A PARABOLIC ANALOGUE OF FINN'S MAXIMUM PRINCIPLE

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Abstract. We obtain a parabolic analogue of the well-known maximum principle established by R. Finn for solutions of the minimal surface equation.

1. Introduction and main results

The aim of this article is to discuss certain properties of the solutions of the equation

$$u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{u_{x_i}}{\sqrt{1 + |\nabla_x u|^2}} \right), \tag{1}$$

which are parabolic analogues of the well-known maximum principle established by Finn [2] for solutions of the minimal surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{u_{x_i}}{\sqrt{1 + |\nabla_x u|^2}} \right) = 0. \tag{2}$$

It is known that while in some respects solutions of the minimal surface equation behave in complete accordance with the general theory of elliptic equations, in other respects these solutions reveal completely unexpected properties. Examples of the latter case are the well-known theorem of Bernstein [1] on the non-existence of non-trivial solutions of equation (2) in the whole space, for $n = 2$, and Finn's maximum principle [2].

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Let $R_{a,b}$ be the ring $\{(x_1, ..., x_n) \in \mathbb{R}^n, a < r < b\}, r = \sqrt{x_1^2 + ... + x_n^2}$, $n \geq 2, 0 < a < b < \infty$, and D an arbitrary domain in $\mathbb{R}^n$. Also let $K(r, a) = \cosh^{-1} \frac{r}{a}, K(r, a) \leq 0$.

The function $K(r, a)$ determines a supersolution (in the case $n = 2$ a solution) of the minimal surface equation (2) in the exterior of the sphere $\{r = a\}$ that vanishes on this sphere. It is easy to see that this supersolution cannot be extended inside the sphere $\{r = a\}$.

Let us state Finn's maximum principle in one of its simplest versions (see [6]).

Let c be an arbitrary fixed constant.

Theorem (Finn). *Let $D \subseteq R_{a,b}$, and let $u(x)$ be a solution of the minimal surface equation (2) in D, continuous in $\overline{D}$ and such that $u - K(r, a) \leq c$ on $\partial D \setminus \{r = a\}$. Then $u - K(r, a) \leq c$ in D.*

We note that the essential difference between the behaviour of solutions of the minimal surface equation and solutions of linear uniformly-elliptic equations is particularly well seen in the case when part of the boundary of the domain D lies on the sphere $\{r = a\}$, for example, when $D = R_{a,b}$.

In view of the fact that each solution of the minimal surface equation is a solution of (1), it is clear that the second solutions inherit one or other properties of the first solutions, and hence it is reasonable to assume that the situation described above for solutions of the minimal surface equation will also be seen in the case of equation (1), namely: while in some respects the solutions of equation (1) behave in complete accordance with the general theory of parabolic equations, in other respects these solutions reveal completely unexpected properties (see, for example, [4-5]). However, sometimes these completely unexpected properties are fully expected, and, moreover, solutions of equation (1) in fact inherit one or other properties of solutions of the minimal surface equation. The main aim of this article is to give another confirmation of this intuitive hypothesis.

Suppose, as above, that D is an arbitrary domain in $\mathbb{R}^n$, which in general may coincide with the whole space, $D_T = (0, T) \times D$, and $\partial D_T = D \cup \{[0, T) \times \partial D\}$ is the parabolic part of the boundary of D_T. If $D = \mathbb{R}^n$ we take ∂D to be empty.

Definition. *By a subsolution (supersolution) of equation (1) in D_T we mean a function $u(t, x)$ which belongs to the space $C(0, T; Lip_{loc}(D))$, whose derivative u_t belongs to the space $L_1(0, T; L_{1,loc}(D))$, and which satisfies the following integral inequality:*

$$\int_{D_T} \left[u_t \varphi + \sum_{i=1}^{n} \varphi_{x_i} \frac{u_{x_i}}{\sqrt{1 + |\nabla_x u|^2}} \right] dt dx \leq 0 \qquad (\geq 0) \tag{3}$$

for an arbitrary nonnegative function $\varphi \in C(0, T; \overset{\circ}{Lip}(D))$.

Let E be the exterior of the sphere $\{r = a\}$, $E_T = (0, T) \times E, \partial_F D_T = D \cup \{[0, T) \times \{\partial D \setminus \{r = a\}\}\}$, and let $c_1 \geq 1$ and c be arbitrary fixed constants.

