UNIFORM ASYMPTOTIC NORMAL STRUCTURE, THE UNIFORM SEMI-OPIAL PROPERTY, AND FIXED POINTS OF ASYMPTOTICALLY REGULAR UNIFORMLY LIPSCHITZIAN SEMIGROUPS. PART II

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ABSTRACT. In this part of our paper we present several new theorems concerning the existence of common fixed points of asymptotically regular uniformly lipschitzian semigroups.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. A mapping $T : C \to C$ is said to be uniformly k-lipschitzian if for each $x, y \in C$ and every natural number n, $\|T^n x - T^n y\| \leq k \|x - y\|$. If $k = 1$, then the mapping T is called nonexpansive. These definitions can also be introduced in metric spaces. The class of uniformly lipschitzian mappings on C is completely characterized as the class of those mappings on C which are nonexpansive with respect to some metric on C which is equivalent to the norm [16]. In this part of our paper we use the new geometric coefficients introduced in its first part to study the existence of (common) fixed points for this class of mappings.

2. BASIC NOTATIONS AND FACTS

Throughout this paper we will use the notations from the first part of our paper [6]. However, before we recall several known fixed point theorems we need a few additional notations.

Let $(X, \|\cdot\|)$ denote a Banach space. For $x \in X$ and a bounded sequence $\{x_n\}$ the asymptotic radius of $\{x_n\}$ at x is the number $r_a(x, \{x_n\})$ $= \limsup_n \|x - x_n\|$. Now for a nonempty closed convex subset C of X the asymptotic radius of $\{x_n\}$ in C is the number $r(C, \{x_n\}) = \inf \{r_a(x, \{x_n\}) : x \in C\}$. The asymptotic center of $\{x_n\}$ in C [13] is the set $Ac(C, \{x_n\}) = \{x \in C : r_a(x, \{x_n\}) = r(C, \{x_n\})\}$. For more details see [1], [16] and [17].

Let $\overline{B}(0, 1)$ be the closed unit ball in X. The modulus of convexity of X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2}; x, y \in \overline{B}(0, 1), \|x - y\| \geq \epsilon \right\}$$

[8]. The characteristic of convexity of X is the number $\epsilon_0(X) = \sup \{\epsilon : \delta(\epsilon) = 0\}$ [16]. When $\epsilon_0(X) = 0$ X is called a uniformly convex space [8].

The Lifshitz characteristic $\kappa(M)$ of a metric space (M, ρ) is the supremum of all positive real numbers b such that there exists $a > 1$ such that for each $x, y \in M$ and $r > 0$ with $\rho(x, y) > r$ there exists $z \in M$ satisfying $\overline{B}(x, br) \cap \overline{B}(y, ar) \subset \overline{B}(z, r)$ [23]. It is obvious that $\kappa(M) \geq 1$. In a Banach space $(X, \|\cdot\|)$ we denote by $\kappa_0(X)$ the infimum of the numbers $\kappa(C)$ where C is a closed, convex, bounded and nonempty subset of X. It is known [2], [16] that

(1) $$\frac{1}{1 - \delta(1)} \leq \kappa_0(X) \leq N(X)$$

and $\epsilon_0(X) < 1$ if and only $\kappa_0(X) > 1$ [12]. Therefore $\kappa_0(X) \leq \sqrt{2}$ [2].

Unfortunately, we know the exact value of $\kappa_0(X)$ or some lower bounds for $\kappa_0(X)$ in special spaces only [2]. Therefore it is convenient to introduce a new coefficient which plays a role similar to the one of $\kappa_0(X)$. In [10] T. Domínguez Benavides and H.K. Xu introduced such a new constant $\kappa_\omega(X)$ in Banach spaces. We give here a slightly different definition of $\kappa_\omega(X)$ from the one given in [10]. Namely, if $(X, \|\cdot\|)$ is a Banach space and C is a nonempty bounded closed convex subset of X, then

(a) a number $b \geq 0$ has property (P_ω) with respect to C if there exists some $a > 1$ such that for all $x, y \in C$ and $r > 0$ with $\|x - y\| \geq r$ and each weakly convergent sequence $\{z_n\}$ with elements in C such that $\limsup_n \|x - z_n\| \leq ar$ and $\limsup_n \|y - z_n\| \leq br$, there exists $z \in C$ such that $\liminf_n \|z - z_n\| \leq r$;

(b) $\kappa_\omega(C) = \sup \{b > 0 : b \text{ has property } (P_\omega) \text{ with respect to } C\}$;

(c) $\kappa_\omega(X) = \inf \{\kappa_\omega(C) : C \text{ is a nonempty bounded closed convex subset of } X\}$.

It is clear that $\kappa_\omega(C) \geq \kappa(C)$ for all nonempty bounded closed convex subsets $C \subset X$. Next let us observe that

$$\kappa_\omega(X) = \inf \{\kappa_\omega(C) : C \text{ is a convex weakly compact subset of } X\}.$$

Hence we get that $\kappa_\omega(X) \leq WCS(X)$ [2].

Let (M, ρ) be a metric space, where M is not a singleton and $T : M \to M$. Then we will use the symbol $|T|$ to denote the exact Lipschitz constant of T, i.e.,

$$|T| = \sup \left\{ \frac{\rho(Tx, Ty)}{\rho(x, y)} : x, y \in M, x \neq y \right\}.$$

Let X be a Banach space, C a nonempty bounded closed convex subset of X, G an unbounded subset of $[0, \infty)$ such that

(2) $t + h \in G$ for all $t, h \in G$,

(3) $t - h \in G$ for all $t, h \in G$ with $t \geq h$,

and $\mathbf{T} = \{ \mathfrak{T}_{\mathfrak{t}} : \mathfrak{t} \in \mathfrak{G} \}$ a family of self-mappings on C. $\mathbf{T}$ is called a semigroup of mappings on C if

  (i) $T_{s+t}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$,
 (ii) for each $x \in C$, the mapping $t \to T_t x$ from G into C is continuous when G has the relative topology of $[0, \infty)$.

Let us observe that in the particular case $G = \mathbb{N}$ we get the semigroups of iterates $\mathbf{T} = \{ \mathfrak{T}_{\mathfrak{t}} : \mathfrak{t} \in \mathfrak{G} \} = \{ \mathfrak{T}_{\mathfrak{1}}^{\mathfrak{n}} : \mathfrak{n} \in \mathbb{N} \}$.

