THE τ-FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. Let X be a Banach space and τ a topology on X. We say that X has the τ-fixed point property (τ-FPP) if every nonexpansive mapping T defined from a bounded convex τ-sequentially compact subset C of X into C has a fixed point. When τ satisfies certain regularity conditions, we show that normal structure assures the τ-FPP and Goebel-Karlovitz's Lemma still holds. We use this results to study two geometrical properties which imply the τ-FPP: the τ-GGLD and $M(\tau)$ properties. We show several examples of spaces and topologies where these results can be applied, specially the topology of convergence locally in measure in Lebesgue spaces. In the second part we study the preservence of the τ-FPP under isomorphisms. In order to do that we study some geometric constants for a Banach space X such that the τ-FPP is shared by any isomorphic Banach space Y satisfying that the Banach-Mazur distance between X and Y is less than some of these constants.

1. INTRODUCTION

Let (M, d) be a metric space. We recall that a mapping $T : M \to M$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in M$. We say that a Banach space X has the fixed point property (FPP) if every nonexpansive mapping T defined from a convex bounded closed subset C of X into C has a fixed point. In 1965 Browder [5,6] and Kirk [33] proved that X has the FPP if X is either a Hilbert space, or a uniformly convex space

1991 *Mathematics Subject Classification.* 47H09, 47H10.

Key words and phrases. Fixed point, nonexpansive mapping, stability of the fixed point property, uniformly Lipschitzian mappings, asymptotically regular mappings, local convergence in measure, Lebesgue spaces.

This research is partially supported by D.G.I.C.Y.T. PB 96-1338-C01-C02 and J.A. FQM 0127

Received: March 4, 1998

or a reflexive space with normal structure. From this starting point, many authors have studied different geometric conditions on X, assuring the FPP. (See, for instance, [19] and references therein). However, c_0 is an easy example of a Banach space without the FPP [25]. The failure of the FPP in c_0 is a consequence of the noncompactness of its unit ball in the weak topology. This fact has originated that many authors study the existence of fixed points for a nonexpansive mapping, under the stronger assumption: C is a weakly compact convex subset of X. (See, for instance, [20] and references therein). If every nonexpansive mapping $T : C \to C$, C as above, has a fixed point, we say that X has the weak fixed point property (w-FPP). In several papers, the weak topology is replaced by some other topologies. For instance, in [1,13,14,28,29,38,40,42] the existence of fixed points is studied when C is a weak* compact convex set of a dual space. In [3,36,37] the same problem is studied for the topology of convergence in measure. More general topologies are considered in [7,15,16,17,31,32,34,35]. In this paper we consider that τ is any topology on X and C is a convex subset of X which is sequentially compact for the topology τ. If every nonexpansive mapping $T : C \to C$ has a fixed point, we say that X has the τ-FPP. Normal structure and Goebel-Karlovitz's lemma are the most usual tools to prove the existence of fixed points for nonexpansive mappings. When τ satisfies certain regularity conditions, we show that normal structure still assures the τ-FPP and Goebel-Karlovitz's lemma still holds. In Section 2, we use these tools to study two geometrical properties which imply the τ-FPP: the τ-GGLD and the $M(\tau)$ properties. We show several examples of spaces and topologies where these results can be applied.

When X has the τ-FPP the following question can be interesting: If Y is a Banach space isomorphic to X and the Banach-Mazur distance $d(X, Y)$ is "small", does Y have the τ-FPP? This problem, originated by Bynum [8], has been widely studied for the weak topology (see, for instance, [10,11,18,23,30,39]) and, occasionally, for the weak* topology [42]. However we do not know any previous result for a general topology. According to the assumptions on the topology τ, we can obtain a scale of stability bounds. When τ is an arbitrary topology we can use the fixed point theory for uniformly Lipschitzian and asymptotically regular mappings to obtain stability bounds for nonexpansive mappings by means of the coefficient $\kappa_\tau(X)$. We show some classes of spaces and topologies where this coefficient is computed. In particular, for $L_1(\Omega)$ and the topology of locally convergence in measure we obtain the best possible stability bound. When X is a separable Banach space, τ is a linear topology and the norm is a τ-sequentially lower semicontinuous (τ-slsc) function we define the coefficient $\tau CS(X)$, which becomes Bynum's weakly convergent sequences coefficient when τ is the weak topology. However, we show that we cannot use a similar definition as that given by Bynum. This coefficient is used, in two different ways, to obtain stability bounds of the τ-FPP. In special classes of spaces, we compare these bounds and those obtained using $\kappa_\tau(X)$. Finally, assuming stronger regu-

larity properties of the norm we obtain better stability bounds using the coefficient $M_\tau(X)$, inspired in [10]. In particular, if $X = L_p(\Omega)$, $1 \leq p < \infty$ and τ is the topology of convergence locally in measure, we obtain stability bounds which are identical to those which appear in [10] for the FPP in ℓ_p-spaces.

2. Geometric Conditions Implying the τ-FPP

Throughout this paper X will be a Banach space and τ an arbitrary topology on X. In some theorems additional conditions will be assumed on τ.

Definition 1. We say that X has the τ-FPP if every nonexpansive mapping T defined from a convex norm-bounded τ-sequentially compact subset C of X into C has a fixed point.

From the Eberlein-Smulian Theorem , it is clear that the τ-FPP is the w-FPP when τ is the weak topology.

Definition 2. We say that X has normal structure with respect to τ (τ-NS) if for every convex norm-bounded τ-sequentially compact $C \subset X$ with diam$C > 0$, there exists an element $x \in C$ which is not diametral, i.e. $\sup\{\|x - y\| : y \in C\} <$diam (C).

Theorem 1. *Suppose that $\|\cdot\|$ is a τ-slsc function. If X has τ-NS then X has the τ-FPP.*

Proof. It is an easy consequence of Theorem 1 in [34]. Indeed, let C be a convex norm-bounded τ-sequentially compact subset of X and $T : C \to C$ a nonexpansive mapping. Denote by S the family of all convex τ-sequentially closed subsets of C. Then S is countably compact, stable under arbitrary intersections, and normal (see definitions in [34]). Furthermore it is easy to prove that the τ-sequential lower semicontinuity of the norm implies that every closed ball B is τ-sequentially closed. Thus, S contains $B \cap C$ for every closed ball B. Theorem 1 in [34] implies that T has a fixed point. ∎

Definition 3. We shall say that X has the τ-GGLD property if

$$\lim_n \|x_n\| < \lim_{n,m;n\neq m} \|x_n - x_m\|$$

for every norm-bounded τ-null sequence such that both limits exist and $\lim_n \|x_n\| \neq 0$.

