q\|\|y_n-x_n\|+\beta_n\|x_n-q\|^2+2\langle Qq-q, y_n-q\rangle]$, and $r_n = 2M_3\alpha_n$. Then (3.91) is rewritten as

$$a_{n+1} \le (1-s_n)a_n + s_nt_n + r_n. \tag{3.92}$$

It is easy to see that $\sum_{n=0}^{\infty} s_n = \infty$, $\limsup_{n\to\infty} t_n \le 0$ and $\sum_{n=0}^{\infty} r_n < \infty$. Thus by Lemma 3.12, we obtain $x_n \to q$. This completes the proof. □

Corollary 3.15. *Let $Q : \mathscr{H}_1 \to \mathscr{H}_1$ be a contractive mapping with coefficient $\rho \in [0,1)$ and $S : \mathscr{H}_1 \to \mathscr{H}_1$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap (\nabla f)^{-1}0 \neq \emptyset$. Assume that $0 < \lambda < 2/\|A\|^2$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences in $\mathscr{H}_1$ generated by*

$$\begin{aligned} x_0 &= x \in \mathscr{H}_1 \text{ chosen arbitrarily}, \\ y_n &= \beta_n Q x_n + (1-\beta_n) S(x_n - \lambda \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} &= (1-\gamma_n) y_n + \gamma_n S(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{aligned} \tag{3.93}$$

where the sequences of parameters $\{\alpha_n\} \subset (0,\infty)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \mathrm{Fix}(S) \cap (\nabla f)^{-1}0$, which is a unique solution of the following variational inequality

$$\langle (I-Q)q, q-p \rangle \leq 0, \quad \forall p \in \mathrm{Fix}(S) \cap (\nabla f)^{-1}0. \tag{3.94}$$

Proof. In Theorem 3.14, putting $C = \mathscr{H}_1$, we deduce that $P_{\mathscr{H}_1} = I$ the identity mapping, $\Gamma = \mathrm{VI}(\mathscr{H}_1, \nabla f) = (\nabla f)^{-1}0$ and

$$\begin{aligned} x_0 &= x \in C(= \mathscr{H}_1) \text{ chosen arbitrarily}, \\ y_n &= \beta_n Q x_n + (1-\beta_n) S P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)) = \beta_n Q x_n + (1-\beta_n) S(x_n - \lambda \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} &= (1-\gamma_n) y_n + \gamma_n S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)) = (1-\gamma_n) y_n + \gamma_n S(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \end{aligned} \tag{3.95}$$

for every $n \geq 0$. Then, by Theorem 3.14, we obtain the desired result. □

Remark 3.16. Theorem 3.14 improves and develops [25, Theorem5.7], [26, Theorem 3.1], and [9, Theorem 3.1] in the following aspects.

(a) The corresponding iterative algorithm in [9, Theorem 3.1] is extended for developing our composite extragradient-like algorithm with regularization in Theorem 3.14.

(b) The technique of proving strong convergence in Theorem 3.14 is very different from those in [25, Theorem 5.7], [26, Theorem 3.1], and [9, Theorem 3.1] because our technique depends on Lemmas 3.12 and 3.13.

(c) Compared with [25, Theorem 5.7], and [26, Theorem 3.1], two weak convergence results, Theorem 3.14 is a strong convergence result. Thus, Theorem 3.14 is quite interesting and very valuable.

(d) In [9, Theorem 3.1], Jung actually introduced the following composite iterative algorithm:

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrarily}, \\ y_n &= \beta_n Q x_n + (1-\beta_n) S P_C (x_n - \lambda_n A x_n), \\ x_{n+1} &= (1-\gamma_n) y_n + \gamma_n S P_C (y_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{aligned} \tag{3.96}$$

where A is inverse-strongly monotone and S is nonexpansive. Now, via replacing $\lambda_n A$ by $\lambda \nabla f_{\alpha_n}$, we obtain the composite extragradient-like algorithm in Theorem 3.14. Consequently, this algorithm is very different from Jung's algorithm.

Furthermore, utilizing Jung [9, Theorem 3.1], we can immediately obtain the following strong convergence result.

Theorem 3.17. *Let $Q : C \to C$ be a contractive mapping with coefficient $\rho \in [0,1)$ and $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following composite extragradient-like algorithm:*

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrarily}, \\ y_n &= \beta_n Q x_n + (1-\beta_n) S P_C (x_n - \lambda_n \nabla f(x_n)), \\ x_{n+1} &= (1-\gamma_n) y_n + \gamma_n S P_C (y_n - \lambda_n \nabla f(y_n)), \quad \forall n \geq 0, \end{aligned} \tag{3.97}$$

where the sequences of parameters $\{\lambda_n\} \subset (0, 2/\|A\|^2)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2/\|A\|^2$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iii) $\limsup_{n\to\infty} \gamma_n < 1$ *and* $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \mathrm{Fix}(S) \cap \Gamma$, which is a unique solution of the following variational inequality:

$$\langle (I-Q)q, q-p \rangle \leq 0, \quad \forall p \in \mathrm{Fix}(S) \cap \Gamma. \tag{3.98}$$

Remark 3.18. It is not hard to see that ∇f is $(1/\|A\|^2)$-ism. Thus, Theorem 3.17 is an immediate consequence of Jung [9, Theorem 3.1].

Acknowledgments

In this research, the first and second authors were partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Leading Academic Discipline Project of Shanghai Normal University (DZL707). Third author was partially supported by the Grant NSC 101–2115-M-037-001.

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