If $\mathbf{T}$ satisfies i. - ii. and in addition there exists $k > 0$ such that

$$\|T_t x - T_t y\| \leq k \|x - y\|$$

for all $x, y \in C$ and t in G, then we say that $\mathbf{T}$ is a uniformly lipschitzian (k-lipschitzian) semigroup of mappings on C.

If $\mathbf{T}$ satisfies i. - ii. and for each $x \in C$, $h \in G$,

$$\lim_{t \to \infty} \|T_{t+h}x - T_t x\| = 0,$$

then $\mathbf{T}$ is said to be asymptotically regular. The concept of asymptotic regularity is due to F.E. Browder and W.V. Petryshyn [5].

Let us observe that the notions of the asymptotic radius and the asymptotic center can be formulated in an obvious way for $\{x_t\}_{t \in G}$, where G satisfies (2) and (3).

The first positive result about fixed points of uniformly lipschitzian mappings is due to K. Goebel and W.A. Kirk.

Theorem 2.1. [14] *Let X be a Banach space with $\epsilon_0(X) < 1$ and let C be a nonempty bounded closed convex subset of X. Suppose $T : C \to C$ is uniformly lipschitzian with a constant $k < \gamma$, where $\gamma > 1$ satisfies the equation*

(4) $$\gamma \left(1 - \delta \left(\frac{1}{\gamma} \right) \right) = 1.$$

Then T has a fixed point in C.

By (1) it is obvious that the constant γ from (4) is strictly less than $\kappa_0(X)$ [23].

In [30] K.-K. Tan and H.-K. Xu proved the following theorem which is formulated in the same spirit as the previous one.

Theorem 2.2. [30] *Let X be a uniformly convex Banach space, C a nonempty bounded closed convex subset of X and $T : C \to C$ a k-lipschitzian mapping with $k < \gamma_1$, where $\gamma_1 > 1$ is the solution of the equation*

$$(5) \qquad \gamma_1 \left[1 - \delta \left(\frac{N(X)}{\gamma_1^2} \right) \right] = 1.$$

Then T has a fixed point in C.

The constant γ_1 given by formula (5) is always bigger than the constant γ defined by (4).

In [23] Lifshitz extended the result of Goebel and Kirk in the following way:

Theorem 2.3. [23] *Let (M, ρ) be a complete metric space and $T : M \to M$ a uniformly lipschitzian mapping with constant $k < \kappa(M)$. If there exists $x_0 \in M$ such that the orbit $\{T^n x_0\}$ is bounded, then T has a fixed point in M.*

Here we must note that in the case of a Banach space we do not know in general how to compare the constant γ_1 given by (5) to the constant $\kappa(X)$ or $\kappa_0(X)$.

In each Banach space we have $\kappa_0(X) \leq N(X)$ (see (1)) but in particular cases we can have $\kappa_0(X) < \sqrt{N(X)}$ [7]. Therefore the following result is important.

Theorem 2.4. [7] *Let X be a Banach space X with uniform normal structure and C a nonempty bounded closed convex subset of X. If $T : C \to C$ is a uniformly k-lipschitzian mapping with $k < \sqrt{N(X)}$, then T has a fixed point.*

Now we recall a fixed point theorem in which the coefficients $\kappa_0(X)$ and $N(X)$ appear simultaneously.

Theorem 2.5. [9] *Let X be a Banach space, C a nonempty bounded closed convex subset of X and $T : C \to C$. If*

$$\liminf_n |T^n| < \frac{1 + \sqrt{1 + 4 \cdot N(X) \cdot (\kappa_0(X) - 1)}}{2},$$

then T has a fixed point.

For a discussion of this theorem see [9].

For asymptotically regular mappings we have the following results.

Theorem 2.6. [20] *Let X be an infinite dimensional uniformly convex Banach space, C a nonempty bounded closed convex separable subset of X and $T : C \to C$ an asymptotically regular mapping with $\liminf_n |T^n| < \gamma_2$, where $\gamma_2 > 1$ is the solution of the equation*

$$(6) \qquad \gamma_2 \left[1 - \delta \left(\frac{WCS(X)}{\gamma_2^2} \right) \right] = 1.$$

Then T has a fixed point in C.

Theorem 2.7. [2, 10] *Suppose X is a Banach space. Suppose also C is a convex weakly compact subset of X and $T : C \to C$ is asymptotically regular. If $\liminf_n |T^n| < \kappa_\omega (C)$, then T has a fixed point.*

Theorem 2.8. [2, 10] *Suppose X is a Banach space such that $WCS (X) > 1$, C is a convex weakly compact subset of X and $T : C \to C$ is asymptotically regular. If $\liminf_n |T^n| < \sqrt{WCS (X)}$, then T has a fixed point.*

Theorem 2.9. [9] *Let X be a reflexive Banach space, C a nonempty bounded closed convex subset of X and $T : C \to C$ an asymptotically regular mapping. If*

$$\liminf_n |T^n| < \frac{1 + \sqrt{1 + 4 \cdot WCS (X) \cdot (\kappa_\omega (X) - 1)}}{2},$$

then T has a fixed point.

For a discussion of the connections among the above results see [9, 10].

In [11], [20], [21], [30], [31] and [32] some of the above results were reformulated in terms of semigroups. It is also worthwhile to see [4], [15], [18], [19], [22], [24], [25], [26], [27], [28], [29], [33], [34].

3. Existence of fixed points of lipschitzian semigroups of mappings

We begin with a generalization of Theorem 2.2.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space with $\epsilon_0 < 1$, C a nonempty bounded closed convex subset of X and $\mathbf{T} = \{\mathfrak{T}_t : t \in \mathfrak{G}\}$ a uniformly lipschitzian semigroup of mappings on C with $\sup_t |T_t| = k < \overline{\gamma}$, where $\overline{\gamma} > 1$ is the solution of the equation*

$$(7) \qquad \overline{\gamma} \left[1 - \delta \left(\frac{N (X)}{\overline{\gamma}} \right) \right] = 1.$$

Then there exists a common fixed point of $\mathbf{T}$.