Remark 1. If τ is the weak topology, the τ-GGLD property is implied by the GGLD property [22], which implies weakly normal structure and hence the w-FPP. Our main goal in this section is to prove an equivalent result when τ is an arbitrary topology. The standard arguments to prove that GGLD implies w-NS [22] are strongly based upon a well known fact: Convex norm-closed subsets of X are weakly closed. This fact is not true

for arbitrary topologies on X. For instance, we consider $X = L_1([0,1])$ endowed with the topology of convergence in measure and the sequence $f_n = n\chi_{[0,\frac{1}{n}]}$ which converges to the null function in measure. However, $0 \notin \bar{co}\{f_n\}$ because $\|g\|_1 = 1$ for every $g \in co\{f_n\}$.

Since the standard approach does not work when τ is not the weak topology, we need to use different arguments.

Theorem 2. *Let X be a separable Banach space and τ a linear topology on X. If X has the τ-GGLD property then X has τ-NS.*

Proof. Suppose that X fails to have τ-NS. Then, there exists a convex, norm-bounded, τ-sequentially compact subset C with $\operatorname{diam}(C) = 1$, which is a diametral set. We shall find a sequence $\{x_n\} \subset C$ such that $\lim_n \|x_n - x\| = 1$ for every $x \in C$. ∎

Let $\{y_k\}$ be a dense sequence in C. Inspired by [37] we are going to construct, by induction, a sequence $\{x_n\} \subset C$ such that $\lim_n \|x_n - y_k\| = 1$ for every $k \in \mathbb{N}$.

Let $x_1 = y_1$ and suppose that we have found $x_2, \dots, x_{n-1} \in C$ such that $\|x_m - y_k\| \geq 1 - \frac{1}{m}$ for $k \leq m$, $m = 1, \dots, n-1$.

Denote by b the geometric center of $y_1, y_2, \dots, y_n$, that is:

$$b = \sum_{i=1}^{n} \frac{y_i}{n}.$$

Since C is a diametral set, there exists $x_n \in C$ such that $\|x_n - b\| \geq 1 - \frac{1}{n^2}$. Then if $k \leq n$ we have

$$\begin{aligned} 1 - \frac{1}{n^2} \leq \|x_n - b\| &= \left\| \sum_{i=1}^{n} \left(\frac{x_n}{n} - \frac{y_i}{n} \right) \right\| \\ &\leq \frac{1}{n} \|x_n - y_k\| + \sum_{i=1; i \neq k}^{n} \frac{1}{n} \|x_n - y_i\| \leq \frac{1}{n} \|x_n - y_k\| + \frac{n-1}{n}, \end{aligned}$$

which implies $\|x_n - y_k\| \geq 1 - \frac{1}{n}$ for $k = 1, \dots, n$.

Consequently, for every $k \in \mathbb{N}$ we get

$$\lim_n \left(1 - \frac{1}{n}\right) \leq \liminf_n \|x_n - y_k\| \leq \limsup_n \|x_n - y_k\| \leq 1,$$

and therefore, $\{x_n\}$ satisfies the required condition.

Notice that if we define the functions $\varphi_1(x) = \liminf_n \|x_n - x\|$, $\varphi_2(x) = \limsup_n \|x_n - x\|$, both are continuous and constant equal to 1 in a dense subset of C. Thus, $\lim_n \|x_n - x\| = 1$ for every $x \in C$.

On the other hand, C is a τ-sequentially compact set. Hence, there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a vector $x \in C$ such that $\{x_{n_k}\}$ converges

to x in the topology τ. We can also assume that $\lim_{k,j;k\neq j}\|x_{n_k}-x_{n_j}\|$ exists (see [2, Theorem III.1.5]). Notice that the condition satisfied by $\{x_n\}$ implies that this limit must be equal to 1. Finally, consider the sequence $y_k = x_{n_k} - x$ which converges to 0 in τ because τ is a linear topology. Moreover, $\lim_k \|y_k\| = 1$ and $\lim_{k,j;k\neq j}\|y_k - y_j\| = 1$. Thus X does not satisfy the τ-GGLD property.

The converse of Theorem 2 is not true even if τ is the weak topology.

Example 1. In the space $L_1([0,1])$ we introduce the equivalent norm

$$|||x|||^2 = \|x\|_1^2 + \left(\sum_{k=1}^{\infty}\frac{|x(k)|}{2^k}\right)^2$$

where $x = \sum_{k=1}^{\infty} x(k)e_k$, $\{e_k\}$ is a Schauder basis in $L_1([0,1])$ (for instance, the Haar system) and $\|\cdot\|_1$ is the usual norm on $L_1([0,1])$.

Since $(L_1([0,1]),\|\cdot\|_1)$ fails to have the w-FPP, it is known [22] that $L_1[0,1]$ does not have the w-GGLD property. Then, there exists a weakly null sequence $\{x_n\}$ such that $\lim_n \|x_n\|_1 = \lim_{n,m;n\neq m}\|x_n - x_m\|_1$.

It is easy to check that if $\{x_n\}$ is a weakly null sequence then

$$\lim_n \sum_{k=1}^{\infty}\frac{|x_n(k)|}{2^k} = 0.$$

Thus

$$\frac{\lim_{n,m;n\neq m}|||x_n - x_m|||^2}{\lim_n |||x_n|||^2} = \frac{\lim_{n,m;n\neq m}\|x_n - x_m\|_1^2}{\lim_n \|x_n\|_1^2} = 1.$$

But $(L_1([0,1]),|||\cdot|||)$ is U.C.E.D. [9, Corollary 6.9, page 66] and consequently this space has the w-NS.

From Theorem 1 and 2 the following result is clear

Theorem 3. *Let X be a separable Banach space and τ a linear topology on X. If X has the τ-GGLD property and the norm is a τ-slsc function, then X has the τ-FPP.*

Let $\{x_n\}$ be a τ-null sequence which is norm-bounded. The function

$$\phi_{(x_n)}(x) = \limsup_n \|x - x_n\|$$

will be called a function of τ-null type. A basic tool in Fixed Point Theory for nonexpansive mappings is Goebel-Karlovitz's Lemma. Its proof is based upon the weak lower semicontinuity of the functions of w-null type and the following fact: if X does not have the w-FPP, there exists a convex

weakly compact subset C of X and a fixed point free nonexpansive mapping $T : C \to C$ such that C is minimal, i.e., if K is a nonempty weakly compact convex subset of C which is invariant under T, then $K = C$.