Theorem 1. *Suppose that $u(t,x)$ is a subsolution of equation (1) in D_T, $D_T \subseteq E_T$, continuous on the set $D_T \cup \partial_F D_T$ and such that $u - c_1 K(r,a) \leq c$ on $\partial_F D_T$. Then $u - c_1 K(r,a) \leq c$ in D_T.*

Similar results hold for supersolutions of equation (1).

Theorem 1′. *Suppose that $v(t,x)$ is a supersolution of equation (1) in D_T, $D_T \subseteq E_T$, continuous on the set $D_T \cup \partial_F D_T$ and such that $v + c_1 K(r,a) \geq c$ on $\partial_F D_T$. Then $v + c_1 K(r,a) \geq c$ in D_T.*

Apparently the most impressive example is when the domain D coincides with the ring $R_{a,b}$ or with the exterior of the sphere $\{r = a\}$. In the first case we have a direct parabolic analogue of the Finn's maximum principle [2].

Corollary 1. *Suppose that $u(t,x)$ is a subsolution of equation (1) in $(0,T)\times R_{a,b}$, continuous on the set $[0,T)\times\{a < r \leq b\}$ and such that $u - K(r,a) \leq c$ on $R_{a,b} \cup \{[0,T) \times \{r = b\}\}$. Then $u - K(r,a) \leq c$ in $(0,T) \times R_{a,b}$.*

Corollary 2. *Suppose that $u\,(t,x)$ is a subsolution of equation (1) in E_T, continuous on E and such that $u - c_1 K(r,a) \leq c$ on E. Then $u - c_1 K(r,a) \leq c$ in E_T.*

The above assertions form a working tool for studying the qualitative properties of solutions of equation (1) in the same way as Finn's maximum principle for solutions of the minimal surface equation (2). We give the simplest of them.

Corollary 3. *Suppose that $u(t,x)$ is a subsolution of equation (1) in $(0,T)\times\{0 < r < b\}$, continuous on the set $[0,T) \times \{0 < r \leq b\}$ and such that $u - K(r,a) \leq c$ on $\{0 < r < b\} \cup \{[0,T) \times \{r = b\}\}$. Then $\limsup_{(\tau,x)\to(t,0)} u(t,x) \leq c$ for all $(\tau,x) \in (0,T) \times \{0 < r < b\}$ and all $t \in (0,T)$.*

Proof. For fixed r, $\lim_{a\to 0} a\cosh^{-1}\frac{r}{a} = 0$. ∎

Questions on the solubility of initial-value problems for equation (1) acquire a special interest in connection with what has been said above. It is easy to construct special cases of insolubility by using the stated assertions.

The basic idea of the proof of Theorem 1 (Theorem 1′) consists in comparing the unknown subsolution (supersolution) of equation (1) with a definite fixed supersolution (subsolution) of the same equation whose properties have been thoroughly studied. An application of this fundamental approach is possible based on the following assertion.

Theorem 2 (Comparison Principle). *Let $u\,(t,x)$ and $v\,(t,x)$ be respectively a subsolution and a supersolution of equation (1) in D_T, continuous on the set $D_T \cup \partial D_T$ and such that $u \leq v$ on ∂D_T. Then $u \leq v$ in D_T.*

We note that the comparison principle stated here has no direct analogues in the framework of the linear theory of parabolic equations. A confirmation of this is well-known fact that a non-trivial classical solution exists of the Cauchy problem for the heat conduction equation with zero initial data [7].

2. Proofs of the main results

Proof of Theorem 2. Let $u(t,x)$ and $v(t,x)$ be respectively a subsolution and a supersolution of equation (1) in D_T such that $u \le v$ on ∂D_T.

It follows then from (3) that for an arbitrary nonnegative function $\varphi \in C(0,T;\overset{\circ}{Lip}(D))$ the relations

$$\int_{D_T}\left[u_t\varphi + \sum_{i=1}^{n}\varphi_{x_i}\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}}\right]dtdx \le 0 \tag{3'}$$

and

$$\int_{D_T}\left[v_t\varphi + \sum_{i=1}^{n}\varphi_{x_i}\frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}}\right]dtdx \ge 0 \tag{3''}$$

hold.