PROOF. Without loss of generality we can assume that $(X, \|\cdot\|)$ is uniformly convex and $k > 1$. If $0 < \epsilon_0 < 1$ we need only make minor changes in the following proof. Let us denote $N (X)$ by N. For each $x \in C$ let z be the unique element of $Ac (C, \{T_t x\})$. Let us assume that

$$(8) \qquad \frac{d (x)}{N} = \frac{diam_a \{T_t x\}}{N} \geq r = r (C, \{T_t x\}) > 0$$

and $d (z) = diam_a \{T_t z\} > 0$. Then we can find sequences $\{s_n\}$ and $\{t_n\}$ such that

$$\lim_n s_n = \lim_n t_n = \infty$$

and

$$\lim_n \|T_{t_n} z - T_{s_n} z\| = diam_a \{T_t z\} = d (z).$$

Next we have

$$r_a (T_{s_n} z, \{T_t x\}) \leq k \cdot r$$

and
$$r_a\left(T_{t_n}z, \{T_t x\}\right) \le k \cdot r.$$

Hence
$$r \le \liminf_n r_a\left(\frac{T_{s_n}z + T_{t_n}z}{2}, \{T_t x\}\right) \le \left[1 - \delta\left(\frac{d(z)}{k \cdot r}\right)\right] \cdot k \cdot r,$$

$$1 \le \left[1 - \delta\left(\frac{d(z)}{k \cdot r}\right)\right] \cdot k \le \left[1 - \delta\left(\frac{N \cdot d(z)}{d(x) \cdot k}\right)\right] \cdot k,$$

and finally

$$(9) \qquad d(z) \le \left[\frac{k}{N} \cdot \delta^{-1}\left(1 - \frac{1}{k}\right)\right] \cdot d(x).$$

Let us denote $\frac{k}{N} \cdot \delta^{-1}\left(1 - \frac{1}{k}\right)$ by a. Directly from our assumption we get $a < 1$. Now we define a sequence $\{x_n\}_{n=0}^\infty$ in C in the following way: x_0 is an arbitrarily chosen element of C and x_{n+1} is the unique element of $Ac\left(C, \{T_t x_n\}\right)$ for $n = 0, 1, 2, \dots$. By (9) we obtain

$$d(x_n) \le a^n \cdot d(x_0),$$

and the inequalities

$$\|x_n - x_{n+1}\| \le \|x_n - T_j x_{n-1}\| + \|T_j x_{n-1} - T_i x_n\| + \|T_i x_n - x_{n+1}\|$$

$$\le \|x_n - T_j x_{n-1}\| + k\|T_{j-i} x_{n-1} - x_n\| + \|T_i x_n - x_{n+1}\|,$$

which are valid for $i < j$, lead to

$$\|x_n - x_{n+1}\| \le (1+k)\, d(x_{n-1}) + d(x_n) \le \left[(1+k)\, a^{n-1} + a^n\right] \cdot d(x_0).$$

This, in turn, yields the conclusion that $\{x_n\}_{n=0}^\infty$ is norm Cauchy and hence strongly convergent. Let $x = \lim_n x_n$. Then by (8) for each $s \in G$ we have

$$\|x - T_s x\| \le \lim_n \|x - x_{n+1}\| + \limsup_n \limsup_t \|x_{n+1} - T_{t+s} x_n\|$$

$$+ \limsup_n \limsup_t \|T_{t+s} x_n - T_s x_{n+1}\| + \lim_n \|T_s x_{n+1} - T_s x\|$$

$$\le \lim_n (1+k)\left[\|x - x_{n+1}\| + \frac{d(x_n)}{N}\right] = 0.$$

This completes the proof. ∎

Remark 3.1. *The constant $\overline{\gamma}$ given by (7) is bigger than the constant γ defined by (5).*

For our next results we need the following simple fact.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a Banach space.*

1) For each $0 < \frac{1}{\theta} < w\text{-}SOC(X)$ and every asymptotically regular sequence $\{x_n\}$ with a weakly compact $\overline{conv}\{x_n\}$ there exists a weakly convergent to w subsequence $\{x_{n_i}\}$ such that

   (i) $r(w, \{x_{n_i}\}) \le \theta \cdot diam_a(\{x_n\})$,

   (ii) $\|w - y\| \le r_a(y, \{x_{n_i}\})$ *for every $y \in X$.*

2) For each $0 < \frac{1}{\theta} < w\text{-}AN(X)$ and every asymptotically regular sequence $\{x_n\}$ with a weakly compact $\overline{conv}\{x_n\}$ there exists a subsequence $\{x_{n_i}\}$ and a point $w \in \overline{conv}\{x_n\}$ such that

$$r(w, \{x_{n_i}\}) \leq \theta \cdot diam_a(\{x_n\}).$$

PROOF. 1) ii. It is sufficient to apply the lower semicontinuity of $\|\cdot\|$ with respect to the weak topology because we have

$$r_a(y, \{x_{n_i}\}) = \limsup_i \|y - x_{n_i}\|$$

and

$$y - x_{n_i} \rightharpoonup y - w. \ \blacksquare$$

We will use the above lemma in the proofs of the next three theorems.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space with $\epsilon_0 < 1$, C a nonempty bounded closed convex subset of X and $\mathbf{T} = \{\mathfrak{T}_\mathfrak{t} : \mathfrak{t} \in \mathfrak{G}\}$ a uniformly k-lipschitzian asymptotically regular semigroup of mappings on C with $k < \overline{\overline{\gamma}}$, where $\overline{\overline{\gamma}}$ satisfies*

$$\overline{\overline{\gamma}}^2 > \max\left(1, \frac{w\text{-}AN(X)}{2}\right)$$

and is the solution of the equation

$$(10) \qquad \overline{\overline{\gamma}}\left[1 - \delta\left(\frac{w\text{-}AN(X)}{(\overline{\overline{\gamma}})^2}\right)\right] = 1.$$

Then there exists in C a common fixed point of $\mathbf{T}$.