In order to assure the existence of a minimal set in the setting of the τ-FPP we need that τ-sequentially compact sets are τ-compact. This property is not only satisfied by either the weak topology or a metric topology. In the case that the Banach space X is separable, any topology τ which is weaker than the norm topology also verifies that τ-sequentially compact sets are τ-compact. Indeed, X is Lindelöf for the norm topology because there is a countable basis of open sets and the same is true for the topology τ because it is weaker than the norm topology. Thus τ-sequentially compact sets are countably compact and Lindelöf, which implies that they are τ-compact.

On the other hand, if T is a nonexpansive mapping defined from a convex norm-bounded subset C of X into C we can always find an approximated fixed point sequence (a.f.p.s.) for T in C (i.e. $\lim \|x_n - Tx_n\| = 0$). For that, we extend the mapping T, in a continuous way, to the set $\bar{C}^{\|\cdot\|}$. This extension is also a nonexpansive mapping. Hence, Banach's Contractive Mapping Principle let us find an a.f.p.s. for the extension of T in $\bar{C}^{\|\cdot\|}$. Finally, by an approximation argument, we can construct an a.f.p.s. for T in C.

If we replace the weak topology by a linear topology τ on X such that τ-sequentially compact sets are τ-compact and the functions of τ-null type are τ-slsc, it is easy to prove that Goebel-Karlovitz's Lemma still holds (see [32,41]). Thus, we can state:

Lemma 1. *Let X be a Banach space and τ a linear topology on X such that τ-sequentially compact sets are τ-compact and the functions of τ-null type are τ-slsc. Let C be a τ-sequentially compact convex set, $T : C \to C$ a nonexpansive mapping and assume that C is minimal, i. e. if K is a nonempty τ-sequentially compact convex subset of C which invariant under T, then $K = C$. If $\{x_n\}$ is an approximated fixed point sequence for T in C then $\lim_n \|x - x_n\| = diam\ (C)$, for all $x \in C$.*

Definition 4. A Banach space X is said to have the property $M(\tau)$ if the functions of τ-null type are constant on the spheres, i.e.

$$\phi_{(x_n)}(x) = \phi_{(x_n)}(y)$$

whenever $\|x\| = \|y\|$.

Remark 2. If τ is the weak topology, $M(\tau)$ is known as property M of Kalton (see [26,27]), which implies the w-FPP [18]. We are going to prove that property $M(\tau)$ implies the τ-FPP.

Lemma 2. *Let X be a Banach space with the property $M(\tau)$ where τ is a linear topology on X. Then the norm and the functions of τ-null type are τ-slsc.*

Proof. We are going to prove that the unit ball B_X is τ-sequentially closed which easily implies that the norm is τ-slsc. Let $\{x_n\}$ be a sequence in B_X which is τ-convergent to x_0. Since $\{x_n - x_0\}$ is τ-null we obtain from property $M(\tau)$

$$1 \geq \limsup_n \|x_n\| = \limsup_n \|x_n - x_0 + x_0\| = \limsup_n \|x_n - 2x_0\|.$$

Hence,

$$\|2x_0\| \leq \limsup_n \|x_n - 2x_0\| + \limsup_n \|x_n\| \leq 2$$

and we obtain that $x_0 \in B_X$.

To prove the second assertion, we shall first prove that functions of τ-null type are nondecreasing with respect to $\|x\|$. Bearing in mind that X satisfies property $M(\tau)$ we only need prove that $\phi_{(x_n)}(tx)$ is a nondecreasing function with respect to t in $[0, \infty)$. Indeed, assume $0 < t_1 < t_2$. Then, there exists $\beta \in (0, 1)$ such that

$$t_1 x = \beta(-t_2)x + (1 - \beta)t_2 x.$$

The convexity of $\phi_{(x_n)}$ implies

$$\phi_{(x_n)}(t_1 x) \leq \beta\phi_{(x_n)}(-t_2 x) + (1 - \beta)\phi_{(x_n)}(t_2 x) \quad (*)$$

From property $M(\tau)$ we derive

$$\phi_{(x_n)}(-t_2 x) = \phi_{(x_n)}(t_2 x) \quad (**)$$

and (*) and (**) imply that $\phi_{(x_n)}(t_1 x) \leq \phi_{(x_n)}(t_2 x)$.

Now, we are going to prove that for every $\alpha > 0$ the set

$$A_\alpha := \{x \in X : \phi_{(x_n)}(x) \leq \alpha\}$$

is τ-sequentially closed, which will imply that functions of τ-null type are τ-slsc. Assume that $\{y_m\}$ is a sequence in A_α which is τ-convergent to $y \in X$. Since

$$\|y_m\| - \phi_{(x_n)}(0) \leq \phi_{(x_n)}(y_m) \leq \alpha$$

we know that $\{y_m\}$ is bounded by $\alpha + \phi_{(x_n)}(0)$. The τ-sequential lower semicontinuity of the norm implies $\|y\| \leq \liminf_m \|y_m\|$. Thus, given $\varepsilon > 1$ there exists $m_0 \in \mathbb{N}$ such that $\|y\| \leq \varepsilon\|y_{m_0}\|$, which implies

$$\phi_{(x_n)}(y) \leq \phi_{(x_n)}\left(\varepsilon y_{m_0}\right).$$

Hence, we have

$$\phi_{(x_n)}\left(\varepsilon y_{m_0}\right) \leq \limsup_n \|x_n - y_{m_0}\| + (\varepsilon - 1)\|y_{m_0}\| = \phi_{(x_n)}(y_{m_0}) + (\varepsilon - 1)\|y_{m_0}\|$$

and

$$\phi_{(x_n)}(y) \leq \alpha + (\varepsilon - 1)(\alpha + \phi_{(x_n)}(0)).$$

Taking limits as ε goes to 1 we obtain the result. ∎

From Lemma 1 we can prove the following Proposition. For the weak topology the proof can be found in [24, Proof of Theorem 1].

Proposition 1. *Assume that X, T and C are as in Lemma 1 and that T has no fixed point in C. If $diam(C) = 1$, $\{x_n\}$ is a τ-null approximated fixed point sequence in C, for every $\varepsilon > 0$ and $t \in [0,1]$ there exists a sequence $\{z_n\}$ in C such that*

(a.-) $\{z_n\}$ is $\tau-$convergent, say to $z \in C$.
(b.-) $\|z_n\| > 1 - \varepsilon$ for any positive integer n.
(c.-) $\limsup_n \limsup_m \|z_n - z_m\| \leq t$.
(d.-) $\limsup_n \|z_n - x_n\| \leq 1 - t$.