By subtracting the latter from the former we obtain the inequality

$$\int_{D_T}\left[(u-v)_t\,\varphi + \sum_{i=1}^{n}\varphi_{x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}}\right)\right]dtdx \le 0. \tag{4}$$

Let $0 < r < R$, $s > 1$ and $p > 1$ be arbitrary real numbers. Set $\varphi = \exp(-t)\,\psi^s(x)\,w^p$ in (4), where $w(t,x) = \exp(-t)\,(u-v)^+(t,x)$, $\psi \in \overset{\circ}{C}{}^\infty(\{x : |x| < R\})$, $0 \le \psi(x) \le 1$, $\psi(x) = 1$ at $\{x : |x| \le r\}$. Then

$$\begin{aligned} &\int_{D_T}(u-v)_t\,\psi^s w^p \exp(-t)\,dtdx \\ &\quad + p\int_{D_T}\sum_{i=1}^{n} w_{x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}}\right) \\ &\quad \cdot \exp(-t)\,\psi^s w^{p-1} dtdx \\ &\quad + s\int_{D_T}\sum_{i=1}^{n} \psi_{x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}}\right) \\ &\quad \cdot \exp(-t)\,\psi^{s-1} w^p dtdx \\ &\equiv I_1 + I_2 + I_3 \le 0. \end{aligned} \tag{5}$$

Estimate further the integrals I_1, I_2, I_3. Since

$$(u-v)_t^+ = \begin{cases} (u-v)_t, & \text{if } u - v > 0, \\ 0, & \text{if } u - v \le 0, \end{cases}$$

the following continued equality

$$
\begin{aligned}
I_1 &= \int_{D_T} (w \exp(t))_t \, \psi^s w^p \exp(-t) \, dtdx \\
&= \int_{D_T} w_t \psi^s w^p dtdx + \int_{D_T} \psi^s w^{1+p} dtdx \\
&= \frac{1}{1+p} \int_{D_T} \left(w^{1+p}\right)_t \psi^s dtdx + \int_{D_T} \psi^s w^{1+p} dtdx
\end{aligned}
\tag{6}
$$

takes place.

It follows from the inequality $u \le v$ on D that

$$\int_D \psi^s(x) \, w^{1+p}(\tau, x) \, dx \to 0 \qquad at \qquad \tau \to 0.$$

Since the integrand $\psi^s w^{1+p}$ is not negative in D_T, we have

$$I_1 \ge \int_{D_T} \psi^s w^{1+p} dtdx. \tag{7}$$

Next it can be easily understood that $I_2 \ge 0$ as

$$\sum_{i=1}^n (u-v)_{x_i} \left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla\_x v|^2}} \right) \ge 0.$$

Then from (5)–(7) we obtain

$$\int_{D_T} \psi^s w^{1+p} dtdx \le |I_3|. \tag{8}$$

We further estimate $|I_3|$ in (8). As far as

$$
\begin{aligned}
|I_3| \le s \int_{D_T} & \exp(-t) \, \psi^{s-1} w^p |\nabla \psi| \\
& \cdot \left(\sum_{i=1}^n \left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}} \right)^2 \right)^{1/2} dtdx
\end{aligned}
$$

and

$$\sum_{i=1}^n \left(\frac{u_{x_i}}{\sqrt{1+|\nabla_x u|^2}} - \frac{v_{x_i}}{\sqrt{1+|\nabla_x v|^2}} \right)^2 \le 4,$$

then

$$|I_3| \le 2s \int_{D_T} \exp(-t) \psi^{s-1} w^p |\nabla \psi| dtdx. \tag{9}$$

Now from inequalities (8) – (9) we conclude that

$$\int_{D_T} \psi^s w^{1+p} dtdx \leq \int_{D_T} 2s\exp(-t)\psi^{s-1} w^p |\nabla\psi| dtdx. \tag{10}$$

Taking $s = 1+p$, such that, $(s-1)\,\frac{1+p}{p} = s$, and estimating the right–hand side in (10) by using of Young's inequality

$$AB \leq \frac{1}{2} A^{\frac{\beta}{\beta-1}} + 2^{\beta-1} B^{\beta},$$

where $A = (\psi w)^{s-1}$, $B = 2s\exp(-t)|\nabla\psi|$ and $\beta = s$, we get the following inequality

$$\int_{D_T(R)} \psi^s w^s dtdx$$
$$\leq \frac{1}{2} \int_{D_T(R)} \psi^s w^s dtdx + 2^{2s-1} s^s \int_{D_T(R)} |\nabla\psi|^s \exp(-st) dtdx.$$

Henceforth $D_T\,(R) = (0,T) \times \{D \cap \{x : |x| < R\}\}$.