PROOF. Without loss of generality we can assume that $(X, \|\cdot\|)$ is uniformly convex, $k^2 > \frac{w\text{-}AN(X)}{2}$ and $k > 1$. If $0 < \epsilon_0 < 1$ we need only make minor changes in our proof. First we choose θ such that $1 < \frac{1}{\theta} < w\text{-}AN(X)$ and

$$k\left[1 - \delta\left(\frac{1}{\theta k^2}\right)\right] < 1.$$

Next we fix $t_0, h \in G$ with $h > 0$ and define $t_n = t_0 + nh$ for $n = 1, 2, \ldots$ Let us observe that for each $x \in C$ the sequence $\{T_{t_n}x\}$ is asymptotically regular and by Lemma 3.1 for each $x \in C$ there exists a weakly convergent subsequence $\{T_{t_{n_i}}x\}$ with $w \in \overline{conv}\{T_{t_n}x\}$ such that

$$(11) \qquad r_a\left(w, \left\{T_{t_{n_i}}x\right\}\right) \leq \theta \cdot diam_a(\{T_{t_n}x\}).$$

In other words, for each $x \in C$ we can find a subsequence $\left\{T_{t_{n_i}}x\right\}$ and $w(x) \in C$ which satisfy the inequality (11). Hence, if $z(x)$ is the unique element of $Ac\left(C, \left\{T_{t_{n_i}}x\right\}\right)$, then we have

(12)

$$r(x) = r_a\left(z(x), \left\{T_{t_{n_i}}x\right\}\right) \leq r_a\left(w(x), \left\{T_{t_{n_i}}x\right\}\right) \leq \theta \cdot diam_a(\{T_{t_n}x\}).$$

The asymptotic regularity of $\mathbf{T}$ leads to

$$r_a\left(T_{t_m}z\left(x\right),\left\{T_{t_{n_i}}x\right\}\right) \leq r_a\left(T_{t_m}z\left(x\right),\left\{T_{t_{n_i}+t_m}x\right\}\right)$$
$$+\limsup_i\left\|T_{t_{n_i}+t_m}x - T_{t_{n_i}}x\right\|$$

$$(13) \qquad = r_a\left(T_{t_m}z\left(x\right),\left\{T_{t_{n_i}+t_m}x\right\}\right) \leq k\cdot r_a\left(z\left(x\right),\left\{T_{t_{n_i}}x\right\}\right)$$

for every m. Now we set

$$R\left(x\right) = r_a\left(x,\left\{T_{t_n}x\right\}\right) = \limsup_n\left\|x - T_{t_n}x\right\|.$$

We observe that by the asymptotic regularity of $\mathbf{T}$ and since

$$t_l = t_0 + lh \qquad \text{for } l = 1,2,...,$$

we have

$$diam_a\left(\left\{T_{t_n}x\right\}\right) = \lim_{\tilde{m}}\sup_{\tilde{m}<m<n}\left\|T_{t_m}x - T_{t_n}x\right\|$$

$$= \lim_{\tilde{m}}\sup_{\tilde{m}<m}\left(\max_{1\leq n\leq n_0}\left\|T_{t_m}x - T_{t_{m+n}}x\right\|,\sup_{l>0}\left\|T_{t_m}x - T_{t_{m+n_0+l}}x\right\|\right)$$

$$= \lim_{\tilde{m}}\left[\sup_{\tilde{m}<m}\left(\sup_{l>0}\left\|T_{t_m}x - T_{t_{m+n_0+l}}x\right\|\right)\right]$$

$$\leq \limsup_{\tilde{m}}\left(\sup_{\tilde{m}<m}\left\|T_{t_m}x - T_{mh}x\right\|\right)$$

$$+ \limsup_{\tilde{m}}\left[\sup_{\tilde{m}<m}\left(\sup_{l>0}\left\|T_{mh}x - T_{t_{m+n_0+l}}x\right\|\right)\right]$$

$$\leq k\cdot\sup_{l>0}\left\|x - T_{t_{n_0+l}}x\right\| = k\cdot\sup_{n>n_0}\left\|x - T_{t_n}x\right\|$$

for $n_0 \geq 1$. Hence

$$(14) \qquad diam_a\left(\left\{T_{t_n}x\right\}\right) \leq k\cdot\limsup_n\left\|x - T_{t_n}x\right\| = k\cdot R\left(x\right).$$

We construct the sequence $\{x_m\}$ in the following way: $x_0 \in C$ is arbitrary and

$$x_{m+1} = z\left(x_m\right)$$

for $m = 0,1,...$. By (12) and (14) we get

$$(15) \qquad r_m = r\left(x_m\right) \leq \theta\cdot k\cdot R\left(x_m\right).$$

Now we have to consider two cases. If for some m_0 we have $r_{m_0} = 0$, then we get

$$R\left(z\left(x_{m_0}\right)\right) = \limsup_n\left\|z\left(x_{m_0}\right) - T_{t_n}z\left(x_{m_0}\right)\right\|$$

$$\leq \limsup_n\limsup_i\left[\left\|z\left(x_{m_0}\right) - T_{t_{n_i}}x_{m_0}\right\|\right.$$

$$\left.+ \left\|T_{t_{n_i}}x_{m_0} - T_{t_n+t_{n_i}}x_{m_0}\right\| + \left\|T_{t_n+t_{n_i}}x_{m_0} - T_{t_n}z\left(x_{m_0}\right)\right\|\right] = 0.$$

In the second case we have $r_m > 0$ for all $m \geq 0$. For each m we first choose an arbitrary $0 < \epsilon < R(x_{m+1})$ and then j such that

$$\left\| T_{t_j} x_{m+1} - x_{m+1} \right\| \geq R(x_{m+1}) - \epsilon.$$

Now we find i_0 such that for all $i \geq i_0$ we have

$$\left\| T_{t_{n_i}} x_m - x_{m+1} \right\| \leq r_m + \frac{\epsilon}{2}$$

(the subsequence $\{t_{n_i}\}$ depends on x_m here) and (see (13))

$$\left\| T_{t_{n_i}} x_m - T_{t_j} x_{m+1} \right\| \leq k \cdot (r_m + \epsilon).$$

It follows that

$$\left\| T_{t_{n_i}} x_m - \frac{1}{2} \left(x_{m+1} + T_{t_j} x_{m+1} \right) \right\|$$

$$\leq \quad k \cdot (r_m + \epsilon) \cdot \left[1 - \delta \left(\frac{R(x_{m+1}) - \epsilon.}{k \cdot (r_m + \epsilon)} \right) \right].$$

Letting i tend to infinity we obtain

$$r_m \leq k \cdot (r_m + \epsilon) \cdot \left[1 - \delta \left(\frac{R(x_{m+1}) - \epsilon.}{k \cdot (r_m + \epsilon)} \right) \right].$$