Proof. From Lemma 1 we know that $\lim_n \|w_n\| = 1$ if $\{w_n\}$ is an a.f.p.s. in C. Then, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x\| > 1 - \varepsilon$ if $x \in C$ and $\|Tx - x\| < \delta(\varepsilon)$. Indeed, otherwise, there exists $\varepsilon_0 > 0$ such that we can find $x_n \in C$ with $\|Tx_n - x_n\| < \frac{1}{n}$ and $\|x_n\| \leq 1 - \varepsilon_0$ for every $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ is an approximated fixed point sequence in C, but $\limsup_n \|x_n\| \leq 1 - \varepsilon_0$.

Let $\varepsilon > 0$ and $t \in [0,1]$. Choose $\gamma > 0$ such that $\gamma < \min\{1, \delta(\varepsilon)\}$ and a null decreasing sequence $\{\eta_n\}$ with $\gamma + \eta_n < \min\{1, \delta(\varepsilon)\}$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we define the contractive mapping $S_n : \bar{C}^{\|\cdot\|} \to \bar{C}^{\|\cdot\|}$ by

$$S_n(x) = (1-\gamma)\bar{T}x + \gamma t x_n \qquad \text{for every } x \in \bar{C}^{\|\cdot\|}$$

where $\bar{T}$ is the unique nonexpansive extension of T to the set $\bar{C}^{\|\cdot\|}$. Using Banach's Contractive Mapping Principle, there exists $y_n \in \bar{C}^{\|\cdot\|}$ such that $y_n = (1-\gamma)\bar{T}y_n + \gamma t x_n$. Since $\bar{T}y_n$ is defined as the limit of the images under T of any sequence in C which converges to the vector y_n we can find $z_n \in C$ with

$$\|z_n - (1-\gamma)Tz_n - \gamma t x_n\| < \eta_n.$$

In addition, from the τ-sequential compactness of C we can assume that the sequence $\{z_n\}$ τ-converges to a vector $z \in C$. Following now the arguments in the proof of Theorem 1 in [24] we obtain the proposition. ∎

Theorem 4. *Let X be a Banach space which satisfies the property $M(\tau)$ where τ is a linear topology on X such that τ-sequentially compact sets are τ-compact and the functions of τ-null type are τ-slsc. Then X has the $\tau-FPP$.*

Proof. Assume that X does not satisfy the τ-FPP. Using standard arguments we can prove that there exist a τ-sequentially compact norm-bounded convex subset K of X with $diam(K) = 1$ and a fixed point free nonexpansive mapping $T : K \to K$ such that K is minimal and contains a τ-null approximated fixed point sequence $\{x_n\}$. For $\varepsilon \in (0, 1/3)$ and $t = 1/2 + \varepsilon/2$, we can construct $\{z_n\}$ as in Proposition 1. Notice that the choice of ε and t implies $t < 1 - \varepsilon$ and $1 - t = t - \varepsilon$. Since the norm is τ-slsc (Lemma 2) we have

$$\|z\| \leq \liminf_n \|z_n - x_n\| \leq 1 - t.$$

On the other hand, let η be a positive number such that $t+\eta < 1-\varepsilon$. Since $\limsup_n \limsup_m \|z_n - z_m\| \leq t$ we can assume $\limsup_n \|z_n - z_m\| \leq t+\eta$ for n large enough. Again, using that the norm is τ-slsc we obtain

$$\limsup_n \|z_n - z\| \leq \limsup_n \liminf_m \|z_n - z_m\| \leq t+\eta < 1-\varepsilon.$$

This implies that $\limsup_n \|z_n - z\| \geq t-\varepsilon$ because otherwise we obtain the contradiction

$$1-\varepsilon \leq \limsup_n \|z_n\| \leq \limsup_n \|z_n - z\| + \|z\| < t-\varepsilon+1-t = 1-\varepsilon$$

Let $d = \limsup_n \|z_n - z\|$. Then

$$\|z\| \leq 1-t = t-\varepsilon \leq \limsup_n \|z_n - z\| = d$$

Consider the sequence $y_n := (z - z_n)/d$. This sequence is τ-null and $\limsup_n \|y_n\| = 1$. Since $\|z/d\| \leq 1$, we have for any m

$$\phi_{(y_n)}(z/d) \leq \phi_{(y_n)}(y_m/\|y_m\|) \leq \phi_{(y_n)}(y_m) + |\|y_m\| - 1|$$

which implies

$$\phi_{(y_n)}(z/d) \leq \limsup_m \phi_{(y_n)}(y_m) = \limsup_m \limsup_n \|y_m - y_n\|.$$

Thus, we obtain the contradiction

$$1-\varepsilon \leq \limsup \|z_n\| \leq d \limsup \|(z_n - z)/d + z/d\|$$

$$= d\phi_{(y_n)}(z/d) \leq \limsup_m \limsup_n \|z_n - z_m\| \leq t. \blacksquare$$

We are going to show some examples, inspired from [31], where the above results can be applied. In fact, these examples satisfy the following stronger property than $M(\tau)$.

Definition 5. Let X be a Banach space and τ a topology on X. We say that X satisfies the property $L(\tau)$ if there exists a function $\delta : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying

(1).- δ is continuous

(2).- $\delta(.,s)$ is nondecreasing.

(3).- $\delta(r,.)$ is nondecreasing

(4).- $\delta(\limsup_n \|x_n\|, \|y\|) = \limsup_n \|x_n - y\|$ for every $y \in X$ and every τ-null sequence.

It is clear that $L(\tau)$ implies $M(\tau)$.