It follows directly from the previous inequality that

$$\int_{D_T(R)} \psi^s w^s dtdx \leq (4s)^s \int_{D_T(R)} |\nabla\psi|^s \exp(-st) dtdx.$$

In order to complete the proof of the theorem we need to consider the two following cases.

If the domain D is bounded we choose r to be sufficiently large, so that, $D_T\,(r) = D_T$. It follows then from the fact that $|\nabla\psi| = 0$ a.e. in D_T and the previous inequality that $w = 0$ a.e. in D_T, and consequently $u \leq v$ a.e. in D_T.

If the domain D is unbounded, we take s to be sufficiently large, so that $s > n$, and minimize the right–hand side of the obtained inequality over all admissible functions $\psi\,(x)$ of the type indicated above (that is equivalent to the calculation of the s–capacity for a certain condenser (see, for example, [3]). As a result we obtain inequality

$$\int_{D_T(r)} w^s dtdx \leq \frac{\sigma_n}{s} (4s)^s \left(\frac{s-n}{s-1}\right)^{s-1} \left(1 - \left(\frac{R}{r}\right)^{\frac{n-s}{s-1}}\right)^{1-s} R^{n-s}. \tag{11}$$

Here σ_n is the $(n-1)$–dimensional volume of the surface of the unit ball in $\mathbb{R}^n$.

Now, taking $R = 2r$ and passing to the limit as $R \to \infty$ in (11), we get the statement of Theorem 2 immediately. ∎

Proof of Theorem 1. Set

$$\epsilon = \sup_{a' \le r} |c_1 K(r,a) - c_1 K(r,a')|$$

for any fixed $a' > a$. Since by the hypotheses of the theorem, it follows that

$$\lim(u(t,x) - c_1 K(r,a')) \le c + \epsilon \tag{12}$$

for all sequences of points $(t,x) \in D_T$, tending to the parabolic part of the boundary of the domain D_T, lying on the set $[0,T) \times \{a' \le r\}$. Now we want to establish that

$$u(t,x) \le c_1 K(r,a') + c + \epsilon \tag{13}$$

throughout the domain $D'_T = [0,T) \times \{D \cap \{a' < r\}\}$. The desired result immediately follows from (13), if we turn a' to a.

To prove (13) it is sufficient by the comparison principle (Theorem 2) to show that (12) holds on the parabolic part of the boundary of the domain D'_T. Since it is true for all points of the parabolic part of the boundary of the domain D_T, lying on the set $[0,T) \times \{a' \le r\}$, it is sufficient to show that (13) holds for all interior points of D_T, lying on the set $(0,T) \times \{r = a'\}$.

Suppose that this is not the case. Then the function $u - c_1 K(r,a)$ takes its maximum value $M > c + \epsilon$ at some point $(t', x') = (t', x'_1, ..., x'_n)$, lying on the set $(0,T) \times \{r = a'\}$ and which is interior to the domain D_T. By the comparison principle (Theorem 2) $u(t,x) - c_1 K(r,a') \le M$ throughout the domain D'_T.

On the other hand $\sup |\nabla u|$ is bounded at this point, whereas $\frac{\partial K}{\partial r}|_{r=a'} = -\infty$. Since $u(t,x) - c_1 K(r,a') = M$ at the point (t', x'), there must then be points of D'_T near (t', x') at which $u(t,x) - c_1 K(r,a') > M$.

Hence, our assumption that (13) does not hold leads us to the contradiction, which completes the proof of the theorem. ∎

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References

1. S. N. Bernstein, *Collected Works*, **III**, Izdat. Akad. Nauk SSSR, Moscow, (1960), 251–258 (Russian).
2. R. Finn, *Remarks relevant to minimal surfaces of prescribed mean curvature*, J. Anal. Math. **14** (1965), 139-160.
3. J. Frehse, *Capacity methods in the theory of partial differential equations*, Jahresber. Deutsch. Math.-Verein. **84** (1982), 1–44.
4. A. S. Kalashnikov, *Some problems of the qualitative theory of second-order nonlinear degenerate parabolic equations*, Uspekhi Mat. Nauk **42** (1987), no. 2(254), 135–176 (Russian).
5. V. Kurta, *Comparison principle for solutions of parabolic inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), 1175–1180.

6. R. Osserman, *Minimal surfaces*, Uspekhi Mat. Nauk **22** (1967), no. 4(136), 55–136 (Russian).
7. A. N. Tikhonov, *Uniqueness theorems for the heat equation*, Mat. Sb. **42** (1935), no. 2, 199–215 (Russian).

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