Taking now ϵ to 0 we get

$$r_m \leq k \cdot r_m \cdot \left[1 - \delta \left(\frac{R(x_{m+1})}{k \cdot r_m} \right) \right]$$

which after applying the inequality (15) implies that

$$R(x_{m+1}) \leq \left[\theta \cdot k^2 \cdot \delta^{-1} \left(1 - \frac{1}{k} \right) \right] \cdot R(x_m).$$

Let us observe that

$$0 < \alpha = \theta \cdot k^2 \cdot \delta^{-1} \left(1 - \frac{1}{k} \right) < 1$$

and therefore

(16) $$R(x_m) \leq \alpha^m \cdot R(x_0)$$

for $m = 1, 2, \ldots$. Hence by (15) and (16) we deduce that

$$\|x_{m+1} - x_m\| \leq \limsup_i \left\| x_{m+1} - T_{t_{n_i}} x_m \right\| + \limsup_i \left\| T_{t_{n_i}} x_m - x_m \right\|$$

$$\leq r_a \left(x_{m+1}, \left\{ T_{t_{n_i}} x_m \right\} \right) + R(x_m) = r_m + R(x_m)$$

$$\leq (\theta \cdot k + 1) \cdot R(x_m) \leq (\theta \cdot k + 1) \cdot \alpha^m \cdot R(x_0)$$

(the subsequence $\{t_{n_i}\}$ depends on x_m here) and therefore the sequence $\{x_n\}$ is strongly convergent to some $x \in C$. By the inequality

$$|R(x) - R(y)| \leq (1 + k) \cdot \|x - y\|,$$

which is valid for all $x, y \in C$, we have $R(x) = 0$.

Thus in both cases we can find $y \in C$ with $R(y) = 0$. This means that

$$y = \lim_n T_{t_n} y$$

and by the asymptotic regularity of $\mathbf{T}$ we have

$$T_t y = \lim_n T_{t+t_n} y = \lim_n T_{t_n} y = y$$

for each $t \in G$. This completes the proof. ∎

Theorem 3.3. *Let C be a convex weakly compact subset of a Banach space X with $w\text{-}SOC(X) > 1$. Then every asymptotically regular uniformly k-lipschitzian semigroup $\mathbf{T} = \{\mathfrak{T}_\mathfrak{t} : \mathfrak{t} \in \mathfrak{G}\}$ of mappings on C with $k < [w\text{-}SOC(X)]^{\frac{1}{2}}$ has a common fixed point.*

PROOF. Let $k \geq 1$ and let us fix θ such that

$$k^2 < \frac{1}{\theta} < w\text{-}SOC(X).$$

We choose $t_0, h \in G$ with $h > 0$ and consider the sequence $\{t_n\} = \{t_0 + nh\}$. For each $x \in C$ we define $R(x)$ by

$$R(x) = r_a(x, \{T_{t_n} x\}) = \limsup_n \|x - T_{t_n} x\|.$$

Let us observe that for each $x \in C$ the sequence $\{T_{t_n} x\}$ is asymptotically regular and by Lemma 3.1 for each $x \in C$ there is a weakly convergent to w subsequence $\left\{T_{t_{n_i}} x\right\}$ such that

(i) $r_a\left(w, \left\{T_{t_{n_i}} x\right\}\right) \leq \theta \cdot diam_a\left(\{T_{t_n} x\}\right)$,

(ii) $\|y - w\| \leq r_a\left(y, \left\{T_{t_{n_i}} x\right\}\right)$ for every $y \in X$.

By i., ii., (14) and the asymptotic regularity of $\mathbf{T}$ we obtain

$$r_a\left(T_{t_m} w, \left\{T_{t_{n_i}} x\right\}\right) \leq r_a\left(T_{t_m} w, \left\{T_{t_{n_i}+t_m} x\right\}\right) + \limsup_i \left\|T_{t_{n_i}+t_m} x - T_{t_{n_i}} x\right\|$$

$$\leq k \cdot r_a\left(w, \left\{T_{t_{n_i}} x\right\}\right) \leq k \cdot \theta \cdot diam_a\left(\{T_{t_n} x\}\right) \leq \theta \cdot k^2 \cdot R(x),$$

$$\|T_{t_m} w - w\| \leq r_a\left(T_{t_m} w, \left\{T_{t_{n_i}} x\right\}\right) \leq \theta \cdot k^2 \cdot R(x)$$

for each m, and finally

$$R(w) \leq \theta \cdot k^2 \cdot R(x) = \alpha \cdot R(x),$$

where $0 \leq \alpha = \theta \cdot k^2 < 1$. Consequently, for each $x \in C$ there is $w(x)$ such that

$$T_{t_{n_i}} x \rightharpoonup w(x),$$

$$r_a\left(w(x), \left\{T_{t_{n_i}} x\right\}\right) \leq \theta \cdot k \cdot R(x)$$

and

$$R(w) \leq \alpha \cdot R(x).$$

This allows us to construct a sequence $\{x_m\}$ which is convergent to a fixed point of $\mathbf{T}$. We simply choose the first element x_1 arbitrarily and next we set

$$x_m = w(x_{m-1})$$

for $m = 2, 3, \dots$. Now it is sufficient to observe that

$$R(x_m) \leq \alpha^{m-1} \cdot R(x_1)$$

and repeat the arguments from the end of the proof of Theorem 3.2 to finish the present proof. ∎

In the next theorem we employ $\kappa_\omega(X)$ and $w\text{-}SOC(X)$.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space with $w\text{-}SOC(X) > 1$, C a convex weakly compact subset of X and $\mathfrak{T} = \{\mathfrak{T}_\mathfrak{t} : \mathfrak{t} \in \mathfrak{G}\}$ an asymptotically regular uniformly k-lipschitzian semigroup of mappings on C. If*

$$\text{(17)} \qquad k < \frac{1 + \sqrt{1 + 4 \cdot [w\text{-}SOC(X)] \cdot (\kappa_\omega(X) - 1)}}{2},$$

then $\mathbf{T}$ has a common fixed point.