Example 2. Let (Ω, Σ, μ) be a positive σ-finite measure space. For every $1 \leq p < +\infty$, consider the Banach space $L_p(\Omega)$ with the usual norm. Let $(\Omega_k)_{k=1}^{\infty}$ be a σ-finite partition of Ω. We consider τ as the topology generated by the metric:

$$d(f,g) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{1}{\mu(\Omega_k)} \int_{\Omega_k} \frac{|f-g|}{1+|f-g|} d\mu \quad \text{for all } f, g \in L_p(\mu).$$

This topology is known as the topology of the convergence locally in measure (clm). It is clear that $L_p(\Omega)$ endowed with the clm topology is a topological vector space and this topology is weaker than the norm topology. The following result can be derived from [4], because every clm-null sequence $\{f_n\}$ has a subsequence convergent to the null function a.e.: If $\{f_n\}$ is a clm-null sequence in $L_p(\Omega)$, $1 \leq p < \infty$, and f is a function in $L_p(\Omega)$, then

$$\limsup_n \|f_n - f\|_p^p = \|f\|_p^p + \limsup_n \|f_n\|_p^p.$$

Thus $L_p(\Omega)$ endowed with the clm topology satisfies the property $L(\tau)$ with $\delta(r,s) = (r^p + s^p)^{1/p}$. In particular, if Ω is N with the cardinal measure and $p > 1$ we obtain ℓ_p with the weak topology. For ℓ_1, the clm-topology is generated by the metric

$$d(x,0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x(k)|}{1+|x(k)|}$$

where x denotes the vector $x = \{x(k)\}_{k\geq 1}$ in ℓ_1. It is an easy exercise to check that the clm topology coincides with the weak* topology in norm-bounded subsets of $\ell_1 = c_0^\star$. (We should note that this equivalence is not true for unbounded sets. Indeed, consider the sequence $x_n = ne_n$ where e_n denotes the basic vector. For this sequence

$$d(x_n, 0) = \frac{1}{2^n} \frac{n}{1+n} \to_{n\to\infty} 0.$$

However, considering the vector $y = \{\frac{1}{k}\}_{k\geq 1} \in c_0$, we have $y(x_n) = 1$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ does not converge to zero in the weak*-topology).

Example 3. Consider the weak topology in c_0. If $\{x_n\}$ is weakly null, it is easy to prove that

$$\limsup_n \|x_n - y\|_\infty = \max\{\limsup_n \|x_n\|_\infty, \|y\|_\infty\}$$

for every $y \in c_0$. Thus c_0 satisfies the property $L(w)$ with $\delta(r,s) = \max\{r,s\}$.

Example 4. Assume that X is a reflexive Banach space with a duality mapping J_ϕ which is weakly sequentially continuous, associated to a continuous increasing function ϕ which satisfies $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Let $\varphi(t) = \int_0^t \phi(x)dx$. Then,

$$\varphi(\|x + y\|) = \varphi(\|x\|) + \int_0^1 < y, J_\phi(x + ty) > dt$$

for every $x, y \in X$. If $\{x_n\}$ is a weakly null sequence and $x \in X$, we have

$$\varphi(\limsup_n \|x_n + x\|) = \varphi(\limsup_n \|x_n\|) + \varphi(\|x\|)$$

Thus X satisfies the property $L(w)$ with $\delta(r, s) = \varphi^{-1}(\varphi(r) + \varphi(s))$.

3.Stability of the τ-FPP

For a general topology τ we can obtain stability results by using some fixed point results for asymptotically regular and uniformly Lipschitzian mappings. We recall a definition

Definition 6. Let M be a bounded convex subset of X.

(i) A number $b \geq 0$ is said to have the property (P_τ) with respect to M if there exists $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| \geq r$ and each τ-convergent sequence $\{\xi_n\} \subset M$ for which $\limsup_n \|\xi_n - x\| \leq ar$ and $\limsup_n \|\xi_n - y\| \leq br$, there exists some $z \in M$ such that $\liminf_n \|\xi_n - z\| \leq r$.

(ii) $\kappa_\tau(M) = \sup\{b > 0 : b$ has property (P_τ) with respect to $M\}$.

(iii) $\kappa_\tau(X) = \inf\{\kappa_\tau(M) : M$ is as above $\}$.

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space, τ an arbitrary topology on X. Assume that $|\cdot|$ is an equivalent norm on X such that*

$$|x| \leq \|x\| \leq d|x|$$

for every $x \in X$ and some $d < \kappa_\tau(X)$. Then $Y = (X, |\cdot|)$ has the τ-FPP.

Proof. Let C be a τ-sequentially compact norm-bounded convex subset of X, and $T : C \to C$ a $|\cdot|$-nonexpansive mapping. By [19, Theorem 9.4] we know that $S = (I + T)/2$ is a $|\cdot|$-nonexpansive and asymptotically regular mapping from C into C. It is easy to check that S is d-uniformly Lipschitzian for the norm $\|\cdot\|$. By [12, Theorem 3.1], S has a fixed point which is also a fixed point of T. ∎

Remark 3. The definition of $\kappa_\tau(X)$ makes difficult its computation. In some classes of spaces we can use an easier definition. We recall that X is said to satisfy the τ-uniform Opial condition if for every $c > 0$ there exists $r > 0$ such that $\liminf_n \|x + x_n\| \geq 1 + r$ for every $x \in X$ such that $\|x\| > c$

and every τ-null sequence $\{x_n\}$ in X such that $\liminf_n \|x_n\| \geq 1$. It is easy to check that this condition is satisfied, for instance, if X verifies property $L(\tau)$ and $\delta(1,\cdot)$ is an increasing function. This happens in Example 2 and 4. Using Theorem 2.1 and Remark 2.2 in [12] we can deduce that

$$\begin{aligned}\kappa_\tau(M) = \sup\{b>0 : \forall z,y\in M, \forall r>0 \quad \text{with} \quad \|z-y\|\geq r \quad \text{and every} \\ \text{sequence} \quad \{\xi_n\}\subset M \quad \tau\text{-convergent to} \quad z\in M \quad \text{such} \\ \text{that} \quad \limsup_n \|\xi_n - y\| \leq br, \text{ we have} \quad \liminf_n \|\xi_n - z\| \leq r\}\end{aligned}$$

when the norm is τ-slsc and X satisfies the τ-uniform Opial condition.

Lemma 3. *Assume that τ is a linear topology and X satisfies property $L(\tau)$ and the τ-uniform Opial property. Then $\kappa_\tau(X) \geq \delta(1,1)$. If, in addition, there exists a τ-null sequence which is not norm-convergent, then $\kappa_\tau(X) = \delta(1,1)$*

Proof. By multiplication and translation we can assume that $r = 1$ and $z = 0$ in the equivalent definition of $\kappa_\tau(M)$ in Remark 3. Assume $b < \delta(1,1)$. If $\{\xi_n\}$ is τ-null sequence, $\|y\| \geq 1$ and $\limsup_n \|\xi_n - y\| \leq b$ we have $\limsup_n \|\xi_n\| \leq 1$, because otherwise we obtain the contradiction

$$b \geq \limsup \|\xi_n - y\| = \delta(\limsup_n \|\xi_n\|, \|y\|) \geq \delta(1,1).$$

Thus $\kappa_\tau(X) \geq \delta(1,1)$. To prove the second assertion, let $\{x_n\}$ be a τ-null normalized sequence and $M = co(\{x_n\}\cup\{0\})$. Consider $y = x_1$ and $\xi_n = x_n$, $n \geq 2$. Then $\|y\| = 1$, $\limsup_n \|\xi_n - y\| = \delta(1,1)$ and $\liminf_n \|\xi_n\| = 1$. Thus $\kappa_\tau(X) \leq \delta(1,1)$. ∎

As a consequence of Theorem 5 and Lemma 3 we have:

Corollary 1. *Assume that Y is a Banach space isomorphic to $L_p(\Omega)$, $p \geq 1$ such that $d(Y, L_p(\Omega)) < 2^{1/p}$. Then Y has the clm-FPP.*

Remark 4. In particular, from Corollary 1 we deduce the following: If Y is any Banach space isomorphic to ℓ_1 and $d(Y,\ell_1) < 2$, then Y has the τ-FPP where τ is the $\sigma(\ell_1, c_0)$-topology. Notice that this result was proved in [42] whenever Y is a dual space isomorphic to ℓ_1.