PROOF. The proof is based on ideas presented in [9]. Let us denote $S = w\text{-}SOC(X)$ and $\kappa = \kappa_\omega(X)$. Without loss of generality we may assume that $S < +\infty$ (see Theorem 3.2 in [6]) and $k > 1$. The inequality (17) and $k > 1$ imply $\kappa > 1$. Next we observe that

$$\frac{1 + \sqrt{1 + 4 \cdot S \cdot (\kappa - 1)}}{2} \leq S < +\infty,$$

because (see Section 2)

$$\kappa \leq WCS(X) \leq S < +\infty.$$

The inequality

$$k < \frac{1 + \sqrt{1 + 4 \cdot S \cdot (\kappa - 1)}}{2}$$

implies the inequality

$$\frac{k}{S} < \frac{\kappa - 1}{k - 1}.$$

Directly from the definition of κ we find $a > 1$ and $1 < b < \kappa$ such that

$$\text{(18)} \qquad \frac{k}{S} < \frac{\frac{b}{a} - 1}{k - 1},$$

and for all $x, y \in C$ and $r > 0$ with $\|x - y\| \geq r$ and each weakly convergent sequence $\{z_n\}$ in C for which $\limsup_n \|x - z_n\| \leq a \cdot r$ and $\limsup_n \|y - z_n\| \leq b \cdot r$ there exists some $z \in C$ such that $\liminf_n \|z - z_n\| \leq r$. Next we choose $\epsilon > 0$ such that

$$\frac{1 + 2\epsilon}{a} = \alpha < 1.$$

Similarly as in the proofs of the previous theorems we consider the sequence $\{t_n\} = \{t_0 + nh\}$, where $t_0, h \in G$ and $h > 0$. For $x \in C$ we define $\overline{R}(x)$ as follows

$$\overline{R}(x) = \inf \left\{ r > 0 : \exists_{y \in C} \liminf_n \|x - T_{t_n} y\| \leq r \right\}.$$

First we will show that $\overline{R}(\widetilde{x}) = 0$ for some $\widetilde{x} \in C$. To this end we take an arbitrary $x \in C$. Assume that $\overline{R}(x) > 0$. Then we can find $y \in C$ with

$$\liminf_n \|x - T_{t_n} y\| < \overline{R}(x) \cdot (1 + \epsilon).$$

There are two possibilities; either

$$(19) \qquad \sup_n \|x - T_{t_n}(x)\| \leq \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{k \cdot a}$$

or

$$(20) \qquad \left\| x - T_{t_j}(x) \right\| > \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{k \cdot a}$$

for some j.

Let us take a look at the first case. By (14) and (19) we get

$$diam_a\left(\{T_{t_n}x\}\right) \leq k \cdot \limsup_n \|x - T_{t_n}(x)\| \leq \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{a}$$

and after applying the definition of $w\text{-}SOC(X)$ we obtain

$$S \cdot \inf \left\{ r_a\left(w, \left\{T_{t_{n_i}}x\right\}\right) : \left\{T_{t_{n_i}}x\right\} \text{ is weakly convergent to } w \right\}$$
$$\leq \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{a}.$$

This implies that

$$\inf \left\{ r_a\left(w, \left\{T_{t_{n_i}}x\right\}\right) : \left\{T_{t_{n_i}}x\right\} \text{ is weakly convergent to } w \right\} \leq \frac{\overline{R}(x) \cdot (1 + \epsilon)}{a}.$$

Hence there exists a subsequence $\left\{T_{t_{n_i}}x\right\}$ which weakly converges to $w \in C$ such that

$$(21) \qquad \lim_i \left\| w - T_{t_{n_i}}x \right\| < \frac{\overline{R}(x) \cdot (1 + 2\epsilon)}{a} = \alpha \overline{R}(x)$$

and therefore

$$(22) \qquad \overline{R}(w) < \alpha \overline{R}(x).$$

Next we see that

$$\|w - x\| \leq \left\| w - T_{t_{n_i}}x \right\| + \left\| x - T_{t_{n_i}}x \right\|$$

and by (19) and (21) this yields

$$\|w - x\| \leq \lim_i \left\| w - T_{t_{n_i}}x \right\| + \limsup_i \left\| x - T_{t_{n_i}}x \right\|$$
$$\leq \alpha \cdot \overline{R}(x) + \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{k \cdot a}$$
$$(23) \qquad \leq \alpha \cdot \overline{R}(x) + \frac{S \cdot \alpha \cdot \overline{R}(x)}{k} = \alpha \cdot \left(1 + \frac{S}{k}\right) \cdot \overline{R}(x).$$

Let us now consider the second case. By (20) we have

$$\left\| x - T_{t_j}x \right\| > \frac{S \cdot \overline{R}(x) \cdot (1 + \epsilon)}{k \cdot a}$$

for some j. Let us choose a weakly convergent $\left\{T_{t_{n_i}}y\right\}$ such that

$$\left\| x - T_{t_{n_i}}y \right\| < \overline{R}(x) \cdot \left(1 + \frac{\epsilon}{2}\right)$$

and

$$\sum_{m=1}^{t_j} \left\| T_{t_{n_i}+m-1}y - T_{t_{n_i}+m}y \right\| < \overline{R}(x) \cdot \frac{\epsilon}{2}$$

for each i. This implies

(24)

$$\left\| x - T_{t_{n_i}+t_j}y \right\| < \left\| x - T_{t_{n_i}}y \right\| + \sum_{m=1}^{t_j} \left\| T_{t_{n_i}+m-1}y - T_{t_{n_i}+m}y \right\| < \overline{R}(x) \cdot (1+\epsilon)$$

and

$$(25) \qquad \left\| T_{t_j}x - T_{t_{n_i}+t_j}y \right\| \le k \cdot \overline{R}(x) \cdot (1+\epsilon)$$

for every i. By (18) we can choose λ such that

$$(26) \qquad \frac{k}{S} < \lambda < \frac{\frac{b}{a}-1}{k-1}.$$

Then by (24), (25) and (26) we have

$$\left\| \lambda T_{t_j}x + (1-\lambda)x - T_{t_{n_i}+t_j}y \right\|$$

$$\le \lambda \left\| T_{t_j}x - T_{t_{n_i}+t_j}y \right\| + (1-\lambda) \left\| x - T_{t_{n_i}+t_j}y \right\|$$

$$\le \lambda \cdot k \cdot \overline{R}(x) \cdot (1+\epsilon) + (1-\lambda) \cdot \overline{R}(x) \cdot (1+\epsilon)$$

$$(27) \qquad = [\lambda(k-1)+1] \cdot \overline{R}(x) \cdot (1+\epsilon) < b \cdot \frac{\overline{R}(x) \cdot (1+\epsilon)}{a}$$

and next by (20) and (26),

$$\left\| \lambda T_{t_j}x + (1-\lambda)x - x \right\| = \lambda \left\| T_{t_j}x - x \right\|$$

$$(28) \qquad > \lambda \cdot \frac{S \cdot \overline{R}(x) \cdot (1+\epsilon)}{k \cdot a} > \frac{\overline{R}(x) \cdot (1+\epsilon)}{a}.$$

Directly from the definition of b and by (24), (27) and (28) there exists $w \in C$ such that

$$(29) \qquad \left\| w - T_{t_{n_i}+t_j}y \right\| \le \frac{\overline{R}(x) \cdot (1+\epsilon)}{a} \le \alpha \cdot \overline{R}(x)$$

for every i. Therefore by the asymptotic regularity of $\mathbf{T}$ and (29) we get

$$\overline{R}(w) \le \limsup_i \left\| w - T_{t_{n_i}}y \right\|$$

$$\le \limsup_i \left\| w - T_{t_{n_i}+t_j}y \right\| + \limsup_i \left\| T_{t_{n_i}+t_j}y - T_{t_{n_i}}y \right\|$$

$$(30) \qquad = \limsup_i \left\| w - T_{t_{n_i}+t_j}y \right\| \le \alpha \cdot \overline{R}(x),$$

and by (24) and (29) we obtain

$$\| w - x \| \le \left\| w - T_{t_{n_i}+t_j}y \right\| + \left\| x - T_{t_{n_i}+t_j}y \right\|$$

$$(31) \qquad \le \alpha \cdot \overline{R}(x) + \overline{R}(x) \cdot (1+\epsilon) = (1+\epsilon+\alpha) \cdot \overline{R}(x).$$

Thus in both cases for every $x \in C$ we can find $w \in C$ for which the following inequalities are valid:

(32) $$\overline{R}(w) \le \alpha \cdot \overline{R}(x)$$

and

(33)
$$\|w - x\| \le \max\left\{1 + \epsilon + \alpha, \alpha \cdot \left(1 + \frac{S}{k}\right)\right\} \cdot \overline{R}(x) = M \cdot \overline{R}(x)$$

(by (22), (23), (30) and (31)). This allows us to define a function $f : C \to C$ by $f(x) = w$, where w satisfies (32) and (33). We introduce a sequence $\{x_n\}$ as follows: x_0 is chosen in an arbitrary way and next $x_n = f(x_{n-1})$ for $n = 1, 2, \dots$. The sequence $\{x_n\}$ is a Cauchy sequence. Indeed, the inequalities (32) and (33) lead to

$$\|x_n - x_{n-1}\| \le M \cdot \overline{R}(x_{n-1}) \le M \cdot \alpha^{n-1} \cdot \overline{R}(x_0).$$

Setting $\widetilde{x}$ as the limit of $\{x_n\}$ and applying the inequality

$$\left|\overline{R}(u) - \overline{R}(v)\right| \le \|u - v\|,$$

which is valid for all $u, v \in C$, we get $\overline{R}(\widetilde{x}) = 0$.

Now we prove that $\overline{R}(\widetilde{x}) = 0$ implies that x is a common fixed point of **T**. To prove this fact it is sufficient to observe that for every $y \in C$, $t \in G$ and every natural n we have

$$\|\widetilde{x} - T_t\widetilde{x}\| \le \|\widetilde{x} - T_{t_n}y\| + \|T_{t_n}y - T_{t_n+t}y\| + \|T_{t_n+t}y - T_t\widetilde{x}\|$$

$$\le (1 + k)\|\widetilde{x} - T_{t_n}y\| + \|T_{t_n}y - T_{t_n+t}y\|.$$

Applying the definition of $\overline{R}$ and the asymptotic regularity of **T** we complete the proof. ∎

4. EXAMPLES

Example 4.1. *Let us consider the space X_β^p [6], where $1 < p < \infty$ and $1 < \beta < 4^{\frac{1}{p}}$. Then for $2^{\frac{1}{p}} \le \beta < 4^{\frac{1}{p}}$ we have $\kappa_\omega\left(X_\beta^p\right) = WCS\left(X_\beta^p\right) = N\left(X_\beta^p\right) = 1$ [2], but $w\text{-}AN\left(X_\beta^p\right) = w\text{-}SOC\left(X_\beta^p\right) = \frac{4^{\frac{1}{p}}}{\beta} > 1$ [6].*

Example 4.2. *Let us observe that for the space X_β^2 with $1 < \beta < \frac{\sqrt{5}}{2}$, we get $WCS\left(X_\beta^2\right) = \max\left(1, \frac{\sqrt{2}}{\beta}\right)$ [2,3], $w\text{-}AN\left(X_\beta^2\right) = w\text{-}SOC\left(X_\beta^2\right) = \sqrt{2}$ [6] and $\kappa_\omega(X) \ge \kappa_0(X)$ [10], where*

$$\kappa_0\left(X_\beta^2\right) = \left(1 + \frac{1}{\beta^2} - \frac{2}{\beta^2}\sqrt{\beta^2 - 1}\right)^{\frac{1}{2}}$$

[9] (see also [35]). Hence for β sufficiently close to 1 the constants

$$\frac{1 + \sqrt{1 + 4 \cdot WCS\left(X_\beta^2\right) \cdot \left(\kappa_0\left(X_\beta^2\right) - 1\right)}}{2}$$

$$< \frac{1 + \sqrt{1 + 4 \cdot \left[w\text{-}SOC\left(X_\beta^2 \right) \right] \cdot \left(\kappa_0\left(X_\beta^2 \right) - 1 \right)}}{2}$$

are strictly bigger than $\sqrt{w\text{-}SOC\left(X_\beta^2 \right)} = 2^{\frac{1}{4}}$. *Let us observe in addition that* [16]

$$\epsilon_0\left(X_\beta^2 \right) = \begin{cases} 2\left(\beta^2 - 1 \right)^{\frac{1}{2}} & \text{for } 1 < \beta \leq \sqrt{2} \\ 2 & \text{for } \sqrt{2} < \beta < 2 \end{cases}$$

and therefore for $1 < \beta < \frac{\sqrt{5}}{2}$ *the constant* $\overline{\overline{\gamma}}$ *given by* (10) *is strictly bigger than the constant* γ_2 *defined by* (6).