On the other hand, it must be noted that 2 is the best possible stability bound for ℓ_1 with respect to the $\sigma(\ell_1, c_0)$-topology (and so, for $L_1(\Omega)$ with respect to the *clm* topology). Indeed, in [38] it is shown that there exists a nonexpansive fixed point free mapping defined from a weakly* compact convex subset K of the Bynum's space $\ell_{1,\infty}$ into K and the Banach-Mazur distance between ℓ_1 and $\ell_{1,\infty}$ is 2.

To study the stability of the fixed point property, Bynum [8] defined some normal structure coefficients. In particular, the weakly convergent sequence coefficient is defined by

$$WCS(X) = \sup\{M \geq 1 : \ M \cdot r_a(\{x_n\}) \leq \mathrm{d}iam_a(\{x_n\})\}$$

where the supremum is taken over all the sequences $\{x_n\}$ in X which are weakly convergent, $r_a(\{x_n\}) = \inf\{\limsup_n \|x_n - y\| : y \in co(\{x_n\})\}$ and $\operatorname{diam}_a(\{x_n\}) = \lim_k \sup\{\|x_n - x_m\| : n, m \geq k\}$.

For an arbitrary topology τ we shall consider a different definition.

Definition 7. Let X be a Banach space and τ a topology on X and assume that there exists a τ-null sequence which is not norm-convergent. Then, we define the following coefficient:

$$\tau CS(X) = \inf\left\{ \frac{\lim_{n,m;n\neq m} \|x_n - x_m\|}{\lim_n \|x_n\|} \right\}$$

where the infimum is taken over all norm-bounded sequences which converge to 0 in τ, both limits exist and $\lim_n \|x_n\| \neq 0$.

It must be noted that we can always suppose that the Banach space X has τ-convergent sequences which are not norm-convergent because otherwise X has the τ-FPP. Indeed, if T is a nonexpansive mapping defined from a convex norm-bounded τ-sequentially compact subset C of X into C we consider an a.f.p.s. $\{x_n\}$ for T in C. In addition, due to the τ-sequential compactness of C we can assume that $\{x_n\}$ is τ-convergent, say to $x \in C$. If $\{x_n\}$ were also norm-convergent, x would be a fixed point for T in C.

Remark 5. If τ is the weak topology then $\tau CS(X)$ coincides to the weakly convergent sequence coefficient $WCS(X)$ (see [2, Lemma VI.3.8]). In spite of this coincidence for the weak topology, when we extend the definition of $WCS(X)$ to an arbitrary topology, it is more useful to consider Definition 7 instead of the straightforward extension of $WCS(X)$. Indeed, consider the space $L_1([0,1])$ endowed with the convergence in measure topology. Take the sequence $f_n = n\chi_{[0,1/n]}$ in $L_1([0,1])$ which converges to zero in measure. Since $\|f_n - f_m\| = 2 - 2\frac{m}{n}$ for every nonnegative integers $n, m; n > m$, we have $\operatorname{diam}_a(\{f_n\}) = 2$. On the other hand, let f be a mapping in $co(\{f_n\})$, say $f = \sum_{k=1}^m \alpha_k f_k$, , $\alpha_k \geq 0$, $k = 1, ..., m$; $\sum_{k=1}^m \alpha_k = 1$. If $n > m$, it is easy to check that $\|f - f_n\| = 2 - 2\sum_{k=1}^m \alpha_k \frac{k}{n}$. Thus $\limsup_n \|f - f_n\| = 2$ and $r_a(\{f_n\}) = 2$. Hence, the straightforward extension would lead to a τ-convergent sequence coefficient value of 1. However, we will show in Corollary 2 that using our definition, $\tau CS(L_1([0,1])) = 2$ when τ is the convergence in the measure topology.

The coefficient $WCS(X)$ produces stability of the w-FPP in the following sense: Assume that X, Y are isomorphic Banach spaces with $d(X,Y) < WCS(X)$, where $d(X,Y)$ is the Banach-Mazur distance. Then Y has the w-FPP [8]. Since it is clear that a Banach space X has the τ-GGLD property if $\tau CS(X) > 1$, from Theorem 3 we obtain a similar stability result for the τ-FPP. Notice that if X and Y are isomorphic Banach spaces, we can consider that Y is X endowed with an equivalent norm and it is easy to check that $\tau CS(X) \leq d(X,Y)\tau CS(Y)$

Theorem 6. *Let X be a separable Banach space and τ a linear topology on X. Assume that $|\cdot|$ is a norm defined on X such that*

$$|x| \leq \|x\| \leq d|x|$$

for every $x \in X$. Let $Y = (X, |\cdot|)$. Then Y has the τ-FPP if $|\cdot|$ is a τ-slsc function and $d < \tau CS(X)$.

We do not know a general relationship between $\kappa_\tau(X)$ and $\tau CS(X)$. However, in some classes of spaces and topologies we can compare both coefficients. When τ is the weak topology, it is known [2, Theorem IX.2.7] that $\kappa_\omega(X) \leq WCS(X)$ (it must be noted [2, Example IX.2]) that this inequality can be strict) and the arguments in the proof of the above inequality equally work when the functions of τ-null type are τ-slsc. In particular we have that $\kappa_\tau(X) \leq \tau CS(X)$ when X satisfies property $M(\tau)$. When X satisfies the τ-uniform Opial condition we have the same result.

Proposition 2. *(1) Assume that X satisfies the τ-uniform Opial condition. Then $\kappa_\tau(X) \leq \tau CS(X)$.*

(2) Assume that X satisfies property $L(\tau)$. Then $\tau CS(X) = \delta(1,1)$.