5. ACKNOWLEDGMENTS

REFERENCES

[1] A. G. AKSOY and M. A. KHAMSI, Nonstandard methods in fixed point theory, Springer-Verlag, New York, 1990.

[2] J. M. AYERBE TOLEDANO, T. DOMÍNGUEZ BENAVIDES and G. LOPEZ ACEDO, Measures of noncompactness in metric fixed point theory, Birkhäuser, Basel, 1997.

[3] J. M. AYERBE and H.-K. XU, On certain geometric coefficients of Banach spaces related to fixed point theory, Panamer. Math. J. **3** (1993), 47–59.

[4] J.-B. BAILLON, Quelques aspects de la théorie des points fixes dans les espaces de Banach, I, Séminaire d'Analyse Fonctionnelle de l'Ecole Polytechnique, Paris, no. VII, 1978–1979.

[5] F. E. BROWDER and W. V. PETRYSHYN, The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. **72** (1966), 571–576.

[6] M. BUDZYŃSKA, T. KUCZUMOW and S. REICH, Uniform asymptotic normal structure, the uniform semi-Opial property and fixed points of asymptotically regular uniformly lipschitzian semigroups, Part I, Abstr. Appl. Anal. **3** (1998), 133–151.

[7] E. CASINI and E. MALUTA, Fixed points of uniformly lipschitzian mappings in spaces with uniformly normal structure, Nonlinear Anal. **9** (1995), 103–108.

[8] J. A. CLARKSON, Uniformly convex spaces, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[9] T. DOMÍNGUEZ BENAVIDES, Fixed point theorems for uniformly lipschitzian mappings and asymptotically regular mappings, Nonlinear Anal. **32** (1998), 15–27.

[10] T. DOMÍNGUEZ BENAVIDES and H.-K. XU, A new geometrical coefficient for Banach spaces and its applications in fixed point theory, Nonlinear Anal. **25** (1995), 311–325.

[11] D. J. DOWNING and W. O. RAY, Uniformly lipschitzian semigroups in Hilbert space, Canad. Math. Bull. **25** (1982), 210–213.

[12] D. J. DOWNING and B. TURETT, Some properties of the characteristic of convexity relating to fixed point theory, Pacific J. Math. **104** (1983), 343–350.

[13] M. EDELSTEIN, The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. **78** (1972), 206–208.

[14] K. GOEBEL and W. A. KIRK, A fixed point theorem for transformations whose iterates have uniform Lipschitz constant, Studia Math. **47** (1973), 135–140.

[15] K. GOEBEL, W. A. KIRK and R. L. THELE, Uniformly lipschitzian families of transformations in Banach spaces, Canad. J. Math. **26** (1974), 1245–1256

[16] K. GOEBEL and W. A. KIRK, Topics in metric fixed point theory, Cambridge Univ. Press, Cambridge, 1990.

[17] K. GOEBEL and S. REICH, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Marcel Dekker, New York, 1984.

[18] J. GÓRNICKI, A fixed point theorem for asymptotically regular mappings, Colloq. Math. **64** (1993), 55–57.

[19] J. GÓRNICKI, Fixed points of asymptotically regular mappings in metric spaces, Demonstratio Math. **29** (1996), 615–620.

[20] J. GÓRNICKI, Fixed points of asymptotically regular semigroups in Banach spaces, Rend. Circ. Mat. Palermo **46** (1997), 89–118.

[21] H. ISHIHARA and W. TAKAHASHI, Fixed point theorems for uniformly lipschitzian semigroups in Hilbert spaces, J. Math. Anal. Appl. **127** (1987), 206–210.

[22] T. H. KIM and W. A. KIRK, Fixed point theorems for lipschitzian mappings in Banach spaces, Nonlinear Anal. **26** (1996), 1905-1911.

[23] E. A. LIFSHITZ, Fixed point theorems for operators in strongly convex spaces, Voronez. Gos. Univ. Trudy Mat. Fak. **16** (1975), 23–28.

[24] T. C. LIM, Fixed point theorems for uniformly lipschitzian mappings in L^p spaces, Nonlinear Anal. **7** (1983), 555–563.

[25] T. C. LIM, Some L^p inequalities and their applications to fixed point theorems of uniformly lipschitzian mappings, Proc. Sympos. Pure Math. **45** (1986), 119–125.

[26] T. C. LIM and H.-K. XU, Uniformly lipschitzian mappings in metric spaces with uniform normal structure, Nonlinear Anal. **25** (1995), 1231–1235.

[27] T. C. LIM, H.-K. XU and Z.-B. XU, An L^p inequality and its applications to fixed point theory and approximation theory, Progress in Approximation Theory, (P. Nevai, A. Pinkus, Eds.), Academic Press, (1991), 609–624.

[28] B. PRUS and R. SMARZEWSKI, Strongly unique best approximations and centers in uniformly convex spaces, J. Math. Anal. Appl. **121** (1987), 10–21.

[29] R. SMARZEWSKI, Strongly unique minimization of functionals in Banach spaces with applications to the theory of approximation and fixed points, J. Math. Anal. Appl. **115** (1986), 155–172 .

[30] K.-K. TAN and H.-K. XU, Fixed point theorems for lipschitzian semigroups in Banach spaces, Nonlinear Anal. **20** (1993), 395–404.

[31] D. TINGLEY, Noncontractive uniformly lipschitzian semigroups in Hilbert space, Proc. Amer. Math. Soc. **92** (1984), 355–361.

[32] H.-K. XU, Fixed point theorems for uniformly lipschitzian semigroups in uniformly convex spaces, J. Math. Anal. Appl. **152** (1990), 391–398.

[33] H.-K. XU, Random fixed point theorems for nonlinear uniformly lipschitzian mappings, Nonlinear Anal. **26** (1996), 1301–1311.

[34] L.-C. ZENG, On the existence of fixed points and nonlinear ergodic retractions for lipschitzian semigroups without convexity, Nonlinear Anal. **24** (1995), 1347–1359.

[35] W. ZHAO, Geometrical coefficients and measures of noncompactness, Ph.D. Dissert., Univ. Glasgow, 1992.

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