Proof. (1) Let ε be an arbitrary positive number and $\{x_n\}$ a normalized τ-null sequence such that $\lim_{n,m;n\neq m} \|x_n - x_m\| < \tau CS(X) + \varepsilon$. Choose k large enough such that $\limsup_n \|x_n - x_k\| \leq \tau CS(X) + 2\varepsilon$. Denote $y_n = (1+\varepsilon)x_n$ and consider the set $M = co(\{y_n\} \cup \{0\})$. The conditions $\limsup_n \|y_k - y_n\| \leq (\tau CS(X) + 2\varepsilon)(1+\varepsilon)$, $\|y_k\| = 1 + \varepsilon$ and $\liminf \|y_n\| = 1 + \varepsilon$ imply $\kappa_\tau(M) \leq (1+\varepsilon)(\tau CS(X) + 2\varepsilon)$. Since ε is arbitrary we obtain $\kappa_\tau(X) \leq \tau CS(X)$.

(2) If X satisfies property $L(\tau)$ and $\{x_n\}$ is a τ-null sequence such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists and $\lim_n \|x_n\| = 1$, it is clear that $\lim_{n,m;n\neq m} \|x_n - x_m\| = \delta(1,1)$, which implies the result. ∎

Corollary 2. *Let (Ω, Σ, μ) as in Example 2. For $p \geq 1$ we have*

$$(clm)CS(L_p(\Omega)) = 2^{1/p}.$$

The following example will show that Theorem 6 can give a stronger stability result than Theorem 5.

Example 5. Let (Ω, Σ, μ) be as in Example 2, and assume that Ω_1 is an atom. For $p > 1$ consider in $L_1(\Omega)$ the equivalent norm

$$|||f||| = \left\{ \left(\int_{\Omega_1} |f| \right)^p + \left(\int_{\Omega - \Omega_1} |f| \right)^p \right\}^{1/p}.$$

If $\{f_n\}$ is clm-null we have that $\int_{\Omega_1} |f_n| \to 0$. From Corollary 2 we can easily deduce that $(clm)CS(X) = 2$ if $X = (L_1(\Omega), |||\cdot|||)$. However, we shall prove

that $\kappa_{clm}(X) \leq 2^{1/p}$. Indeed, consider the functions $f_n = \chi_{\Omega_n}/\mu(\Omega_n)$, $n \geq 2$ and $f = \chi_{\Omega_1}/\mu(\Omega_1)$. Then $\{f_n\}$ is a *clm*-null sequence, $|||f_n - f||| = 2^{1/p}$ and $|||f_n||| = 1$ for any $n \geq 2$, and $|||f||| = 1$. If $b > 2^{1/p}$ and $r = 2^{1/p}/b$ we have $|||f||| > r$, $|||f_n - f||| = br$ but $|||f_n||| > r$. Since X satisfies the *clm*-uniform Opial condition, we deduce from Remark 3 that $\kappa_{clm}(X) \leq 2^{1/p}$. In fact, it can be proved that $\kappa_{clm}(X) = 2^{1/p}$.

The main limitation of Theorem 6 is that we only obtain stability for norms which are τ-slsc. However, the following example shows that this condition is not preserved by isomorphisms.

Example 6. Consider in R^2 the vectors $e_1 = (1, 0)$ and $e_2 = (-a, \sqrt{1-a^2})$ where $a \in (0, 1)$. Define $\|(x_1, x_2)\|_e = \|x_1 e_1 + x_2 e_2\|_2$, where $\|\cdot\|_2$ denotes the euclidean norm.

In ℓ_1 we define the norm

$$|x| = \left\| \left(|x(1)|, \sum_{n=2}^{+\infty} |x(n)| \right) \right\|_e .$$

It is clear that this norm is equivalent to the usual norm in ℓ_1. In fact, the Banach-Mazur distance between the two spaces is close to $\sqrt{2}$ if a is small enough. Consider the sequence $x_n = e_1 + ae_n$, where $\{e_n\}$ denotes the basis sequence in ℓ_1. Then $x_n \to e_1$ in the $\sigma(\ell_1, c_0)$ topology, but $|x_n| = \|(1, a)\|_e = \sqrt{1-a^2} < 1 = |e_1|$. From this, the norm $|\cdot|$ is not a $\sigma(\ell_1, c_0)$-slsc function.

In [12, Theorem 3.2], it is proved that a k- uniformly Lipschitzian asymptotically regular mapping $T : C \to C$ has a fixed point if C is a weakly compact convex set and $k < \sqrt{WCS(X)}$. Since the arguments in the proof equally work for $\tau CS(X)$ if τ is a topology weaker than the norm topology and the norm is τ-slsc, we can state the following theorem.

Theorem 7. *Let X be a Banach space and τ a linear topology on X such that τ is weaker than the norm topology and the norm is τ-slsc. If Y is an isomorphic Banach space and $d(X, Y) < \sqrt{\tau CS(X)}$, then Y has the τ-FPP.*

Example 5 and the following Example 7 show the different scopes of Theorems 5, 6 and 7.

Example 7. Consider the Banach space $X = \mathbb{R} \times L_1[0, 1]$ equipped with the norm

$$\|(\lambda, f)\|_X = (|\lambda|^p + \|f\|_1^p)^{1/p} \qquad \text{with } p > 2$$

where $\|\cdot\|_1$ denotes the usual norm in $L_1[0, 1]$. Let τ be the product topology of the usual topology on $\mathbb{R}$ and the topology of the convergence in measure on $L_1[0, 1]$. It is easy to check that $\tau CS(X) = 2$ and $\kappa_\tau(X) = 2^{1/p}$.

Having chosen $a \in (0, 1)$, we define the Banach space $Y = (X, \|\cdot\|_Y)$ where $\|\cdot\|_Y$ is the following equivalent norm on X:

$$\|(\lambda, f)\|_Y = (|\lambda|^p + |||f|||^p)^{1/p}$$

with $|||f||| = \|(\int_0^{1/2}|f|, \int_{1/2}^1 |f|)\|_m$ and $\|(\cdot,\cdot)\|_m$ is the norm on $\mathbb{R}^2$ whose unit ball is the balanced convex hull of the set $\{(0,1),(1+a,a)\}$. Trivially, the norm $\|\cdot\|_X$ is τ-slsc, however, we shall check that $\|\cdot\|_Y$ is not τ-slsc. For that, we consider the sequence $\{(0,f_n)\} \subset Y$ with $f_n = 2\chi_{[0,1/2]} + an\chi_{[\frac{n-1}{n},1]}$. Then $(0,f_n) \to_n (0, 2\chi_{[0,\frac{1}{2}]})$ in the topology τ but

$$\|(0,f_n)\|_Y = \|(1,a)\|_m = \frac{a^2+1}{1+a} < 1 = \|(1,0)\|_m = \|(0,2\chi_{[0,1/2]})\|_Y.$$

It is not difficult to prove that $d(X,Y) = 1+2a$. Then, when a tends to 0, $d(X,Y)$ tends to 1. Therefore, there does not exist, in general, an upper bound of the Banach-Mazur distance between two isomorphic Banach spaces such that the norm on Y is τ-slsc if the norm on X is τ-slsc and $d(X,Y)$ is less than that bound.

Since $\|\cdot\|_Y$ is not τ-slsc we cannot apply Theorem 6 to assure the τ-FPP in Y. However, we can deduce from Theorem 7 that Y has the τ-FPP if $1+2a < \sqrt{2}$. It must be noted that we cannot apply Theorem 6 either if $2^{1/p} \leq 1+2a$.

For topologies such that the functions of τ-null type are τ-slsc we can obtain better stability results using a coefficient inspired in the definition of $M(X)$ in [10].

Definition 8. Let X be a Banach space and τ a topology on X. For any nonnegative number a we define the coefficient

$$R_\tau(a,X) = \sup\{\liminf \|x_n + x\|\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all τ-null sequences in the unit ball such that $\lim_{n,m;n\neq m} \|x_n - x_m\| \leq 1$. We define the coefficient $M_\tau(X)$ as

$$\sup\left\{\frac{1+a}{R_\tau(a,X)} : a \geq 0\right\}.$$

Theorem 8. *Let X be a Banach space and τ a linear topology such that τ-sequentially compact sets are τ-compact. Let $|\cdot|$ be an equivalent norm on X such that*

$$|x| \leq \|x\| \leq d|x|$$

for every $x \in X$, and let $Y = (X, |\cdot|)$. If $d < M_\tau(X)$ and the functions of τ-null type are τ-slsc in Y, then Y has the τ-FPP.

Proof. We shall use Proposition 1 to give an elemental proof. Since $d < M_\tau(X)$ there exists $a \geq 0$ such that $dR_\tau(a,X) < 1+a$. Since $R_\tau(\cdot,X)$ is a continuous function we can assume that $a > 0$. Let $t = (1+a)^{-1} \in (0,1)$ and choose a positive $\varepsilon < 1 - (1+a)^{-1}dR_\tau(a,X)$. If Y does not have

the τ-FPP, a standard argument proves that there exist T and C as in Proposition 1. Consider the sequence there defined. Taking a subsequence, if necessary, we can assume that $\lim_{n,m\,,\,n\neq m}\|z_n - z_m\|$ exists and $\{z_n\}$ is τ-convergent, say to z. Furthermore, $\lim_{n,m\,,\,n\neq m}\|z_n - z_m\| \leq dt$ because $\limsup_n \limsup_m |z_n - z_m| \leq t$ and $\|z\| \leq d(1-t)$ because $|z| \leq 1-t$. Let $0 < \eta < (1-\varepsilon)/R_\tau(a,X) - dt$. Then we can assume that the sequence $\{(z_n - z)/(dt+\eta)\}$ belongs to the unit ball in Y. Thus, we obtain the contradiction

$$\frac{1-\varepsilon}{dt+\eta} < \frac{|z_n|}{dt+\eta} \leq \left|\frac{z_n - z}{dt+\eta} + \frac{z}{dt+\eta}\right| \leq R_\tau(a,X). \blacksquare$$

Theorem 8 is of special interest when every τ-convergent sequence is weakly convergent, because in this case the functions of τ-null type are τ-slsc and this condition is preserved under isomorphisms. That happens for $L_p(\Omega)$, $p > 1$, if τ is the *clm* topology (see, for instance, [21], page 207). If $p > 1$ we know that $L_p(\Omega)$ has the *clm*-FPP, because every convex, norm-bounded, *clm*-compact subset of $L_p(\Omega)$ is weakly compact and it is well known that $L_p(\Omega)$ has the w-FPP if $p > 1$. In fact, it is known that $WCS(L_p(\Omega)) \geq \min\{2^{1/p}, 2^{1-1/p}\}$. (The equality holds if either $p > 2$ or μ is not purely atomic [2, Theorem VI.6.3]). However, better stability bounds can be obtained when the *clm*-FPP is considered.

Theorem 9. *Let X be a Banach space. Assume that τ is a linear topology such that τ-sequentially compact sets are τ-compact. Assume that τ is sequentially stronger than the weak topology and X satisfies $L(\tau)$. If Y is an isomorphic Banach space and*

$$d(X,Y) < \sup\left\{\frac{1+a}{\delta\left(\frac{1}{\delta(1,1)}, a\right)} : a \geq 0\right\}$$

then Y has the τ-FPP. In particular, Y has the τ-FPP if $d(X,Y) < (1+\sqrt{5})/2$.

Proof. We can assume that there exists a norm-bounded τ-null sequence which is not norm-convergent, because otherwise it is clear that Y has the τ-FPP. Using property $L(\tau)$ we can easily deduce that

$$R_\tau(a,X) = \delta\left(\frac{1}{\tau CS(X)}, a\right) = \delta\left(\frac{1}{\delta(1,1)}, a\right).$$

Since δ is an homogeneous function, if we choose $a = 1/\delta(1,1)$ we obtain $M_\tau(X) \geq 1 + (1/\delta(1,1))$. For $a = 0$ we also know that $M_\tau(X) \geq \delta(1,1)$. Noting that

$$\max\left\{1 + \frac{1}{\delta(1,1)}, \delta(1,1)\right\} \geq \frac{1+\sqrt{5}}{2}$$

we obtain the result. $\blacksquare$

From Theorem 9, we can obtain stability bounds in $L_p(\Omega)$ for the *clm*-FPP, which are identical to those obtained in [10] for the FPP in ℓ_p-spaces.

Corollary 3. *Let (Ω, Σ, μ) be a positive σ-finite measure space and $|\cdot|$ a norm on $L_p(\Omega), p > 1$ which satisfies*

$$|f| \leq \|f\|_p \leq d|f|$$

for every $f \in L_p(\Omega)$. Let $X = (L_p(\Omega), |\cdot|)$. If

$$d < \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}},$$

then X has the clm-FPP.

Proof. A standard argument proves that

$$\sup\left\{\frac{(1+a)}{(a^p + 1/2)^{1/p}} : a \geq 0\right\} = \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}. \blacksquare$$

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